UNIT 13

ANGULAR MOMENTUM

Structure

13.1 INTRODUCTION

In Unit 4 the angular momentum operator $\hat{L}$ was derived as the quantum mechanical analogue of the classical angular momentum associated with angular motion. You have also studied about angular momentum in quantum mechanics in the context of spherically symmetric potentials in Units 8 and 9 and obtained the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$. You saw that for a spherically symmetric potential, the Hamiltonian operator $\hat{H}$ commutes with $\hat{L}^2$ and with the components of $\hat{L}$ $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$. The components $\hat{L}_x, \hat{L}_y, \hat{L}_z$, however do not commute among themselves.

In this unit, we start our discussion with the commutation relations for the angular momentum operators and determine the eigenvalues for $\hat{L}^2$ and $\hat{L}_z$. The results we obtain in this unit are startling, because you will find that the eigenvalues of $\hat{L}_z$ could be either integral or half integral multiples of $\hbar$. Startling, because you studied in Unit 8 that L_z could be only an integral multiple of $\hbar$. This is an important result because it paves the way for the concept of the "spin angular momentum", which is a purely quantum mechanical concept.

In Sec. 13.2 we present a brief recap of angular momentum in quantum mechanics as you have studied in Units 8 and 9 of this course and state all the properties of the angular momentum operators. In Sec. 13.3 we introduce the raising and lowering operators for angular momentum, which are similar to the raising and lowering operators you studied in Unit 12 for the simple harmonic oscillator. In Sec. 13.4 we use the raising and lowering operators to determine the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$. We then introduce the total angular momentum

operator $\hat{J}$, which combines both the spin and orbital angular momentum of all particles. In Sec. 13.5 we calculate the matrix elements of the angular momentum operators, in particular we calculate the matrix elements for $j = 1/2$, which as you will see in Unit 14, is the spin angular momentum of the electron. In Sec. 13.6 you will learn how to determine the angular momentum eigenfunctions in the coordinate representation using the raising and lowering operators.

Expected Learning Outcomes

After studying this unit, you should be able to:

- define the raising and lowering operators for angular momentum;
- derive the commutation relations between the raising and lowering operators and the other angular momentum operators;
- determine the action of the raising operators on the angular momentum eigenstates;
- obtain the eigenvalues and eigenfunctions of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$;
- explain the total angular momentum operator $\hat{J}$ and determine the matrix elements for $\hat{J}^2$, $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$; and
- determine the matrix elements for $j = 1/2$.

13.2 ANGULAR MOMENTUM: A RECAP

We start this section with a quick recap of what you have already studied about angular momentum in quantum mechanics in the context of spherically symmetric potentials in Units 8 and 9. Recollect that you solved the following stationary state Schrodinger equation in spherical polar coordinates for the wave function $\psi(r,\theta,\phi)$:

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \tag{13.1}$$

where the Laplacian in spherical polar coordinates is:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{13.2}$$

Writing the wave function as:

$$\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi) \tag{13.3}$$

and defining the opertator $\hat{L}$ through the familiar relation $\vec{L} = \vec{r}\times\vec{p}$, the expression for the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ were written explicitly (in spherical polar coordinates) as:

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{13.4a}$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \qquad (13.4b)$$

We obtained the following general results:

- The operator $\hat{L}^2$ commutes with the Hamiltonian : $\left[\hat{L}^2,\hat{H}\right]=0$.
- The operator $\hat{L}_z$ commutes with $\hat{L}^2$: $\left[\hat{L}_z,\hat{L}^2\right]=0$. Actually, $\hat{L}_x$ and $\hat{L}_y$ also commute with $\hat{L}^2$ $\left[\hat{L}_x,\hat{L}^2\right]=0$ and $\left[\hat{L}_y,\hat{L}^2\right]=0$, but $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ do not commute among themselves and we have:
$$\left[\hat{L}_x,\hat{L}_y\right]=i\hbar\hat{L}_z;\left[\hat{L}_y,\hat{L}_z\right]=i\hbar\hat{L}_x;\left[\hat{L}_z,\hat{L}_x\right]=i\hbar\hat{L}_y \qquad (13.5)$$
- It is possible to obtain a wave function which is an eigenfunction of $\hat{H},\hat{L}_z,\hat{L}^2$ (since $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ do not commute amongst themselves we cannot find simultaneous eigenfunctions of $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ and $\hat{L}^2$.

