UNIT 14

SPIN ANGULAR MOMENTUM

Structure

14.1 INTRODUCTION

In this unit we study about spin angular momentum. In the last unit we saw how the z-component of the angular momentum operator could have values that are either integral and half integral multiples of $\hbar$. So the total angular momentum $\hat{J}$ was written as the sum of the familiar orbital angular momentum, $\hat{L}$, and the spin angular momentum $\hat{S}$. We now know several particles in nature that have this property of spin angular momentum like electrons, protons, neutrons, photons to name a few. How do we know that quantum particles actually possess this property of spin? We know this because it is measureable in experiment.

Since angular momentum is an observable, it is measurable. It is actually related to the magnetic moment of an atom. Therefore, if we are able to measure magnetic moment of a quantum particle, we can measure its angular momentum. One such experiment was devised in 1922 by O. Stern and W. Gerlach to measure the magnetic moment of the electron. Recall that in Unit 9 you were introduced to the three quantum numbers (*n*, *l*, *ml*). However, the quantitative results obtained from the Stern-Gerlach experiment could not be explained with the help of only three quantum numbers. In 1925, Pauli formulated the exclusion principle and the idea of 4 quantum numbers for electron. In the same year, the existence of the spin property of the electron

was postulated by two research students S.A. Goudsmit and G.E. Uhlenbeck to explain observed spectroscopic data. They hypothesized that every electron possesses a spin angular momentum $\pm\frac{\hbar}{2}$.

Those of you who are interested may like to read about the history of the hypothesis of "spin" in Sec. 13.7 of the book: ***Quantum Mechanics: Theory and Applications*** by Ajoy Ghatak and S. Lokanathan, 5th Edition, Macmillan India Ltd.

Although spin is described as an angular momentum, it is not the kind of angular momentum associated with motion in space (like $\vec{L}=\vec{r}\times\vec{p}$). For example, suppose in an experiment, an electron is prepared in a state ψ of zero momentum $\left(-i\hbar\vec{\nabla}\psi=0\right)$. Then $\hat{L}_x\psi=\hat{L}_y\psi=\hat{L}_z\psi=0$ and so if any component of the electron's angular momentum is measured, the result should be 0. That is true if you assume that the angular momentum of the electron is just the quantum mechanical analogue of the classical angular momentum. What you actually get is $\pm\frac{\hbar}{2}$. So, the electron has an "intrinsic" angular momentum, or a quantum degree of freedom, called its "spin" which has nothing to do with its actual motion. The nomenclature of "spin" is misleading because it creates the impression the electron is spinning about its axis, like a top – though that is not the case. Although initially it was believed, that if the electron has an angular momentum when it is not moving in space - it must be rotating about its axis.

Since the spin angular momentum does not have a classical analogue, we cannot follow the standard practice and derive a quantum mechanical spin angular momentum operator from its classical counterpart (by replacing *x* and *p* in the classical expression by their corresponding operators as postulated in Unit 4). We therefore start by establishing the mathematical framework for the spin angular momentum by describing the spin angular momentum of the electron in Sec. 14.2. We do this by extending the angular momentum algebra we studied in Unit 13 to the spin angular momentum operator and assuming that the electron's spin angular momentum can have values $\pm\frac{\hbar}{2}$. We derive the spin angular momentum matrices, show how the spin angular momentum operators can be written in terms of the Pauli spin matrices and also use the two-component Pauli formalism to write the electron spin eigenkets and the complete wave function. The formalism is developed for the spin angular momentum of the electron, with the understanding that the spin of other particles must be studied in the same way.

In Sec 14.3 we derive the magnetic moment associated with the orbital and spin angular momentum. In Sec. 14.4 we describe the Stern Gerlach experiment and explain the results using the idea of the spin magnetic moment of the electron. In Sec. 14.5 we briefly discuss spin dynamics.

Expected Learning Outcomes

After studying this unit, you should be able to:

- state the properties of the spin angular momentum operator;
- derive the spin angular momentum operators for the electron in terms of the eigenkets of $\hat{S}_z$;
- write the spin angular momentum matrices for the electron in terms of the Pauli matrice and derive the eigenkets $\hat{S}_z$;

- write the general spin state of the electron as a two-component spinor and represent the spin angular momentum operators of the electron in terms of the Pauli matrices;
- describe the electron wave function in the spinor formalism;
- derive the orbital and spin magnetic moment of the electron;
- describe the Stern-Gerlach experiment and explain the results; and
- describe the time evolution of the spin state of the electron.

14.2 SPIN ANGULAR MOMENTUM

While we do not know the form of the spin angular momentum operator, we can deduce some of its properties. Let us first look at these.

14.2.1 Properties of the Spin Angular Momentum Operators

We start with the definition of the total angular momentum (Eq. 13.42) of Unit 13:

$$\hat{J} = \hat{L} + \hat{S} \tag{14.1}$$

As we know, $\hat{J}$ and $\hat{L}$ are Hermitian operators, with the components $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$ and $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ respectively. Therefore we can say that $\hat{S}$ is also a Hermitian operator with the components $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$. In component form we can write Eq. (14.1) as:

$$\hat{J}_x = \hat{L}_x + \hat{S}_x; \hat{J}_y = \hat{L}_y + \hat{S}_y; \hat{J}_z = \hat{L}_z + \hat{S}_z \tag{14.2}$$

Given that $\hat{J}$ and $\hat{L}$ satisfy the commutation relations:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z; [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x; [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y \tag{14.3a}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z; [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x; [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{14.3b}$$

We can write:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \Rightarrow [\hat{L}_x + \hat{S}_x, \hat{L}_y + \hat{S}_y] = i\hbar(\hat{L}_z + \hat{S}_z) \tag{14.4}$$

We don't know what $\hat{S}$ looks like at this stage, but we do know that $\hat{L}$ can be written in the coordinate representation and also in terms of the angular variables θ and ϕ (Unit 4, Unit 8) and $\hat{S}$ cannot. Therefore $\hat{L}$ and $\hat{S}$ are commuting operators. With this condition, for Eq. (14.4) to hold, given Eq. (14.3a and b) we must have $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$. In general, the same commutation rules should hold for the spin angular momentum operator $\hat{S}$ (and its components $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$) that hold for $\hat{L}$ and $\hat{J}$:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z; [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x; [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \tag{14.5a}$$

In the vector notation this relation is written as:

$$\hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hbar\hat{\mathbf{S}} \quad \text{or} \quad [\hat{S}_i, \hat{S}_j] = i\hbar\sum_{k=1}^{3}\varepsilon_{ijk}\hat{S}_k \tag{14.5b}$$

and $\hat{S}^2 = \hat{S}_x{}^2 + \hat{S}_y{}^2 + \hat{S}_z{}^2$ (14.6)

Also: $\left[\hat{S}^2, \hat{S}_z\right] = 0$ (14.7)

It is also true that $\left[\hat{S}^2, \hat{S}_x\right] = 0$ and $\left[\hat{S}^2, \hat{S}_y\right] = 0$. However since $\hat{S}_x, \hat{S}_y, \hat{S}_z$ do not commute among themselves, we choose the eigenkets of the spin angular momentum operator to be the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$.

To write down the spin angular momentum operator for the electron (or rather any other spin half particle like the neutron or the proton), we start with the result that the electron spin angular momentum in the *z*-direction has the value $\pm\frac{\hbar}{2}$ (Uhlenbeck and Goudsmit, 1925). So we are looking for an operator $\hat{S}$ for which $\hat{S}_z$ has the eigenvalues $\pm\frac{\hbar}{2}$ and which satisfies the commutation rules of Eq. (14.5) and (14.7). Clearly there will be two eigenkets, and if we label them as $|\phi_1\rangle$ and $|\phi_2\rangle$, we can write:

$$\hat{S}_z|\phi_1\rangle = \frac{\hbar}{2}|\phi_1\rangle \;;\; \hat{S}_z|\phi_2\rangle = -\frac{\hbar}{2}|\phi_2\rangle \quad (14.8a)$$

$|\phi_1\rangle$ and $|\phi_2\rangle$ are eigenkets of $\hat{S}^2$ as well.

Since $\hat{S}_z$ is Hermitian and $|\phi_1\rangle$ and $|\phi_2\rangle$ have distinct eigenvalues, $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal and form a complete set for a two dimensional Hilbert space for the spin angular momentum of the electron. They can be scaled to be normalized and so we can write:

$$\langle\phi_i|\phi_i\rangle = \delta_{ij}, \quad i, j = 1,2 \quad (14.8b)$$

And

$$|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = \hat{\mathbf{I}} \quad (14.8c)$$

Using the spectral representation (SAQ 6a of Unit 10), $\hat{S}_z$ can be written as:

$$\hat{S}_z = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|] \quad (14.8d)$$

You can check that this is the correct form of the operator $\hat{S}_z$.

SAQ 1

Show that for $\hat{S}_z$ defined by Eq. (14.8d), $\hat{S}_z|\phi_1\rangle = \frac{\hbar}{2}|\phi_1\rangle$ and $\hat{S}_z|\phi_2\rangle = -\frac{\hbar}{2}|\phi_2\rangle$.

We next write down the spin angular momentum matrices for the electron.