The eigenfunction was determined as

$$\psi_{nlm_l}(r,\theta,\phi)=R_{n,l}(r)\,Y_{l,m_l}(\theta,\phi)$$

with
$$Y_{l,m_l}(\theta,\phi)=P_l(\theta)\Phi_{m_l}(\phi) \qquad (13.6)$$

We found that $Y_{l,m_l}(\theta,\phi)$ is an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$ and $\Phi_{m_l}(\phi)$ is an eigenfunction of $\hat{L}_z$, such that

$$\hat{L}^2Y_{l,m_l}(\theta,\phi)=l(l+1)\hbar^2\,Y_{l,m_l}(\theta,\phi) \qquad (13.7a)$$

$$\hat{L}_z\Phi_{m_l}(\phi)=m_l\hbar\,\Phi_{m_l}(\phi) \qquad (13.7b)$$

The eigenvalues of $\hat{L}^2$ are discrete and equal to $l(l+1)\hbar^2$ where ***l* is an integer**. The eigenvalues of $\hat{L}_z$ are $m_l\hbar$ and for a given integer *l* there will be following $(2l+1)$ **integer values of m_l :**

$$-l,-l+1,-l+2,...,0,...,l-1,l \qquad (13.7c)$$

And:

$$\hat{H}\psi_{nlm_l}(r,\theta,\phi)=E_n\,R_{n,l}(r)Y_{l,m_l}(\theta,\phi) \qquad (13.7d)$$

We now embark on a slightly different route to understand angular momentum in the context of quantum mechanics. We had earlier derived the commutation of the angular momentum by expressing it in terms of position and momentum. We now start the description purely in terms of the commutators without bothering how the relations came about.

We now derive the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ **without explicitly defining the operators in the coordinate basis**. The method is similar to that followed for the simple harmonic oscillator followed in Unit 12. Before we start let us first introduce the Dirac notation for the eigenfunctions of $\hat{L}_z$, $\hat{L}^2$ and $\hat{H}$ (for a spherically symmetric potential, which are $\Phi_{m_l}(\phi)$, $Y_{l,m_l}(\theta,\phi)$ and $\psi_{nlm_l}(r,\theta,\phi)$, respectively, as:

$$\Phi_{m_l}(\phi)\rightarrow|\beta\rangle \qquad (13.8a)$$

$$Y_{l,m_l}(\theta,\varphi)\rightarrow|\alpha,\beta\rangle \qquad (13.8b)$$

$$\psi_{nlm_l}(r,\theta,\varphi)\rightarrow|E,\alpha,\beta\rangle \qquad (13.8c)$$

We now define the raising and lowering operators.

13.3 RAISING AND LOWERING OPERATORS

We define the following operators:

Raising Operator $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ (13.9a)

Lowering Operator $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ (13.9b)

You can see that the operators $\hat{L}_+$ and $\hat{L}_-$ are the adjoints of each other:

$$\hat{L}_+^\dagger = \hat{L}_x^\dagger - i\hat{L}_y^\dagger = \hat{L}_-$$

and $$\hat{L}_-^\dagger = \hat{L}_x^\dagger + i\hat{L}_y^\dagger = \hat{L}_+$$

since $\hat{L}_x^\dagger = \hat{L}_x$ and $\hat{L}_y^\dagger = \hat{L}_y$

Using the commutation relations in Eq. (13.5), you can show that:

$$\left[\hat{L}_z, \hat{L}_+\right] = \hbar\hat{L}_+ \quad (13.10a)$$

$$\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\hat{L}_- \quad (13.10b)$$

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z \quad (13.10c)$$

$$\left[\hat{L}^2, \hat{L}_+\right] = 0 \quad (13.10d)$$

$$\left[\hat{L}^2, \hat{L}_-\right] = 0 \quad (13.10e)$$

Further, since $\hat{L}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2$, using Eqs. (13.5), (13.9a) and (13.9b) we get:

$$\begin{aligned}\hat{L}_+\hat{L}_- &= \left(\hat{L}_x + i\hat{L}_y\right)\left(\hat{L}_x - i\hat{L}_y\right) = \hat{L}_x{}^2 + \hat{L}_y{}^2 - i\left[\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\right] \\ &= \hat{L}^2 - \hat{L}_z^2 - i\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}^2 - \hat{L}_z^2 - i\left(i\hbar\hat{L}_z\right) \\ &= \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z \quad (13.11a)\end{aligned}$$

And

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z \quad (13.11b)$$

Adding Eqs. (13.11a) and (13.11b) we can write:

$$\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ = 2\left(\hat{L}^2 - \hat{L}_z^2\right) \quad (13.12)$$

And hence

$$\hat{L}^2 = \hat{L}_z^2 + \frac{1}{2}\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) \quad (13.13)$$

You may like to work out some of these relations for yourself.

SAQ 1

Prove the following relations:

i) $\left[\hat{L}_z, \hat{L}_+\right] = \hbar\hat{L}_+$ ii) $\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\hat{L}_-$ iii) $\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z$

iv) $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$ v) $\left[\hat{L}^2, \hat{L}_+\right] = 0$ vi) $\left[\hat{L}^2, \hat{L}_-\right] = 0$

Using just these relations, let us now determine the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$.

13.4 EIGENVALUES OF $\hat{L}^2$ AND $\hat{L}_z$

We have already shown in Sec. 10.4.3 that when two observables commute, they have a complete set of common eigenvectors.

Since the operators $\hat{L}^2$ and $\hat{L}_z$ commute, they have a common set of eigenstates. We can therefore write:

$$\hat{L}^2|\alpha,\beta\rangle = \alpha|\alpha,\beta\rangle \tag{13.14a}$$

$$\hat{L}_z|\alpha,\beta\rangle = \beta|\alpha,\beta\rangle \tag{13.14b}$$

Where α and β are the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ respectively, for the eigenstate $|\alpha,\beta\rangle$, which we now have to determine. Using Eq. (13.10a) we can write:

$$\hat{L}_z\hat{L}_+ = \hat{L}_+\hat{L}_z + \hbar\hat{L}_+ \tag{13.15}$$

Let us now consider the action of this operator $\hat{L}_z\hat{L}_+$ on the state $|\alpha,\beta\rangle$.

$$\hat{L}_z\hat{L}_+|\alpha,\beta\rangle = \left(\hat{L}_+\hat{L}_z + \hbar\hat{L}_+\right)|\alpha,\beta\rangle \tag{13.16}$$

Using Eq. (13.14b) we get:

$$\hat{L}_z\left(\hat{L}_+|\alpha,\beta\rangle\right) = (\beta+\hbar)\left(\hat{L}_+|\alpha,\beta\rangle\right) \tag{13.17}$$

From Eq. (13.17) it is clear that the state $\hat{L}_+|\alpha,\beta\rangle$ is an eigenstate of the operator $\hat{L}_z$ with an eigenvalue $(\beta+\hbar)$. Recall that the eigenvalue of $\hat{L}_z$ for the state $|\alpha,\beta\rangle$ was β. The action of $\hat{L}_+$ on a state for which the eigenvalue of $\hat{L}_z$ is β, is to change it to a state for which the the eigenvalue of $\hat{L}_z$ is increased by $\hbar$ to a value $\beta+\hbar$. $\hat{L}_+|\alpha,\beta\rangle$ is an eigenstate for which, the eigenstate of $\hat{L}_z$ has been "**raised**" from β to $\beta+\hbar$. $\hat{L}_+$ is therefore called a **raising** operator.

Let us check the action of $\hat{L}^2$ on $\hat{L}_+|\alpha,\beta\rangle$. Using Eq. (13.10d) we have $\hat{L}^2\hat{L}_+ = \hat{L}_+\hat{L}^2$ and :

$$\hat{L}^2\left(\hat{L}_+|\alpha,\beta\rangle\right) = \hat{L}_+\hat{L}^2|\alpha,\beta\rangle = \alpha\left(\hat{L}_+|\alpha,\beta\rangle\right) \tag{13.18}$$

So the eigenvalue of $\hat{L}^2$ for the state $\hat{L}_+|\alpha,\beta\rangle$ is the same as for the state $|\alpha,\beta\rangle$, which is α. So the operator $\hat{L}_+$ acting on a state, **raises** the value of $\hat{L}_z$ by $\hbar$, but leaves the eigenvalue of $\hat{L}^2$ unchanged. Thus, continuing the same nomenclature, we would like to name the new eigenket, $|\alpha,\beta+\hbar\rangle$. The normalization factor is yet to be determined. Because if $|\ \rangle$ is an eigenket of an operator, then $a|\ \rangle$, where *a* is a number is also an eigenket.

So we write $\hat{L}_+|\alpha,\beta\rangle \propto |\alpha,\beta+\hbar\rangle$. More specifically

$$\hat{L}_+|\alpha,\beta\rangle = C_+|\alpha,\beta+\hbar\rangle \tag{13.19}$$

where C_+ is a constant. C_+ is to be determined which we do by demanding that $C_+|\alpha,\beta\rangle$ is normalized. For this we will have first understand the algebra of $(\hat{L}_+)^\dagger = \hat{L}_-$. If $\hat{L}_+$ were to act on the state $|\alpha,\beta+\hbar\rangle$, you would get the state $|\alpha,\beta+2\hbar\rangle$ and so on.