14.2.2 Spin Angular Momentum Matrices for the Electron

Recall that in Unit 13, we have written down the angular momentum matrices for $j = 1/2$:

$$J_x = \frac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; J_y = \frac{1}{2}\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}; J_z = \frac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (14.9)$$

You can check for yourself that:

- The eigenvalues of J_z (J_x and J_y also) are $\pm\frac{\hbar}{2}$. So if a measurement of J_z is made (or J_x or J_y) you would get $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.
- The matrices J_x, J_y and J_z satisfy the commutation rules (TQ5, Unit 13):

$$[J_x, J_y] = i\hbar J_z; [J_y, J_z] = i\hbar J_x; [J_z, J_x] = i\hbar J_y .$$

The matrices J_x, J_y and J_z hence are an appropriate representation of the electron spin angular momentum operators. So the spin angular momentum matrices of the electron S_x, S_y and S_z are:

$$S_x = \frac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; \; S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}; \; S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (14.10)$$

We can also write the matrices as:

$$S_x = \frac{\hbar}{2}\sigma_x; S_y = \frac{\hbar}{2}\sigma_y; S_z = \frac{\hbar}{2}\sigma_z \quad (14.11)$$

where $\sigma_x, \sigma_y, \sigma_z$ are the **Pauli spin matrices** defined as

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; \; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}; \; \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (14.12)$$

We can now also write the matrix for the operator S^2 (Eq. 13.64 of Unit 13) as

$$S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad (14.13)$$

And we have:

$$\hat{S}^2|\phi_1\rangle = \frac{3}{4}\hbar^2|\phi_1\rangle \; ; \; \hat{S}^2|\phi_2\rangle = \frac{3}{4}\hbar^2|\phi_2\rangle \quad (14.14)$$

Designating the angular momentum by a value *s* (in place of *j*), we have the general result for $s = \frac{1}{2}$:

$$\hat{S}^2|\phi_i\rangle = s(s+1)\hbar^2|\phi_i\rangle \quad i = 1, 2 \quad (14.15)$$

And

$$\hat{S}_z|\phi_i\rangle = s_z\hbar|\phi_i\rangle \quad i = 1, 2 \quad (14.16)$$

Where the values of s_z are $s_z = -s$ and $s_z = -s + 1 = s$.

Recall that the angular momentum matrices were defined in the basis $|j, m_j\rangle$.

For $j = \frac{1}{2}$ the basis eigenkets were $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$. Notice that both S^2 and S_z are diagonal in this basis and therefore will have simultaneous

eigenvectors. The eigenvectors of the matrices S^2 and S_z can be determined as column vectors and these will be the simultaneous eigenkets $|\phi_1\rangle$ and $|\phi_2\rangle$ of the operators $\hat{S}^2$ and $\hat{S}_z$. You will find (SAQ 2b) that:

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14.17a}$$

And

$$\langle\phi_1| = (1 \quad 0); \quad \langle\phi_2| = (0 \quad 1) \tag{14.17b}$$

SAQ 2

a) Show that in the basis $|\phi_1\rangle$ and $|\phi_2\rangle$ defined in Eq. (14.8a)

$$\hat{S}_x = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|]$$

and

$$\hat{S}_y = \frac{i\hbar}{2}[-|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|]$$

b) Show that:

i) the eigenvalues of S_x, S_y and S_z are $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

ii) the eigenvectors of S_z are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

iii) the eigenvectors of S_z are also the eigenvectors of S^2 with the eigenvalue $\frac{3}{4}\hbar$.

The eigenvectors $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are also referred to as the "spin-up" and "spin-down" states, respectively for a spin-half particle. If the *z*-component of the spin angular momentum is measured for an electron in the state $|\phi_1\rangle$, the result obtained would be $\frac{\hbar}{2}$. If the *z*-component of the spin angular momentum is measured for an electron in the state $|\phi_2\rangle$, the result obtained would be $-\frac{\hbar}{2}$. The states $|\phi_1\rangle$ and $|\phi_2\rangle$ are often represented as $|\uparrow\rangle$ and $|\downarrow\rangle$ or $|+\rangle$ and $|-\rangle$,respectively, to indicate their "**spin-up"** and "**spin-down**" character. For the rest of this unit we designate $|\phi_1\rangle$ and $|\phi_2\rangle$ by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

We next write down the wave function for the electron, which, as you now know, has two spin angular momentum states. For this one uses a formalism called the Pauli two–component spinor formalism (introduced by W. Pauli in 1927).

SAQ 3

Show that the eigenvectors for $\hat{S}_x$ are

$$|S_{x\uparrow}\rangle = \left|S_x = \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

and

$$|S_{x\downarrow}\rangle = \left|S_x = -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

14.2.3 Pauli Two-Component Formalism

Let $|\phi\rangle$ be the general **spin state** of the electron in space. Let us write $|\phi\rangle$ as a linear combination of the eigenkets $|\uparrow\rangle$ and $|\downarrow\rangle$ as:

$$|\phi\rangle = C_1|\uparrow\rangle + C_2|\downarrow\rangle = \langle\uparrow|\phi\rangle|\uparrow\rangle + \langle\downarrow|\phi\rangle|\downarrow\rangle \qquad (14.18)$$

$|\uparrow\rangle$ and $|\downarrow\rangle$ are column vectors and $|\phi\rangle$ is also a column matrix

$$|\phi\rangle = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} \qquad (14.19a)$$

And the adjoint $\langle\phi|$ is a row matrix:

$$\langle\phi| = \begin{pmatrix} C_1^* & C_2^* \end{pmatrix} = \begin{pmatrix} \langle\phi|\uparrow\rangle & \langle\phi|\uparrow\rangle \end{pmatrix} \qquad (14.19b)$$

The column vector of Eq. (14.19a) is called a **two-component spinor**.
Designating the state $|\phi\rangle$ by χ to show that it is a **spinor**, we can write Eqs. (14.19a and b) as:

$$\chi = C_1\chi_\uparrow + C_2\chi_\uparrow \qquad (14.20a)$$

$$\chi^\dagger = C_1^*\chi_\uparrow^\dagger + C_2^*\chi_\downarrow^\dagger \qquad (14.20b)$$

You must understand that these are equivalent descriptions and so:

$$\chi \equiv |\phi\rangle = \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix}; \ \chi_\uparrow \equiv |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \ \chi_\downarrow \equiv |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (14.21)$$

Similarly:

$$\chi^\dagger \equiv \langle\phi| = \begin{pmatrix} \langle\phi|\uparrow\rangle & \langle\phi|\uparrow\rangle \end{pmatrix};$$

$$\chi_\uparrow^\dagger \equiv \langle\uparrow| = (1 \quad 0); \ \chi_\downarrow^\dagger \equiv \langle\downarrow| = (0 \quad 1) \qquad (14.22)$$

Now let us consider the action of $\hat{S}_z$ on $|\phi\rangle$:

$$\hat{S}_z|\phi\rangle = |\phi'\rangle \qquad (14.23)$$

As in Eq. (14.19a), we can write $|\phi'\rangle$ in terms of its components which is another spinor, say χ':

$$|\phi'\rangle = \begin{pmatrix} \langle\uparrow|\phi'\rangle \\ \langle\downarrow|\phi'\rangle \end{pmatrix} \equiv \chi' \qquad (14.24)$$

Using $\hat{\mathbf{I}} = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ we can write

Now $$\langle\uparrow|\phi'\rangle = \langle\uparrow|\hat{S}_z|\phi\rangle = \langle\uparrow|\hat{S}_z\left[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right]|\phi\rangle$$

$$= \langle\uparrow|\hat{S}_z|\uparrow\rangle\langle\uparrow|\phi\rangle + \langle\uparrow|\hat{S}_z|\downarrow\rangle\langle\downarrow|\phi\rangle \quad (14.25a)$$

Similarly

$$\langle\downarrow|\phi'\rangle = \langle\downarrow|\hat{S}_z|\phi\rangle = \langle\downarrow|\hat{S}_z\left[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right]|\phi\rangle$$

$$= \langle\downarrow|\hat{S}_z|\uparrow\rangle\langle\uparrow|\phi\rangle + \langle\downarrow|\hat{S}_z|\downarrow\rangle\langle\downarrow|\phi\rangle \quad (14.25b)$$

So

$$|\phi'\rangle \equiv \chi' = \begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\uparrow|\hat{S}_z|\downarrow\rangle \\ \langle\downarrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} \quad (14.26)$$

Now from Eqs. (14.8a) we can write: $\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$; $\hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$ (SAQ 1)

and

$$\langle\uparrow|\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}; \langle\uparrow|\hat{S}_z|\downarrow\rangle = 0; \langle\downarrow|\hat{S}_z|\uparrow\rangle = 0;$$

$$\langle\downarrow|\hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2} \quad (14.27)$$

So Eq. (14.26) reduces to:

$$|\phi'\rangle \equiv \chi' = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} \quad (14.28)$$

Using the definition of the Pauli spin matrices from Eq. (14.12), Eq. (14.28) is:

$$\chi' = \frac{\hbar}{2}\sigma_z\chi \quad (14.29)$$

So effectively, the operator $\hat{S}_z$ is the matrix $\frac{\hbar}{2}\sigma_z$:

$$\hat{S}_z \Leftrightarrow \frac{\hbar}{2}\sigma_z \quad (14.30a)$$

Similarly it can be shown that:

$$\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\sigma_x \quad (14.30b)$$

$$\hat{S}_y \Leftrightarrow \frac{\hbar}{2}\sigma_y \quad (14.30c)$$

The expectation value of the operator $\hat{S}_z$ in the state $|\phi\rangle$ is

$$\langle\phi|\hat{S}_z|\phi\rangle = \langle\phi|\phi'\rangle$$

$$= \left(\langle\phi|\uparrow\rangle \ \langle\phi|\uparrow\rangle\right) \begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\uparrow|\hat{S}_z|\downarrow\rangle \\ \langle\downarrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} \quad (14.31)$$

Which we can write as:

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2} \chi^\dagger \sigma_z \chi \qquad (14.32)$$

So the expectation value of $\hat{S}_z$ is written in terms of the spinor and Pauli spin matrix in a compact way (Compare Eqs. 14.31 and 14.32).

We can define similar relations for $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$.

Properties of the Pauli Spin matrices

- The Pauli spin matrices anti-commute so

$$\{\sigma_i, \sigma_j\} = 0 \quad \text{where } i, j = x, y, z \qquad (14.33a)$$

- The trace of each Pauli matrix is zero.
- The square of each Pauli matrix is the identity matrix

$$\sigma_i{}^2 = I \quad \text{where } i = x, y, z \text{ and } I \text{ is the } 2 \times 2 \text{ identity matrix.} \qquad (14.33b)$$

- The Pauli matrices follow the commutation rules:

$$[\sigma_x, \sigma_y] = 2i\sigma_z \; ; \; [\sigma_y, \sigma_z] = 2i\sigma_x \; ; [\sigma_z, \sigma_x] = 2i\sigma_y \qquad (14.33c)$$

SAQ 4

a) Show that $\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\sigma_x$ and $\hat{S}_y \Leftrightarrow \frac{\hbar}{2}\sigma_y$.

b) Show that $\{\sigma_x, \sigma_y\} = 0 = \{\sigma_y, \sigma_z\} = \{\sigma_z, \sigma_x\}$.

c) Show that $[\sigma_x, \sigma_y] = 2i\sigma_z ; [\sigma_y, \sigma_z] = 2i\sigma_x ; [\sigma_z, \sigma_x] = 2i\sigma_y$.

d) Show that $\sigma_x\sigma_y = i\sigma_z$; $\sigma_y\sigma_z = i\sigma_x$ and $\sigma_z\sigma_x = i\sigma_y$.

14.2.4 Electron Wavefunction

The probability density of locating a quantum particle at a point *x*, is defined through its wave function $\psi(x) = \langle x | \psi \rangle$ where $|x\rangle$ are the eigenkets of the position operator defined by $\hat{x}|x\rangle = x|x\rangle$. As you have seen in Unit 11, we solve the stationary state Schrodinger equation $\hat{H}|E\rangle = E|E\rangle$, where $\hat{H}$ is typically a function of the position and momentum operators, to determine $\psi_E(x) = \langle x | E \rangle$.

However if the quantum particle is the electron, which also has spin angular momentum, the wave function $\psi(x)$ would not define the spin angular momentum state of the electron at *x*, which could either $\frac{\hbar}{2}$ (spin-up) or $-\frac{\hbar}{2}$ (spin-down) irrespective of its location. You cannot therefore determine, for example, the probability of locating the electron at *x* in the spin up state. So instead of representing the state of the electron at *x by a single wave function* $\psi(x)$ it makes more sense to represent the state by two functions $\psi_\uparrow(x)$ and $\psi_\downarrow(x)$, where

- $\psi_{\uparrow}(x)$ defines the probability of locating the particle at the point *x* in the **spin up** state, that is with the *z*-component of the spin angular momentum $\frac{\hbar}{2}$, and
- $\psi_{\downarrow}(x)$ defines the probability of locating the particle at the point *x* in the **spin down** state, that is with the z-component of the spin angular momentum $-\frac{\hbar}{2}$.

So now the electron is described by a two-component wave function in the coordinate basis. Therefore you can see that the size of the Hilbert space for the electron is double of what it would have been if it had no spin degree of freedom.

The wave function at *x* can be written in the spinor formalism as a two component spinor as:

$$\psi_P(x) = \begin{pmatrix} \psi_{\uparrow}(x) \\ \psi_{\downarrow}(x) \end{pmatrix} \tag{14.34a}$$

$\psi_P(x)$ is a two- component wave function which is a 2×1 matrix, different from the scalar wave function $\psi(x)$ which is just a complex number (or a scalar so to say). In general in three dimensions,

$$\psi_P(x,y,z) = \begin{pmatrix} \psi_{\uparrow}(x,y,z) \\ \psi_{\downarrow}(x,y,z) \end{pmatrix} \tag{14.34b}$$

We denote the column vector of Eq. (14.34a or b) by the spinor χ_P ($\psi_P(x,y,z) \equiv \chi_P$) and we can write:

$$\chi_P \equiv \psi_P(x,y,z) = \psi_{\uparrow}(x,y,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_{\downarrow}(x,y,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_{\uparrow}\chi_{\uparrow} + \psi_{\downarrow}\chi_{\downarrow} \tag{14.35}$$

where

- $\int |\psi_{\uparrow}(x,y,z)|^2 d\tau$ is the probability of obtaining the spin up state $|\uparrow\rangle$ at (x,y,z), and
- $\int |\psi_{\downarrow}(x,y,z)|^2 d\tau$ is the probability of obtaining the spin up state $|\downarrow\rangle$ at (x,y,z).

Further:

$$\int |\psi_{\uparrow}|^2 d\tau + \int |\psi_{\downarrow}|^2 d\tau = 1 \tag{14.36}$$

If the Hamiltonian for the electron depends on the position, momentum and spin operators, the general stationary state Schrodinger equation of the electron is:

$$\hat{H}\psi_P = E\psi_P \Rightarrow \hat{H}\begin{pmatrix} \psi_{\uparrow}(x,y,z) \\ \psi_{\downarrow}(x,y,z) \end{pmatrix} = E\begin{pmatrix} \psi_{\uparrow}(x,y,z) \\ \psi_{\downarrow}(x,y,z) \end{pmatrix} \tag{14.37a}$$

where $\hat{H}$ is in general a 2×2 matrix. Eq. (14.37a) can also be written as the spinor equation:

$$\hat{H}\chi_P = E\chi_P \tag{14.37b}$$

And the time-dependent Schrodinger equation is:

$$i\hbar\frac{\partial}{\partial t}\psi_P(x,y,z) = \hat{H}\psi_P(x,y,z)$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_\uparrow(x,y,z)\\ \psi_\downarrow(x,y,z)\end{pmatrix} = \hat{H}\begin{pmatrix}\psi_\uparrow(x,y,z)\\ \psi_\downarrow(x,y,z)\end{pmatrix} \tag{14.38}$$

Let us illustrate what we have been discussing with the example of the Hydrogen atom. If we consider the Hamiltonian for the problem to be

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2.I + V(r).I \tag{14.39}$$

where I is the 2×2 identity matrix and the only interaction is the Coulomb interaction $V(r)$, the Schrodinger equation is:

$$\left[-\frac{\hbar^2}{2m}\nabla^2.I + V(r).I\right]\psi_P(x,y,z) = E\psi_P(x,y,z) \tag{14.40}$$

The Hamiltonian is a diagonal 2×2 matrix and Eq. (14.40) reduces to identical partial differential equations for $\psi_\uparrow(x,y,z)$ and $\psi_\downarrow(x,y,z)$.

However if the spin orbit interaction is included in the Hamiltonian, then the Hamiltonian operator is:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2.\mathbf{I} + V(r).\mathbf{I} + \alpha\frac{\hat{L}.\hat{S}}{r^3} \tag{14.41}$$

Where

$$\hat{L}.\hat{S} = \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y + \hat{L}_z\hat{S}_z$$

$$= \hat{L}_x\left(\frac{\hbar}{2}\sigma_x\right) + \hat{L}_y\left(\frac{\hbar}{2}\sigma_y\right) + \hat{L}_z\left(\frac{\hbar}{2}\sigma_z\right) \tag{14.42}$$

The Hamiltonian in this case is no longer diagonal and we get two coupled partial differential equations for $\psi_\uparrow(x,y,z)$ and $\psi_\downarrow(x,y,z)$. You will learn the technique of solving such equations using perturbation theory in your third semester course, Quantum Mechanics-II.

SAQ 5

Show that $\hat{L}.\hat{S} = \frac{1}{2}\hbar\begin{pmatrix}\hat{L}_z & \hat{L}_x - i\hat{L}_y\\ \hat{L}_x + i\hat{L}_y & -\hat{L}_z\end{pmatrix}$.

14.3 MAGNETIC MOMENT

The angular momentum of an atom is measured using its magnetic moment. Let us now describe the magnetic moment of the atom in terms of its orbital and spin angular momentum.