We can repeat the same analysis for the operator $\hat{L}_-$. The action of the operator $\hat{L}_z\hat{L}_-$ on the state $|\alpha,\beta\rangle$ is the following (SAQ 2):

$$\hat{L}_z\left(\hat{L}_-|\alpha,\beta\rangle\right) = (\beta-\hbar)\left(\hat{L}_-|\alpha,\beta\rangle\right) \tag{13.20}$$

The action of the operator $\hat{L}_-$ on a state for which the eigenvalue of $\hat{L}_z$ is β, is to change it to a state for which the eigenvalue of $\hat{L}_z$ is decreased by $\hbar$ to a value $\beta-\hbar$. This is why $\hat{L}_-$ is called the "**lowering**" operator. You can also show that:

$$\hat{L}^2\left(\hat{L}_-|\alpha,\beta\rangle\right) = \alpha\left(\hat{L}_-|\alpha,\beta\rangle\right) \tag{13.21}$$

So the action of the lowering operator is to lower the value of $\hat{L}_z$ by $\hbar$, but leave the eigenvalue of $\hat{L}^2$ unchanged. We write this in an equation as:

$$\hat{L}_-|\alpha,\beta\rangle = C_-|\alpha,\beta-\hbar\rangle \tag{13.22}$$

where C_- is a constant. Note that if $\hat{L}_-$ were to act on the state $|\alpha,\beta-\hbar\rangle$, you would get the state $|\alpha,\beta-2\hbar\rangle$ and so on.

You can prove these relations for yourself.

SAQ 2

Prove the following relations:

i) $\hat{L}_z\left(\hat{L}_-|\alpha,\beta\rangle\right) = (\beta-\hbar)\left(\hat{L}_-|\alpha,\beta\rangle\right)$

ii) $\hat{L}^2\left(\hat{L}_-|\alpha,\beta\rangle\right) = \alpha\left(\hat{L}_-|\alpha,\beta\rangle\right)$

From Eqs. (13.17 to 13.21) you can see that given an eigenstate $|\alpha,\beta\rangle$ you can have eigenstates $|\alpha,\beta+\hbar\rangle, |\alpha,\beta+2\hbar\rangle, |\alpha,\beta+3\hbar\rangle$, ……and also $|\alpha,\beta-\hbar\rangle, |\alpha,\beta-2\hbar\rangle, |\alpha,\beta-3\hbar\rangle$, ……, seemingly an infinite number of eigenstates. This suggests that for any given eigenvalue of $\hat{L}^2$ (say α), $\hat{L}_z$ could have a whole range of (discrete not continuous) eigenvalues, $-\infty < \beta < \infty$, with consecutive values differing by $\hbar$. β could be arbitrarily large or small.

In classical mechanics, this is clearly not true, because we must have $|L_z| \le \sqrt{L^2}$. Let us see if quantum mechanics places any restrictions on the values of α and β.

We can write $\hat{L}^2 - \hat{L}_z{}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2$. The expectation value of $\hat{L}^2 - \hat{L}_z{}^2$ in the state $|\alpha,\beta\rangle$ can be written as:

$$\left\langle\alpha,\beta\left|\hat{L}^2 - \hat{L}_z{}^2\right|\alpha,\beta\right\rangle = \left\langle\alpha,\beta\left|\hat{L}_x{}^2 + \hat{L}_y{}^2\right|\alpha,\beta\right\rangle \tag{13.23}$$

which is

$$(\alpha - \beta^2)\langle \alpha,\beta|\alpha,\beta\rangle = \left\langle \alpha,\beta\middle|\hat{L}_x{}^2 + \hat{L}_y{}^2\middle|\alpha,\beta\right\rangle \qquad (13.24)$$

Now, assuming $\hat{L}_x|\alpha,\beta\rangle = |P\rangle$ and $\hat{L}_y|\alpha,\beta\rangle = |Q\rangle$, we can write

$$\langle \alpha,\beta|\hat{L}_x^2|\alpha,\beta\rangle = \langle \alpha,\beta|\hat{L}_x\,\hat{L}_x|\alpha,\beta\rangle$$

$$= \langle \alpha,\beta|\hat{L}_x^{\dagger}\,\hat{L}_x|\alpha,\beta\rangle = \langle P|P\rangle \geq 0 \qquad (13.25a)$$

and $$\langle \alpha,\beta|\hat{L}_y^2|\alpha,\beta\rangle = \langle \alpha,\beta|\hat{L}_y\,\hat{L}_y|\alpha,\beta\rangle$$

$$= \langle \alpha,\beta|\hat{L}_y^{\dagger}\,\hat{L}_y|\alpha,\beta\rangle = \langle Q|Q\rangle \geq 0 \qquad (13.25b)$$

Hence the right hand side of Eq. (13.24) is positive definite:

$$\left\langle \alpha,\beta\middle|\hat{L}_x{}^2 + \hat{L}_y{}^2\middle|\alpha,\beta\right\rangle \geq 0 \qquad (13.25c)$$

So $$\alpha - \beta^2 \geq 0 \Rightarrow \alpha \geq \beta^2 \qquad (13.26)$$

Eq. (13.26) tells us that the maximum value of $|\beta|$ is restricted by the value of α. Let us call this maximum value β_{max}.

For the state $|\alpha,\beta_{max}\rangle$ we can say that,

$$\hat{L}_+|\alpha,\beta_{max}\rangle = 0 \qquad (13.27)$$

The state $|\alpha,\beta_{max}\rangle$ cannot be raised to a higher value of β.