14.3.1 Orbital Magnetic Moment

Classical Analogue

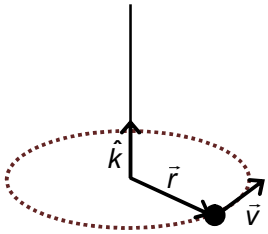

Fig. 14.1

We start with the classical description of the magnetic moment due to a charge q of mass m moving in a circular path of radius r (Fig. 14.1) with a velocity $\vec{v}$. The angular momentum $\vec{L}$ is directed perpendicular to the plane of the path and we can write:

$$\vec{L} = m(\vec{r} \times \vec{v}) = mvr\,\hat{k} \tag{14.43}$$

where $\hat{k}$ is the unit vector in the z-direction. This rotating charge constitutes a current I which is:

$I =$ charge flowng through any point of the path per second

$$= q\left(\frac{v}{2\pi r}\right) \text{ Amperes} \tag{14.44}$$

The magnetic moment μ of this current loop is the product of the current and the area of the loop so:

$$\mu = IA = q\left(\frac{v}{2\pi r}\right)(\pi r^2) = \left(\frac{q}{2m}\right)(mvr) = \left(\frac{q}{2m}\right)L \tag{14.45}$$

Notice also that $\mu/L = \left(\frac{q}{2m}\right)$. From Eq. (14.45), if the charge q is positive, the angular momentum and the magnetic moment point in the same direction, so they are parallel. If q is negative, the angular momentum and the magnetic moment point in the opposite directions and are anti-parallel. We can write Eq. (14.45) as a vector equation:

$$\vec{\mu}_L = \left(\frac{q}{2m}\right)\vec{L} \tag{14.46}$$

The subscript L on the magnetic moment indicates that we are talking about the magnetic moment due a charge moving in an orbit and $\vec{L}$ is its orbital angular momentum. The interaction Hamiltonian H^L_{int} for the magnetic moment $\vec{\mu}$ in an external magnetic field $\vec{B}$ is just its potential energy (U):

$$H^L_{\text{int}} = U = -\vec{\mu}_L.\vec{B} = -\left(\frac{q}{2m}\right)\vec{L}.\vec{B}$$

$$= -\frac{q}{2m}\left[L_xB_x + L_yB_y + L_zB_z\right] \tag{14.47}$$

The force acting on the charge in the magnetic field is just

$$\vec{F} = -\vec{\nabla}U = \vec{\nabla}\left(\vec{\mu}_L.\vec{B}\right) = \left(\frac{q}{2m}\right)\vec{\nabla}\left(\vec{L}.\vec{B}\right) \tag{14.48}$$

An electron moving inside an atom will therefore have a magnetic moment $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$ and experience a force $\vec{F} = -\left(\frac{e}{2m}\right)\vec{\nabla}\left(\vec{L}.\vec{B}\right)$. If the external magnetic field is in the z-direction $\left(\vec{B} = B_z\hat{k}\right)$, then the potential energy of the electron is $U = \left(\frac{e}{2m}\right)L_zB_z$ and the force acting on it is along the z-direction:

$$F \approx \mu_z\left(\frac{\partial B_z}{\partial z}\right) = -\left(\frac{e}{2m}\right)L_z\left(\frac{\partial B_z}{\partial z}\right) \qquad (14.49)$$

Quantum Formalism

The quantum mechanical analogue of the orbital angular momentum is $\vec{L}$ is $\hat{L}$ and we can write the magnetic moment of the electron as:

$$\hat{\mu}_L = -\left(\frac{e}{2m}\right)\hat{L} \qquad (14.50a)$$

Let us now discuss the magnetic moment due to the spin angular momentum.

14.3.2 Spin Magnetic Moment

Goudsmit and Uhlenbeck analysed a large number of spectroscopic lines and found that the results could not be explained with the help of the three quantum numbers n, l and m_l. The most familiar example is the two closely spaced D_1 and D_2 lines in the spectrum of a sodium lamp. If analysed with the three quantum numbers n, l and m_l there should have been just a single line in the sodium spectrum, corresponding the transition from 3p to 3s. To explain the observed spectroscopic data, Goudsmit and Uhlenbeck advanced the hypothesis according to which ***every electron rotates with an angular momentum*** $\frac{\hbar}{2}$***, which is the spin angular momentum, and carries, in addition to its charge, a magnetic moment equal to one Bohr Magneton*** $\left(\mu_B = \frac{e\hbar}{2m_e}\right)$.

Those of you who are interested may like to read about the history of the hypothesis of "spin" in Sec. 13.7 of the book: ***Quantum Mechanics: Theory and Applications*** by Ajoy Ghatak and S. Lokanathan, 5th Edition, Macmillan India Ltd.

This intrinsic angular momentum is not due to the rotation of the electron of the electron about its axis (SAQ 6b). The magnetic moment due to the spin angular momentum is actually

$$\vec{\mu}_S = -g\left(\frac{e}{2m}\right)\vec{S} \qquad (14.50b)$$

where $g\,(\approx 2.0023)$ is the Lande g factor, e is the electronic charge and $\vec{S}$ is the spin angular momentum. Therefore: $\hat{\mu}_S = -g\left(\frac{e}{2m}\right)\hat{S}$. The z-component of the magnetic moment can have values $\mu_z = -g\left(\frac{e}{2m}\right)\frac{\hbar}{2} = \mu_0$ or $\mu_0 = g\left(\frac{e}{2m}\right)\frac{\hbar}{2} = -\mu_0$, corresponding to the two eigenvalues of $\hat{S}_z$ which are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ for the spin up and spin down states of the electron respectively.

The interaction Hamiltonian for the spin magnetic moment and the external magnetic field is

$$\hat{H}^S_{\text{int}} = -\hat{\mu}_S.\vec{B} \qquad (14.51)$$

SAQ 6

a) Calculate the value of the Bohr magneton in SI units.

b) Show that if the spin angular momentum of the electron is attributed to the rotation of the electron about its own axis, the velocity of the electron would be greater than the velocity of light.

With this idea of the magnetic moments associated with the orbital and spin angular momentum we now describe the Stern-Gerlach experiment armed.

14.4 THE STERN-GERLACH EXPERIMENT

The experiment, originally conceived by Otto Stern in 1921, was carried out by him and W. Gerlach in 1922 in Frankfurt. The experiment was designed to verify the notion of space quantization, that is the quantization of the *z*-component of the angular momentum of the electron as proposed in the Bohr-Sommerfeld model (Unit 8).

Silver (Ag) atoms are heated in an oven, in which here is a small hole through which the silver atoms escape. This beam of silver atoms is passed through a collimator and then enters an inhomogenous magnetic field produced by a pair of pole pieces. The standard experimental arrangement is shown in Fig. 14.2. Why we use silver atoms will become clear in the next few paragraphs.

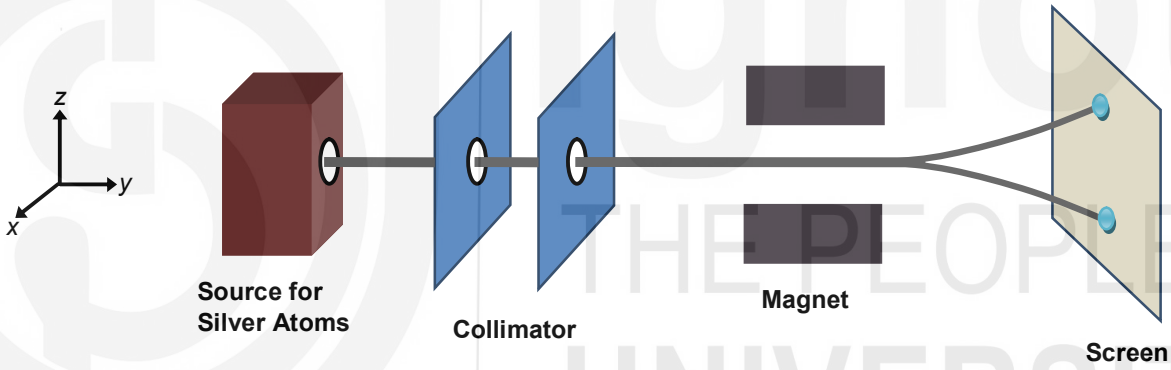

Fig. 14.2: **Stern-Gerlach Experiment.**

The particles with non-zero magnetic moment are deflected from the straight path by the magnetic field gradient before they strike a detector screen, such as a glass slide. The silver atoms are travelling along the *y*-direction as you can see and the magnetic field is predominantly along the *z*-direction. The detector shows a two line pattern indicating that the single beam of silver atoms splits into two. We now analyze the results of this experiment in the classical and quantum mechanical picture

Classical Picture

Assuming that the magnetic moment of each silver atom is μ, the force on each silver atom is $F \approx \mu_z \left(\frac{\partial B_z}{\partial z} \right)$, where μ_z is the magnetic moment of the atom along the magnetic field. The silver atom, as you know has 47 electrons out of which the outermost is a 5s electron. The 46 inner electrons create a spherically symmetrical electron cloud which has no net angular momentum, so the angular momentum of the silver atom is only due to the angular momentum of the single (47^{th} or 5s) electron. Therefore the net magnetic moment of a silver atom is due to the angular momentum of a single electron.

In the classical picture, the magnetic moment of the silver atom is due to the orbital motion of this electron which is $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$ and the force acting in the *z*- direction is $F = -\left(\frac{e}{2m}\right)L_z\left(\frac{\partial B_z}{\partial z}\right)$. So the silver atoms in the beam should be deflected in direct proportion to the z-component of the magnetic dipole moment and by looking at the trace of the deflected particle on a screen, one can measure the *z*-component of the magnetic moment of the electron. So if a beam of silver atoms (atomic dipoles) is sent through the Stern Gerlach apparatus, since the values of the *z*-component of its magnetic moment can vary continuously from $-\mu_z$ to μ_z (depending on the direction of the angular momentum vector), the beam of atoms would fan out and form one continuous trace in the classical picture.

However, since it was found that a single atomic beam produced more than one trace on thescreen *S*, the experiment clearly showed that the inhomogeneous magnetic field resolved one single beam of atoms into **more than one discrete component**. This showed that μ_z and therefore the *z*-component of the angular momentum of the electron can take only discrete values. This experiment therefore confirms the **quantization of the *z*- component of the angular momentum.**

The results were also unusual in the light of quantum mechanical ideas which we now discuss.

Quantum Mechanical Picture

The assumption so far is that the magnetic moment of the electron originates from its orbital motion. In the quantum mechanical picture, we know L_z is quantized and therefore so is μ_z. The angular momentum eigenstates, which are simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$ are characterized by the values of *l* and m_l, respectively, where *l* takes on integer values, and for each value of *l*, m_l has $2l+1$ integer values running from $-l$ to l. So $\mu_z = -\frac{e\hbar}{2m}m_l$ has discrete values corresponding to the discrete values of m_l. In keeping with the idea of quantization of orbital angular momentum a single beam of atomic dipoles will split into $2l+1$ beams corresponding to the $2l+1$ values of m_l. However *l* has integer values, so $2l+1$ is always odd. However in the experiment with silver atoms, the number of traces is an even number, 2. Stern and Gerlach also measured distance between the traces and concluded that the value of m_l does change by one.