Since $\hat{L}_+|\alpha,\beta_{max}\rangle = 0$, acting upon $\hat{L}_+|\alpha,\beta_{max}\rangle$ with the operator $\hat{L}_-$ we get

$$\hat{L}_-\left(\hat{L}_+|\alpha,\beta_{max}\rangle\right) = 0 \qquad (13.28a)$$

From Eq. (13.11b) we know that $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$, so Eq. (13.28a) is

$$\left(\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z\right)|\alpha,\beta_{max}\rangle = 0 \qquad (13.28b)$$

Using Eqs.(13.14a and b) in Eq. (13.28b) we get

$$\left(\alpha - \beta_{max}^2 - \hbar\beta_{max}\right)|\alpha,\beta_{max}\rangle = 0 \Rightarrow \alpha = \beta_{max}(\beta_{max} + \hbar) \qquad (13.29)$$

Let us now consider the action of $\hat{L}_-$ on the state $|\alpha,\beta_{max}\rangle$. Using Eq. (13.22) with the constant $C_- = C_{max}$ let us write

$$\hat{L}_-|\alpha,\beta_{max}\rangle = C_{max}|\alpha,\beta_{max} - \hbar\rangle \qquad (13.30a)$$

Where C_{max} is a constant. We could repeat the process *r* times to get

$$\left(\hat{L}_-\right)^r|\alpha,\beta_{max}\rangle = C_{max-r}|\alpha,\beta_{max} - r\hbar\rangle = C_{max-r}|\alpha,\beta'\rangle \qquad (13.30b)$$

where $\beta' = \beta_{max} - r\hbar$. However there would be a value of *r* for which any further application of $\hat{L}_-$ would give us a value of β' that would violate the inequality of Eq. (13.26). Let us call this value β' as $\beta_{min} = \beta_{max} - r\hbar$ (*r* is an integer). For the state $|\alpha,\beta_{min}\rangle$ we can say that,

$$\hat{L}_-|\alpha,\beta_{min}\rangle = 0 \qquad (13.31)$$

The state $|\alpha,\beta_{min}\rangle$ cannot be lowered to a lower value of β. Acting upon $\hat{L}_-|\alpha,\beta_{min}\rangle$ with the operator $\hat{L}_+$ and using Eq. (13.11b) we get

$$\hat{L}_+\left(\hat{L}_-|\alpha,\beta_{min}\rangle\right)=\left(\hat{L}^2-\hat{L}_z^2+\hbar\hat{L}_z\right)|\alpha,\beta_{min}\rangle=0 \qquad (13.32)$$

which is

$$\hat{L}_-\hat{L}_+=\hat{L}^2-\hat{L}_z^2-\hbar\hat{L}_z$$

$$\left(\alpha-\beta_{min}^2+\hbar\beta_{min}\right)|\alpha,\beta_{min}\rangle=0\Rightarrow\alpha=\beta_{min}(\beta_{min}-\hbar) \qquad (13.33)$$

Comparing Eqs. (13.28) and (13.31) we get:

$$\beta_{min}(\beta_{min}-\hbar)=\beta_{max}(\beta_{max}+\hbar) \qquad (13.34a)$$

From which we get that

$$\beta_{min}^2-\beta_{min}\hbar=\beta_{max}^2+\beta_{max}\hbar$$

$$\Rightarrow \qquad (\beta_{min}+\beta_{max})(\beta_{min}-\beta_{max}-\hbar)=0 \qquad (13.34b)$$

The two possible solutions to Eq. (13.34a) are $\beta_{min}=-\beta_{max}$ and $\beta_{min}=\beta_{max}+\hbar$.

Since β_{min} must be less than β_{max}, the solution $\beta_{min}=\beta_{max}+\hbar$ is clearly unphysical. Therefore the only possible solution is

$$\beta_{min}=-\beta_{max} \qquad (13.35)$$

So for any eigenvalue of α of $\hat{L}^2$, the values of the eigenvalues of $\hat{L}_z$, β have the following properties

- $\beta^2\le\alpha$
- β lies between β_{min} and β_{max}: $\beta_{min}\le\beta\le\beta_{max}$ with $\beta_{min}=-\beta_{max}$
- the interval $\beta_{max}-\beta_{min}=\beta_{max}-(-\beta_{max})=2\beta_{max}$
- β increases from β_{min} to β_{max} in steps of $\hbar$

So the values of β are:

$$-\beta_{max},-\beta_{max}+\hbar,-\beta_{max}+2\hbar,....,\beta_{max}-2\hbar,\beta_{max}-\hbar,\beta_{max} \qquad (13.36)$$

The interval between β_{min} and β_{max} is $2\beta_{max}$ and we go from β_{max} to β_{min} in *r* steps of $\hbar$, so $\beta_{min}=\beta_{max}-r\hbar$:

$$\beta_{max}-\beta_{min}=r\hbar\Rightarrow 2\beta_{max}=r\hbar\Rightarrow\beta_{max}=\frac{r\hbar}{2}\text{ with }r=0,1,2,.... \qquad (13.37)$$

From Eq. (13.37) we see that, because *r* can have all integer values, the value of β_{max} **can be integral or half integral multiples of** $\hbar$**.** Also since $r=\frac{2\beta_{max}}{\hbar}$, the total number of eigenvalues of β for a given value of α is $\frac{2\beta_{max}}{\hbar}+1$.