Now, the valence electron in the silver atom is located in the *s* shell, for this electron $l = m_l = 0$ and the magnetic moment in the *z*-direction is also zero. Therefore if the angular momentum of the electron is only due to its orbital angular momentum, the silver atoms should not be deflected at all by the magnetic field and there should be just one trace, not even two.

So while the Stern-Gerlach experiment showed that the *z*-component of the angular momentum was quantized and that the angular momentum component in the direction of the magnetic field changed in integral multiples of $\hbar$, it could not shed light on the origin of the magnetic moment.

With the introduction of the idea of the spin angular momentum of the electron however, even with $l = 0$, the electron has a finite angular momentum in the z-direction due to its spin angular momentum, $\pm\frac{\hbar}{2}$ and two possible values of the magnetic moment which causes the deflection of the silver atoms in the magnetic field leading to two traces.

With this it seems reasonable to introduce four quantum numbers n, l, m_l and m_s for the electron. The eigenvalue of $\hat{S}^2$ is $s(s+1)\hbar^2$ where $s = \frac{1}{2}$. The eigenvalues of $\hat{S}_z$ are $m_s\hbar$, where $m_s = \frac{1}{2}$ or $-\frac{1}{2}$.

14.4.1 Sequential Stern-Gerlach Experiments

The results of the Stern-Gerlach experiment make it clear that if a measurement of the magnetic moment of the electron were done, the results would be either μ_0 or $-\mu_0$ and after the measurement the electron is either in the spin up state $|\uparrow\rangle$ or in the spin down state $|\downarrow\rangle$.

As we have said earlier, the z-direction is not special in any way and the eigenvalues of $\hat{S}_x$ and $\hat{S}_y$ are also $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. The two beams observed in the experiment therefore correspond to the spin up state $|\uparrow\rangle$ or in the spin down states of $\hat{S}_z$. If instead of applying the magnetic field in the z-direction, you rotate the Stern-Gerlach apparatus so that the magnetic field is in the x-direction, you would again get two traces corresponding to the two eigenvalues of $\hat{S}_x$ and the two beams would correspond to the spin up and spin down states of $\hat{S}_x$.

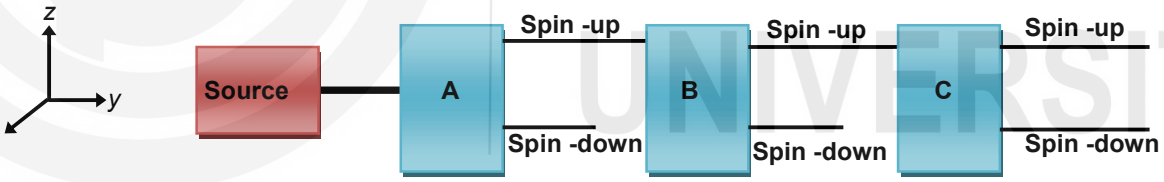

Fig. 14.3: **Sequential Stern-Gerlach experiments.**

Let us now see what the results of a sequence of Stern-Gerlach measurements tell us, when the measurement is carried out sequentially along two orthogonal directions in space. Consider the sequential experiment shown in Fig. 14.3. In the first measurement (Stern-Gerlach apparatus **A**) the magnetic field is along the z-direction and the output has the two states of the $|\uparrow\rangle$ and $|\downarrow\rangle$. Let us now block one of the output beams, say the beam corresponding to $|\downarrow\rangle$ and allow the $|\uparrow\rangle$ beam to be incident on a second Stern-Gerlach apparatus **B** where the magnetic field is along the x-direction. You will find that once again the beam will split into two parts and a measurement of the magnetic moment will yield the values of μ_0 or $-\mu_0$ with equal probability (1/2). If you now block one of the beams in the output, say once again the spin-down state $|S_{x\downarrow}\rangle$, and let the $|S_{x\uparrow}\rangle$ beam be incident on a third Stern-Gerlach apparatus **C** where the magnetic field is along the z-direction (so you are measuring S_z). You will find that once again the beam

will split into two parts and a measurement of the magnetic moment will yield the values of μ_0 or $-\mu_0$ with equal probability (1/2).

So what the sequential SG experiments tell us is that S_x and S_z are incompatible observables and $\left[\hat{S}_z, \hat{S}_x\right] \neq 0$. You would get similar results if you were measuring any pair out of S_x, S_y, S_z.

14.5 SPIN DYNAMICS

Consider an electron at rest in a magnetic field in the z-direction $\vec{B} = B_0\hat{k}$.
The Hamiltonian for the electron is:

$$H = H_{\text{int}}^{S} = -\vec{\mu}_S.\vec{B} = -\mu_z B_0 \qquad (14.52)$$

And $\mu_z = -g\left(\frac{e}{2m}\right)S_z$ where $S_z = \frac{\hbar}{2}$ or $-\frac{\hbar}{2}$. Using $g \approx 2$, we get

$\mu_z \approx -\left(\frac{e}{m}\right)S_z$ and $\hat{H}_{\text{int}}^{S} = \left(\frac{e}{m}\right)\hat{S}_z B_0 = \left(\frac{eB_0}{m}\right)\hat{S}_z = \omega_0\hat{S}_z$.

So for any initial state $|\psi(0)\rangle$, the final state of the system is

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle = e^{-\frac{i\omega_0\hat{S}_z t}{\hbar}}|\psi(0)\rangle \qquad (14.53)$$

Let us use Eq. (14.52) to determine the expectation values of the spin angular momentum operators.

SAQ 7

Given that the initial state of the system is $|S_{x\downarrow}\rangle$ calculate $\langle\hat{S}_x\rangle, \langle\hat{S}_y\rangle, \langle\hat{S}_z\rangle$ at a later time *t*.

14.6 SUMMARY

- The same commutation rules for the spin angular momentum operator $\hat{S}$ (and its components $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$) are the same as for the orbital angular momentum $\hat{L}$ and the total angular momentum $\hat{J}$:

  $$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z; \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x; \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$$

  and

  $$\left[\hat{S}^2, \hat{S}_x\right] = 0; \quad \left[\hat{S}^2, \hat{S}_y\right] = 0; \quad \left[\hat{S}^2, \hat{S}_z\right] = 0$$

  where $\hat{S}^2 = \hat{S}_x{}^2 + \hat{S}_y{}^2 + \hat{S}_z{}^2$.

  For the electron, $\hat{S}_z$ has two eigenvalues $\pm\frac{\hbar}{2}$. The two corresponding eigenkets are $|\uparrow\rangle$ and $|\downarrow\rangle$, which are also the eigenkets of $\hat{S}^2$:

  $$\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \;;\; \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$$

- The spectral representation of the spin angular momentum operators is:

$$\hat{S}_x = \frac{\hbar}{2}\left[|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right]; \quad \hat{S}_y = \frac{i\hbar}{2}\left[-|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right];$$

$$\hat{S}_z = \frac{\hbar}{2}\left[|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right]$$

- The spin angular momentum matrices for the electron are

$$S_x = \frac{\hbar}{2}\sigma_x; S_y = \frac{\hbar}{2}\sigma_y; S_z = \frac{\hbar}{2}\sigma_z$$

where $\sigma_x, \sigma_y, \sigma_z$ are the **Pauli spin matrices** defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And $S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- The eigenvectors are represented by column matrices: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are also referred to as the "spin –up"and "spin-down "states, respectively for a spin half particle.

The general **spin state** of the electron in space, $|\phi\rangle$, is a linear combination of the eigenkets $|\uparrow\rangle$ and $|\downarrow\rangle$:

$$|\phi\rangle = C_1|\uparrow\rangle + C_2|\downarrow\rangle = \langle\uparrow|\phi\rangle|\uparrow\rangle + \langle\downarrow|\phi\rangle|\downarrow\rangle$$

$|\phi\rangle$ is also a column matrix $|\phi\rangle = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix}$

The column vector $|\phi\rangle$ is called a **two-component spinor** χ and we write:

$$\chi = C_1\chi_\uparrow + C_2\chi_\uparrow .$$

The two descriptions are equivalent and

$$\chi \equiv |\phi\rangle = \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix}; \chi_\uparrow \equiv |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \chi_\downarrow \equiv |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this description the spin angular momentum operators are represented using the Pauli spin matrices as follows:

$$\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\sigma_x \;\; ; \;\; \hat{S}_y \Leftrightarrow \frac{\hbar}{2}\sigma_y ; \quad \hat{S}_z \Leftrightarrow \frac{\hbar}{2}\sigma_z$$

- The wave function of the electron can be written in the spinor formalism as a two component spinor:

$$\psi_P(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}$$

where

- $\psi_{\uparrow}(x)$ defines the probability of locating the particle at the point *x* in the **spin up** state, that is with the *z*-component of the spin angular momentum $\frac{\hbar}{2}$, and
- $\psi_{\downarrow}(x)$ defines the probability of locating the particle at the point *x* in the **spin down** state, that is with the z-component of the spin angular momentum $-\frac{\hbar}{2}$.