Using the result of Eq. (13.37) in Eq. (13.29) we get the value of α to be:

$$\alpha=\frac{r\hbar}{2}\left(\frac{r\hbar}{2}+\hbar\right)=\hbar^2\frac{r}{2}\left(\frac{r}{2}+1\right)\text{ with }r=0,1,2,.... \qquad (13.38a)$$

And for every eigenvalue of $\hat{L}^2$: $\alpha = \hbar^2 \frac{r}{2}\left(\frac{r}{2}+1\right)$, there are $\frac{2\beta_{max}}{\hbar}+1$ eigenvalues of $\hat{L}_z$, with $\beta_{max} = \frac{r\hbar}{2}$:

$$-\frac{r\hbar}{2}, -\frac{r\hbar}{2}+\hbar, -\frac{r\hbar}{2}+2\hbar, ..., \frac{r\hbar}{2}-2\hbar, \frac{r\hbar}{2}-\hbar, \frac{r\hbar}{2} \qquad (13.38b)$$

So our result for the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ is different from what we had in Unit 8. In Unit 8 we had derived the eigenvalues of $\hat{L}_z$ to be $m_l\hbar$, where m_l can have $2l+1$ integral values between $-l$ to l, (l itself is an integer) and the eigenvalues of $\hat{L}^2$ were $\hbar^2 l(l+1)$.

Comparing Eqs. (8.39a and b) with Eqs. (13.38a and b), and writing $\frac{r}{2}\left(=\frac{\beta_{max}}{\hbar}\right)$ as j where j can take the values $0, \frac{1}{2}, 1, \frac{3}{2}, ...$ we can write $\alpha = \hbar^2 j(j+1)$. j is different from l, because l has only integral values.

The result of this entire analysis is then the following:

- The eigenstates of $\hat{L}^2$ correspond to the eigenvalues $\hbar^2 j(j+1)$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$
- There are $2j+1$ eigenvalues of $\hat{L}_z$ (β) corresponding to every eigenvalue of $\hat{L}^2$. So for any value of j, eigenvalues of $\hat{L}_z$ run from $-j\hbar, -j\hbar+\hbar, -j\hbar+2\hbar, ..., j\hbar-2\hbar, j\hbar-\hbar, j\hbar$ in steps of $\hbar$.

  We label these eigenvalues by the number m_j with

  $$m_j = -j, -j+1, -j+2, ..., j-2, j-1, j.$$

  So the eigenvalues of $\hat{L}_z$ are $m_j\hbar$ for every value of j.
- $\beta_{max}^2 \left(= j^2\hbar^2\right) < \alpha\left(= j(j+1)\hbar^2\right)$ for all values of j except $j = 0$. For $j = 0$, $\beta_{max}^2 = \alpha$.
- Each eigenstate of $\hat{L}^2$ is also an eigenstate of $\hat{L}_z$ ($\hat{L}^2$ and $\hat{L}_z$ are commuting operators) and has a $(2j+1)$-fold degeneracy, corresponding to the $2j+1$ eigenvalues of $\hat{L}_z$.
- You can label the angular momentum eigenstates by the eigen vales of $\hat{L}^2$ and $\hat{L}_z$, which are α and β respectively, as $\left|\hbar^2 j(j+1), m_j\hbar\right\rangle$. However, it is more convenient to label the eigenstates of the angular momentum by the numbers j and m_j as $\left|j, m_j\right\rangle$ (in analogy with $\left|l, m_l\right\rangle$).
- The eigenkets $\left|j, m_j\right\rangle$ form the basis and are assumed to satisfy the following orthonormality condition:

  $$\left\langle j, m_j \middle| j', m_{j'} \right\rangle = \delta_{jj'}\delta_{m_j m_{j'}} \qquad (13.39)$$

At this point **we choose another symbol $\hat{J}$ for the angular momentum operator** (so long represented by the symbol $\hat{L}$) and write

$$\hat{J}^2 \left| j, m_j \right\rangle = \hbar^2 j(j+1) \left| j, m_j \right\rangle \tag{13.40a}$$

$$\hat{J}_z \left| j, m_j \right\rangle = m_j \hbar \left| j, m_j \right\rangle \tag{13.40b}$$

All the rules written so far for $\hat{L}$ can be written in terms of $\hat{J}$, just by replacing $\hat{L}$ by $\hat{J}$ in each equation. For example:

$$\left[\hat{J}^2, \hat{J}_z\right] = 0 \tag{13.41a}$$

And

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar \hat{J}_z; \left[\hat{J}_y, \hat{J}_z\right] = i\hbar \hat{J}_x; \left[\hat{J}_z, \hat{J}_x\right] = i\hbar \hat{J}_y \tag{13.41b}$$

You may like to apply these rules to a few values of *j*.