- An electron moving inside an atom has a magnetic moment $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$ due to its orbital motion and a magnetic moment $\vec{\mu}_s = -g\left(\frac{e}{2m}\right)\vec{S}$ due to its spin angular momentum $\vec{S}$, where *g* (≈ 2.0023) is the Lande *g* factor .
- The Stern-Gerlach experiment was designed to verify the notion of space quantization, that is the quantization of the *z*-component of the angular momentum of the electron. The experiment showed that the *z*-component of the angular momentum was quantized and that the angular momentum component in the direction of the magnetic field changed in integral multiples of $\hbar$. The complete results of the experiments could be explained only after assuming the existence of the spin angular momentum of electrons.

14.7 TERMINAL QUESTIONS

1. Write down the eigenstates of $\hat{S}_y$ in terms of the eigenkets $|\uparrow\rangle$ and $|\downarrow\rangle$.
2. Determine the eigenstates of the operator $\hat{S}_x \cos\beta + \hat{S}_y \sin\beta$.
3. Show that $(\sigma.A)(\sigma.B) = A.B + i\sigma.(A \times B)$.
4. Given that the initial state of the system is $|\psi(0)\rangle = \sin\beta|\uparrow\rangle + \cos\beta|\downarrow\rangle$, determine $\langle\hat{S}_x\rangle, \langle\hat{S}_y\rangle, \langle\hat{S}_z\rangle$ at a later time *t*.
5. Consider a Stern-Gerlach experiment with a uniform magnetic in the *z*-direction. Calculate $|\psi(t)\rangle$ assuming that the initial state is $|\psi(0)\rangle = \cos\alpha|\uparrow\rangle + \sin\alpha|\downarrow\rangle$.

14.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) $$\hat{S}_z|\phi_1\rangle = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|]|\phi_1\rangle$$

$$= \frac{\hbar}{2}\langle\phi_1|\phi_1\rangle|\phi_1\rangle - \langle\phi_2|\phi_1\rangle|\phi_2\rangle$$

$$= \frac{\hbar}{2}|\phi_1\rangle$$

$$\hat{S}_z|\phi_2\rangle = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|]|\phi_2\rangle$$

$$= \frac{\hbar}{2}[\langle\phi_1|\phi_2\rangle|\phi_1\rangle - |\phi_2\rangle\langle\phi_2|\phi_2\rangle]$$

$$= -\frac{\hbar}{2}|\phi_2\rangle$$

2. a) Using the results of SAQ 6(a), Unit 10

$$\hat{S}_x = (S_x)_{11}|\phi_1\rangle\langle\phi_2| + (S_x)_{12}|\phi_1\rangle\langle\phi_2|$$
$$+ (S_x)_{21}|\phi_2\rangle\langle\phi_1| + (S_x)_{22}|\phi_2\rangle\langle\phi_2|$$

Using $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we get $(S_x)_{11} = (S_x)_{22} = 0$ and $(S_x)_{12} = (S_x)_{21} = \hbar/2$.

So: $\hat{S}_x = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|]$

Similarly,

$$\hat{S}_y = (S_y)_{11}|\phi_1\rangle\langle\phi_1| + (S_y)_{12}|\phi_1\rangle\langle\phi_2|$$
$$+ (S_y)_{21}|\phi_2\rangle\langle\phi_1| + (S_y)_{22}|\phi_2\rangle\langle\phi_2|$$
$$= -\frac{i\hbar}{2}|\phi_1\rangle\langle\phi_2| + \frac{i\hbar}{2}|\phi_2\rangle\langle\phi_1| = \frac{i\hbar}{2}[-|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|]$$

b) i) The characteristic equation for S_x is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow s_x = \pm\frac{\hbar}{2}$$

The characteristic equation for S_y is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \text{ and } s_y = \pm\frac{\hbar}{2}$$

The characteristic equation for S_z is

$$(\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda = \pm 1 \quad \text{and} \quad s_z = \pm\hbar/2$$

ii) Let the eigenvectors of S_z be $|\phi_1\rangle = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Defining

$$\hat{S}_z|\phi_1\rangle = \frac{\hbar}{2}|\phi_1\rangle \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow x_1 = x_1; y_1 = 0$$

$$\therefore \quad |\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

With $\hat{S}_z|\phi_2\rangle = -\frac{\hbar}{2}|\phi_2\rangle \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ -y_2 \end{pmatrix} \Rightarrow x_2 = 0; y_2 = y_2 \Rightarrow |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

iii) Using Eq. (14.13) we write:

$$S^2|\phi_1\rangle = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2 |\phi_1\rangle$$

Similarly $S^2|\phi_2\rangle = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2 |\phi_2\rangle$

3. Let the eigenstates of $\hat{S}_x$ be $|S_{x\uparrow}\rangle$ and $|S_{x\downarrow}\rangle$. Writing the eigenstate $|S_{x\uparrow}\rangle$ in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$ we have: $|S_{x\uparrow}\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ where $c_1^2 + c_2^2 = 1$. Since $\hat{S}_x|S_{x\uparrow}\rangle = \frac{\hbar}{2}|S_{x\uparrow}\rangle$ we get:

$$\hat{S}_x|S_{x\uparrow}\rangle = \frac{\hbar}{2}[c_1|\uparrow\rangle + c_2|\downarrow\rangle] \qquad \text{(i)}$$

Using the definition of $\hat{S}_x$ from SAQ 2a we can also write

$$\hat{S}_x|S_{x\uparrow}\rangle = \frac{\hbar}{2}[|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|]|S_{x\uparrow}\rangle$$

$$= \left[\frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\right]c_1|\uparrow\rangle + \left[\frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\right]c_2|\downarrow\rangle$$

$$= \frac{\hbar}{2}(c_1)|\downarrow\rangle + \frac{\hbar}{2}c_2|\uparrow\rangle \qquad \text{(ii)}$$

Equating the RHS of Eqs. (i) and (ii) we get:

$$\frac{\hbar}{2}[c_1|\uparrow\rangle + c_2|\downarrow\rangle] = \frac{\hbar}{2}(c_1)|\downarrow\rangle + \frac{\hbar}{2}c_2|\uparrow\rangle$$

$\Rightarrow$ $c_1|\uparrow\rangle + c_2|\downarrow\rangle = c_1|\downarrow\rangle + c_2|\uparrow\rangle$

Or $(c_1 - c_2)|\uparrow\rangle + (c_2 - c_1)|\downarrow\rangle = 0 \Rightarrow c_1 = c_2$ (iii)

Using Eq. (iii) in Eq. (i) we get $2c_1^2 = 1$

or $c_1 = \frac{1}{\sqrt{2}} = c_2$ (iv)

$\therefore$ $|S_{x\uparrow}\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ (v)

We follow the same steps to determine $|S_{x\downarrow}\rangle$

We write $|S_{x\downarrow}\rangle = c_1'|\uparrow\rangle + c_2'|\downarrow\rangle$ with:

$$c_1'^2 + c_2'^2 = 1 \qquad \text{(vi)}$$

also $\hat{S}_x|S_{x\downarrow}\rangle = -\frac{\hbar}{2}[c_1'|\uparrow\rangle + c_2'|\downarrow\rangle]$ (vii)

and $\hat{S}_x|S_{x\downarrow}\rangle = \frac{\hbar}{2}c_1'|\downarrow\rangle + \frac{\hbar}{2}c_2'|\uparrow\rangle$ (viii)

equating Eqs. (vii and viii) we get

$$(c_1' + c_2')|\uparrow\rangle + (c_1' + c_2')|\downarrow\rangle = 0 \qquad \Rightarrow c_1' = -c_2'$$

So $c_1'^2 + c_2'^2 = 1 \Rightarrow c_1' = -c_2' = \frac{1}{\sqrt{2}}$

and hence $|S_{x\downarrow}\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$

4. a) Using Eq. (14.26) with $\hat{S}_z$ replaced by $\hat{S}_x$ we get

$$|\phi'\rangle = \hat{S}_x|\phi\rangle = \begin{pmatrix} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix}$$

Using the definition of $\hat{S}_x = \frac{\hbar}{2}|\uparrow\rangle\langle\downarrow| + \frac{\hbar}{2}|\downarrow\rangle\langle\uparrow|$

We get $\hat{S}_x|\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle$ and $\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2}|\downarrow\rangle$

$\therefore$ $\langle\uparrow|\hat{S}_x|\uparrow\rangle = 0 = \langle\downarrow|\hat{S}_x|\downarrow\rangle$ and $\langle\uparrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2} = \langle\downarrow|\hat{S}_x|\uparrow\rangle$

$\therefore$ $|\phi'\rangle = \hat{S}_x|\phi\rangle = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} = \frac{\hbar}{2}\sigma_x|\phi\rangle$

$\therefore$ $\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\sigma_x$

Using (Eq. (14.26) for $\hat{S}_y$, we have

$$|\phi'\rangle = \hat{S}_y|\phi\rangle = \begin{pmatrix} \langle\uparrow|\hat{S}_y|\uparrow\rangle & \langle\uparrow|\hat{S}_y|\downarrow\rangle \\ \langle\downarrow|\hat{S}_y|\uparrow\rangle & \langle\downarrow|\hat{S}_y|\downarrow\rangle \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix}$$

$\hat{S}_y|\uparrow\rangle = \frac{i\hbar}{2}[-|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|]|\uparrow\rangle = \frac{i\hbar}{2}|\downarrow\rangle$ and $\hat{S}_y|\downarrow\rangle = -\frac{i\hbar}{2}|\uparrow\rangle$

$\therefore$ $\langle\uparrow|\hat{S}_y|\uparrow\rangle = \langle\downarrow|\hat{S}_y|\downarrow\rangle = 0$