SAQ 3

Write down the values of *j*, α, m_j, β_{max} and the states $\left| j, m_j \right\rangle$ for $j = 0$, $j = \frac{1}{2}$, $j = 1$ and $j = \frac{3}{2}$. Also state the degeneracy of each angular momentum state.

We have not said anything about the physical implications of the fact that the z-component of the angular momentum may be a half integral multiple of $\hbar$. In the next section we discuss a bit about this new result, which is at different from what we had derived in Unit 8.

13.4.1 Angular Momentum Operator $\hat{J}$

Let us highlight what we have done differently in this Unit to obtain this different result for the eigenvalues of the angular momentum operators.

Recollect that in the derivation of the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ or $\hat{J}^2$ and $\hat{J}_z$ in Sec 13.3, we have **only** invoked the commutation relations of the angular momentum. At no point have we invoked the actual form of the angular momentum operator in the coordinate basis as the components of $\vec{L} = \vec{r} \times \vec{p}$ or used $\hat{L}_z = i\hbar \frac{\partial}{\partial \phi}$. Had we done that, we would have gotten the eigenvalues of the *z*-component of the angular momentum operator to be integral multiples of $\hbar$, as we did in Unit 8.

It appears that the angular momentum $\hat{J}$ can be written as the sum of two operators. For one of these operators the z-component has eigenvalues which are only integral multiples of $\hbar$, for the other operator the z-component can have eigenvalues which are either integral or half integral multiples of $\hbar$. So we state, in a somewhat adhoc fashion that the angular momentum $\hat{J}$ can be written as:

$$\hat{J} = \hat{L} + \hat{S} \tag{13.42}$$

$\hat{J}$ is called the total angular momentum.

Where $\hat{L}$ is called the **orbital angular momentum** which has the classical analogue $\vec{L} = \vec{r} \times \vec{p}$. The eigenvalues of $\hat{L}_z$ are $0, \pm\hbar, \pm 2\hbar, \ldots$. Since $\hat{J}$ satisfies the same commutation relations as $\hat{L}$, $\hat{S}$ which is called the **spin angular momentum** (or just the **spin**) must also satisfy the same commutation relations and $\hat{S}_z$ must have eigenvalues $0, \pm\frac{\hbar}{2}, \pm\hbar, \pm\frac{3\hbar}{2}$. $\hat{S}$ **does not have any classical analogue** nor can its foundation be explained in non-relativistic quantum mechanics. In relativistic quantum mechanics there is a natural explanation for spin. You know off course that there are many particles which have half integral spins, like the electron, the proton etc.

From classical mechanics you know that the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is the generator of rotation. You will study in your next course on Quantum Mechanics in Semester II that the angular momentum operator $\hat{J}$ is defined in such a way that it is the generator of infinitesimal rotations. So $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are respectively the generators of infinitesimal rotations about the *x*, *y* and *z* axes. The commutator relations in the angular momentum (Eq. 13.41) actually reflect the properties of the combinations of infinitesimal rotations and show that rotations about different axes do not commute.

When the wave function is not a scalar function $\psi(x, y, z)$ as we have assumed all through this course so far, but has, say, two or more component field, then the generators of infinitesimal rotations must have both the components *L* and *S*.

Finally, for a system in which the spin angular momentum is zero or can be ignored, the total angular momentum operator is just the orbital angular momentum operator $\hat{J} \equiv \hat{L} = \hat{r} \times \hat{p}$ and we get back the results of Unit 8-9. In the next unit we you study more about the spin angular momentum operator.

We next calculate the constants C_+ and C_- introduced in Eqs. (13.19) and (13.22) respectively.