$\langle\uparrow|\hat{S}_y|\downarrow\rangle = -\frac{i\hbar}{2}; \langle\downarrow|\hat{S}_y|\uparrow\rangle = \frac{i\hbar}{2}$

$\therefore$ Eq. (vii) is: $|\phi'\rangle = \hat{S}_y|\phi\rangle \begin{pmatrix} 0 & \frac{-i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} \langle\uparrow|\phi\rangle \\ \langle\downarrow|\phi\rangle \end{pmatrix} = \frac{\hbar}{2}\sigma_y|\phi\rangle$

$\therefore$ $\hat{S}_y \Leftrightarrow \frac{\hbar}{2}\sigma_y$.

b) $\{\sigma_x, \sigma_y\} = \sigma_x \sigma_y + \sigma_y \sigma_x$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$$

We can calculate similarly for $\{\sigma_y, \sigma_z\}$ and $\{\sigma_z, \sigma_x\}$.

c) $[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\,\sigma_z$$

$[\sigma_y, \sigma_z]$ and $[\sigma_y, \sigma_z]$ can be derived similarly.

d) We have $\{\sigma_x, \sigma_y\} = 0 \Rightarrow \sigma_x\sigma_y = -\sigma_y\sigma_x$

and

$$[\sigma_x, \sigma_y] = 2i\,\sigma_z \Rightarrow \sigma_x\sigma_y - \sigma_y\sigma_x = 2\sigma_x\sigma_y = 2i\,\sigma_z$$

or $\sigma_x\sigma_y = i\,\sigma_z$

Similarly $\{\sigma_y, \sigma_z\} = 0$ and $[\sigma_y, \sigma_z] = 2i\,\sigma_x \Rightarrow \sigma_y\sigma_z = i\,\sigma_x$

and

$\{\sigma_z, \sigma_x\} = 0$ and $[\sigma_z, \sigma_x] = 2i\,\sigma_y \Rightarrow \sigma_z\sigma_x = i\,\sigma_y$.

5. Using Eq. (14.42) we get

$$\hat{L}.\hat{S} = \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y + \hat{L}_z\hat{S}_z$$

$$= \hat{L}_x\left(\frac{\hbar}{2}\sigma_x\right) + \hat{L}_y\left(\frac{\hbar}{2}\sigma_y\right) + \hat{L}_z\left(\frac{\hbar}{2}\sigma_z\right)$$

$$= \frac{\hbar}{2}\hat{L}_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\hat{L}_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\hbar}{2}\hat{L}_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \hat{L}_x \\ \hat{L}_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\hat{L}_y \\ i\hat{L}_y & 0 \end{pmatrix} + \begin{pmatrix} \hat{L}_z & 0 \\ 0 & -\hat{L}_z \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \hat{L}_z & \hat{L}_x - i\hat{L}_y \\ \hat{L}_x + i\hat{L}_y & -\hat{L}_z \end{pmatrix}$$

6. a) Bohr Magneton $\mu_B = \dfrac{e\hbar}{2m} = \dfrac{1.6\times10^{-19}\times1.054\times10^{-34}\,Js}{2\times9.1\times10^{-31}\,kg}$

$$= 9.27\times10^{-24}\ \text{JTesla}^{-1}$$

b) Let the radius and linear velocity of the spinning electron be *r* and *v* respectively. The spinning of the electron about its own axis produces a current of $\frac{ev}{2\pi r}$ and the corresponding magnetic moment is

$$\mu_B = I_A = \left(\frac{ev}{2\pi r}\right)\pi r^2$$

i) Equating μ_B calculated from Eq. (i) to the experimentally observed value $\mu_B = \frac{e\hbar}{2m}$ we get

$$\left(\frac{ev}{2\pi r}\right)\pi r^2 = \frac{e\hbar}{2m} \Rightarrow v = \frac{\hbar}{mr}$$

The classical radius to the electron path is obtained by equating its rest energy mc^2 to its self electrostatic energy $\frac{e^2}{r}$.

$$\therefore \quad \frac{e^2}{r} = mc^2 \Rightarrow r = e^2/mc^2 \qquad \text{(iii)}$$

Substituting for r in Eq. (ii) from Eq. (iii) we get:

$$v = \frac{\hbar}{me^2}\left(mc^2\right) \Rightarrow \frac{v}{c} = \frac{\hbar c}{e^2} = 137$$

The speed of the spinning electron therefore has to be 137c to produce this magnetic moment, which is not possible.

7. From SAQ 4 we have $|\psi(0)\rangle = |S_x\downarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$.

Using Eq. (14.52) we get

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle = e^{-\frac{i\omega_0\hat{S}_z t}{\hbar}}|S_x\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{i\omega_0\hat{S}_z t}{\hbar}}\left(|\uparrow\rangle - |\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{-\frac{i\omega_0 t}{2}}|\uparrow\rangle - e^{\frac{i\omega_0 t}{2}}|\downarrow\rangle\right)$$

$$= \left[e^{-\frac{i\omega_0 t}{2}}\begin{pmatrix}1\\0\end{pmatrix} - e^{\frac{i\omega_0 t}{2}}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\frac{i\omega_0 t}{2}}\\ -e^{\frac{i\omega_0 t}{2}}\end{pmatrix}$$

Therefore: $\langle\psi(t)| = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{\frac{i\omega_0 t}{2}} & -e^{-\frac{i\omega_0 t}{2}}\end{pmatrix}$

Now using $\hat{S}_x = \frac{\hbar}{2}\sigma_x$:

$$\left\langle\psi(t)\middle|\hat{S}_x\middle|\psi(t)\right\rangle = \frac{\hbar}{2}\langle\psi(t)|\sigma_x|\psi(t)\rangle$$

$$= \frac{\hbar}{4}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}} & -e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-\frac{i\omega_0 t}{2}} \\ -e^{\frac{i\omega_0 t}{2}} \end{pmatrix} = -\frac{\hbar}{2}\cos\omega_0 t$$

$$\langle \psi(t)|\hat{S}_y|\psi(t)\rangle = \frac{\hbar}{2}\langle \psi(t)|\sigma_y|\psi(t)\rangle$$

$$= \frac{\hbar}{4}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}} & -e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-\frac{i\omega_0 t}{2}} \\ -e^{\frac{i\omega_0 t}{2}} \end{pmatrix} = -\frac{\hbar}{2}\sin\omega_0 t$$

$$\langle \psi(t)|\hat{S}_z|\psi(t)\rangle = \frac{\hbar}{2}\langle \psi(t)|\sigma_z|\psi(t)\rangle$$

$$= \frac{\hbar}{4}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}} & -e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-\frac{i\omega_0 t}{2}} \\ -e^{\frac{i\omega_0 t}{2}} \end{pmatrix} = 0$$

The values of $\langle \hat{S}_x \rangle, \langle \hat{S}_y \rangle, \langle \hat{S}_z \rangle$ imply that the direction of the spin angular momentum vector rotates about the z-axis with an angular speed ω_0.

Terminal Questions

1. Following SAQ 3, let the eigenstates of $\hat{S}_y$ be $|S_{y\uparrow}\rangle$ and $|S_{y\downarrow}\rangle$, then

$$|S_{y\uparrow}\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \text{ with } c_1^2 + c_2^2 = 1 \quad \text{(i)}$$

$$\hat{S}_y|S_{y\uparrow}\rangle = \frac{\hbar}{2}c_1|\uparrow\rangle + \frac{\hbar}{2}c_2|\downarrow\rangle \quad \text{(ii)}$$

and
$$\hat{S}_y|S_{y\uparrow}\rangle = \frac{i\hbar}{2}[-|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|]c_1|\uparrow\rangle + \frac{i\hbar}{2}[-|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|]c_2|\downarrow\rangle$$

$$= \frac{i\hbar}{2}c_1|\downarrow\rangle - \frac{i\hbar}{2}c_2|\uparrow\rangle \quad \text{(iii)}$$

Equating the RHS of Eqs. (ii) and (iii), we get:

$$(c_1 + i\,c_2)|\uparrow\rangle + (c_2 - i\,c_1)|\downarrow\rangle = 0 \Rightarrow c_2 = i\,c_1$$

Also $$c_1^2 + c_2^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{2}}; c_2 = \frac{i}{\sqrt{2}} \Rightarrow |S_{y\uparrow}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$$

Similarly, $$|S_{y\downarrow}\rangle = c_1'|\uparrow\rangle + c_2'|\downarrow\rangle \text{ with } c_1'^2 + c_2'^2 = 1 \quad \text{(iv)}$$

and $$\hat{S}_y|S_{y\downarrow}\rangle = -\frac{\hbar}{2}c_1'|\uparrow\rangle - \frac{\hbar}{2}c_2'|\downarrow\rangle \quad \text{(v)}$$

and $$\hat{S}_y|S_{y\downarrow}\rangle = \frac{i\hbar}{2}c_1'|\downarrow\rangle - \frac{i\hbar}{2}c_2'|\uparrow\rangle \quad \text{(vi)}$$

Equating the RHS of Eqs. (v) and (vi) we get:

$$(c_1' - i\,c_2')\left|\uparrow\right\rangle + (c_2' + i\,c_1')\left|\downarrow\right\rangle = 0$$

$$\Rightarrow \qquad c_2' = -i\,c_1' \text{ and } c_1 = \frac{1}{\sqrt{2}}; c_2 = \frac{-i}{\sqrt{2}} \Rightarrow \left|S_{y\uparrow}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle - i\left|\downarrow\right\rangle\right)$$

2. Matrix corresponding to $\hat{S}_\beta = \hat{S}_x \cos\beta + \hat{S}_y \sin\beta$ is

$$S_\beta = \frac{\hbar}{2}\cos\beta\sigma_x + \frac{\hbar}{2}\sin\beta\sigma_y$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\beta \\ \cos\beta & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sin\beta \\ i\sin\beta & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\beta} \\ e^{i\beta} & 0 \end{pmatrix}$$

Let us determine the eigenvalues of this operator. The charactertic equation is

$$\left(\lambda^2 - \frac{\hbar^2}{4}\right) = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

Let the eigenvectors of S_β be $|1\rangle = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}$ for $\lambda = \hbar/2$ and $|2\rangle = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for $\lambda = -\hbar/2$. So:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\beta} \\ e^{i\beta} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\Rightarrow x_1 = e^{-i\beta}y_1 \Rightarrow y_1 = e^{i\beta}x_1 \Rightarrow |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\beta} \end{pmatrix}$$

And

$$\frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\beta} \\ e^{i\beta} & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow x_2 = -e^{-i\beta}y_2$$

or $$y_2 = -e^{i\beta}x_2 \Rightarrow |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\beta} \end{pmatrix}$$

3. We write $\sigma.\hat{A} = \sigma_x\hat{A}_x + \sigma_y\hat{B}_y + \sigma_z\hat{B}_z$ and
$\sigma.\hat{B} = \sigma_x\hat{B}_x + \sigma_y\hat{B}_y + \sigma_y\hat{B}_y + \sigma_z\hat{B}_z$.

$$\therefore \qquad (\sigma.\hat{A})(\sigma.B) = (\sigma_x\hat{A}_x + \sigma_y\hat{A}_y + \sigma_z\hat{A}_z)(\sigma_x\hat{B}_x + \sigma_y\hat{B}_y + \sigma_z\hat{B}_z)$$

$$= \sigma_x\hat{A}_x\sigma_x\hat{B}_y + \sigma_y\hat{A}_y\sigma_y\hat{B}_y + \sigma_z\hat{A}_z\sigma_z\hat{B}_z$$

$$+ \sigma_x\hat{A}_x\sigma_y\hat{B}_y + \sigma_x\hat{A}_x\sigma_z\hat{B}_z$$

$$+ \sigma_y\hat{A}_y\sigma_x\hat{B}_x + \sigma_y\hat{A}_y\sigma_z\hat{B}_z$$

$$+ \sigma_z\hat{A}_z\sigma_y\hat{B}_y + \sigma_z\hat{A}_z\sigma_x\hat{B}_x \qquad \text{(i)}$$

Check for yourself that $\sigma_x\hat{A}_x = \hat{A}_x\sigma_x;\ \sigma_y\hat{A}_y = \hat{A}_y\sigma_y;\ \sigma_z\hat{A}_z = \hat{A}_z\sigma_z$

and $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

Also $\sigma_x\sigma_y = i\sigma_z \Rightarrow \sigma_y\sigma_x = -i\sigma_z$. Similarly $\sigma_z\sigma_y = -i\sigma_x$; $\sigma_x\sigma_z = -i\sigma_y$.

$$\therefore \quad (\sigma.\hat{A})(\sigma.\hat{B}) = \hat{A}_x\hat{B}_x + \hat{A}_y\hat{B}_y + \hat{A}_z\hat{B}_z + \hat{A}_x(\sigma_x\sigma_y)\hat{B}_y + \hat{A}_y(\sigma_y\sigma_x)\hat{B}_x + \hat{A}_x(\sigma_x\sigma_z)\hat{B}_z + \hat{A}_z(\sigma_z\sigma_x)\hat{B}_x + \hat{A}_y(\sigma_y\,\sigma_z)\hat{B}_z + \hat{A}_z(\sigma_z\sigma_y)\hat{B}_y \qquad \text{(ii)}$$

From SAQ 4 we have $\sigma_x\sigma_y = i\,\sigma_z$

Therefore Eq. (ii) is:

$$(\sigma.\hat{A})(\sigma.\hat{B}) = \hat{A}.\hat{B} + i\,\sigma_z(\hat{A}_x\hat{B}_y - \hat{A}_y\hat{B}_x) + i\,\sigma_y(\hat{A}_z\hat{B}_x - \hat{A}_x\hat{B}_z) + i\,\sigma_x(A_yB_z - A_z\,B_y) = \hat{A}.\hat{B} + \sigma.(\hat{A}\times\hat{B})$$

4. $|\psi(0)\rangle = \sin\beta|\uparrow\rangle + \cos\beta|\downarrow\rangle$

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle = e^{-i\hat{H}t}\left[\sin\beta|\uparrow\rangle + \cos\beta|\downarrow\rangle\right]$$

$$= e^{-\frac{i\omega_0 t}{2}}\sin\beta|\uparrow\rangle + e^{\frac{i\omega_0 t}{2}}\cos\beta|\downarrow\rangle$$

$$= e^{-\frac{i\omega_0 t}{2}}\sin\beta\begin{pmatrix}1\\0\end{pmatrix} + e^{\frac{i\omega_0 t}{2}}\cos\beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}e^{-\frac{i\omega_0 t}{2}}\sin\beta\\ e^{\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}$$

So $\langle\psi(t)| = \sin\beta e^{\frac{i\omega_0 t}{2}}\langle\uparrow| + \cos\beta e^{-\frac{i\omega_0 t}{2}}\langle\downarrow| = \begin{pmatrix}e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}$

$$\therefore \quad \langle\hat{S}_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$= \frac{\hbar}{2}\begin{pmatrix}e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-\frac{i\omega_0 t}{2}}\sin\beta\\ e^{\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}\begin{pmatrix}e^{\frac{i\omega_0 t}{2}}\cos\beta\\ e^{-\frac{i\omega_0 t}{2}}\sin\beta\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\sin\beta\cos\beta e^{i\omega_0 t} + \cos\beta\sin\beta e^{-i\omega_0 t}\right)$$

$$= \frac{\hbar}{2}\sin\beta\cos\beta\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) \quad = \frac{\hbar}{2}\sin 2\beta\cos(\omega_0 t)$$

Similarly,

$$\langle\hat{S}_y\rangle = \langle\psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \frac{\hbar}{2}\begin{pmatrix}e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{-\frac{i\omega_0 t}{2}}\sin\beta\\ e^{\frac{i\omega_0 t}{2}}\cos\beta\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta \end{pmatrix}\begin{pmatrix} -ie^{\frac{i\omega_0 t}{2}}\cos\beta \\ ie^{-\frac{i\omega_0 t}{2}}\sin\beta \end{pmatrix}$$

$$= \frac{i\hbar}{2}\left(-\sin\beta\cos\beta\, e^{i\omega_0 t} + \sin\beta\cos\beta e^{-i\omega_0 t}\right)$$

$$= \frac{\hbar}{2}\sin 2\beta\, \sin(\omega_0 t)$$

$\therefore$ $\langle \hat{S}_z \rangle = \langle \psi(t) | \hat{S}_z | \psi(t) \rangle$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-\frac{i\omega_0 t}{2}}\sin\beta \\ e^{\frac{i\omega_0 t}{2}}\cos\beta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{\frac{i\omega_0 t}{2}}\sin\beta & e^{-\frac{i\omega_0 t}{2}}\cos\beta \end{pmatrix}\begin{pmatrix} e^{-\frac{i\omega_0 t}{2}}\sin\beta \\ -e^{\frac{i\omega_0 t}{2}}\cos\beta \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\sin^2\beta - \cos^2\beta\right) = \frac{\hbar}{2}\cos 2\beta$$

5. $H_{\text{int}} = -\vec{\mu}_s.\vec{B} = \omega_0 S_y$

$\therefore$ $|\psi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} = e^{-\frac{i\omega_0 \hat{S}_y}{\hbar}}|\psi(0)\rangle$ (i)

From TQ 1, we have:

$$|S_{y\uparrow}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\uparrow\rangle\right) \quad \text{(ii)}$$

$$|S_{y\downarrow}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right) \quad \text{(iii)}$$

From Eqs. (ii) and (iii) we get

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|S_{y\uparrow}\rangle + |S_{y\downarrow}\rangle\right);\ |\downarrow\rangle = -\frac{i}{\sqrt{2}}\left(|S_{y\uparrow}\rangle - |S_{y\downarrow}\rangle\right)$$

$$|\psi(0)\rangle = \cos\alpha\left(\frac{1}{\sqrt{2}}|S_{y\uparrow}\rangle + \frac{1}{\sqrt{2}}|S_{y\downarrow}\rangle\right) + \sin\alpha\left(-\frac{i}{\sqrt{2}}|S_{y\uparrow}\rangle + \frac{i}{\sqrt{2}}|S_{y\downarrow}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha - i\sin\alpha)|S_{y\uparrow}\rangle + \frac{1}{\sqrt{2}}(\cos\alpha + i\sin\alpha)|S_{y\downarrow}\rangle$$

Now $\hat{S}_y|S_{y\uparrow}\rangle = \frac{\hbar}{2};\quad \hat{S}_y|S_{y\uparrow}\rangle = -\frac{\hbar}{2}$

So

$$|\psi(t)\rangle = \left(\frac{\cos\alpha}{\sqrt{2}}e^{-\frac{i\omega_0 t}{2}} - \frac{i}{\sqrt{2}}\sin\alpha e^{-\frac{i\omega_0 t}{2}}\right)|S_{y\uparrow}\rangle + \left(\frac{\cos\alpha}{\sqrt{2}}e^{\frac{i\omega_0 t}{2}} + i\sin\alpha e^{\frac{i\omega_0 t}{2}}\right)|S_{y\downarrow}\rangle$$

FURTHER READINGS

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