13.4.2 Calculating C_+ and C_-

We first rewrite Eqs. (13.19) and (13.22) as:

$$\hat{J}_+ \left| j, m_j \right\rangle = C_+ \left| j, m_j + 1 \right\rangle \qquad (13.43a)$$

$$\hat{J}_- \left| j, m_j \right\rangle = C_- \left| j, m_j - 1 \right\rangle \qquad (13.43b)$$

Let us now define a state $|R\rangle$ as follows:

$$|R\rangle = \hat{J}_+ \left| j, m_j \right\rangle = C_+ \left| j, m_j + 1 \right\rangle \qquad (13.44a)$$

The adjoint of the ket $|R\rangle$ is

$$\langle R| = \left\langle j, m_j \right| \hat{J}_+{}^\dagger = \left\langle j, m_j \right| \hat{J}_- \Rightarrow \langle R| = C_+^* \left\langle j, m_j + 1 \right| \qquad (13.44b)$$

In going from Eq. (13.44a) to Eq. (13.44b) we have used the property $\hat{J}_+^\dagger = \hat{J}_-$. Taking the inner product of the states $|R\rangle$ and $\langle R|$ we get:

$$\langle R|R\rangle = \left\langle j,m_j \mid \hat{J}_-\hat{J}_+ \mid j,m_j \right\rangle = C_+C_+^{*}\left\langle j,m_j+1 \mid j,m_j+1\right\rangle \tag{13.45}$$

Using the orthonormality of the eigenkets given in Eq. (13.39) we can write

$$\left\langle j,m_j \mid \hat{J}_-\hat{J}_+ \mid j,m_j \right\rangle = |C_+|^2 \tag{13.46}$$

Eq. (13.11b) can be rewritten for $\hat{J}$ as

$$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \tag{13.47}$$

Substituting for $\hat{J}_-\hat{J}_+$ from Eq. (13.47) into Eq. (13.46) we get

$$\left\langle j,m_j \left| \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \right| j,m_j \right\rangle = |C_+|^2 \tag{13.48}$$

or
$$|C_+|^2 = \left\langle j,m_j \left|\hat{J}^2\right| j,m_j \right\rangle - \left\langle j,m_j \left|\hat{J}_z^2\right| j,m_j \right\rangle - \left\langle j,m_j \left|\hbar\hat{J}_z\right| j,m_j \right\rangle \tag{13.49}$$

Using the results of Eqs. (13.40 and b) in Eq. (13.49) we can write:

$$\begin{aligned} |C_+|^2 &= j(j+1)\hbar^2\left\langle j,m_j \mid j,m_j\right\rangle - m_j{}^2\hbar^2\left\langle j,m_j \mid j,m_j\right\rangle \\ &\qquad - m_j\hbar^2\left\langle j,m_j \mid j,m_j\right\rangle \\ &= \left[j(j+1) - m_j{}^2 - m_j\right]\hbar^2 \\ &= (j-m_j)(j+m_j+1)\hbar^2 \end{aligned} \tag{13.50}$$

using $\left\langle j,m_j \mid j,m_j\right\rangle = 1$. From Eq. (13.50) we finally get:

$$C_+ = \left[(j-m_j)(j+m_j+1)\right]^{1/2}\hbar \quad \text{but for a phase factor} \tag{13.51}$$

Since the value of C_+ for the state $\left|j,m_j\right\rangle$ depends on the values of both j and m_j we write $C_+ = C_+(j,m_j)$.

And Eq. (13.42) is:

$$\begin{aligned} \hat{J}_+\left|j,m_j\right\rangle &= C_+(j,m_j)\left|j,m_j+1\right\rangle \\ &= \left[(j-m_j)(j+m_j+1)\right]^{1/2}\hbar\left|j,m_j+1\right\rangle \end{aligned} \tag{13.52}$$

It can also be shown that

$$\begin{aligned} \hat{J}_-\left|j,m_j\right\rangle &= C_-(j,m_j)\left|j,m_j-1\right\rangle \\ &= \left[(j+m_j)(j-m_j+1)\right]^{1/2}\hbar\left|j,m_j-1\right\rangle \end{aligned} \tag{13.53}$$

Since the maximum and minimum values of m_j are j and $-j$ respectively, it is also true that

$$\hat{J}_+|j,j\rangle = 0; \hat{J}_-|j,-j\rangle = 0 \tag{13.54}$$

In compact notation we write Eqs. (13.52) and (13.53) as

$$\hat{J}_{\pm}\left|j,m_j\right\rangle = C_{\pm}(j,m_j)\left|j,m_j \pm 1\right\rangle$$

$$= \left[(j \mp m_j)(j \pm m_j + 1)\right]^{1/2}\hbar\left|j,m_j \pm 1\right\rangle \qquad (13.55)$$

SAQ 4

Show that

$$J_{-}\left|j,m_j\right\rangle = C_{-}(j,m_j)\left|j,m_j - 1\right\rangle = \left[(j + m_j)(j - m_j + 1)\right]^{1/2}\hbar\left|j,m_j - 1\right\rangle$$

We now learn how to write down the matrices corresponding to the angular momentum operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ and $\hat{J}^2$.

13.5 MATRIX ELEMENTS OF THE ANGULAR MOMENTUM OPERATORS

We first write down the matrix elements of $\hat{J}^2$ in the basis $\left|\ j,m_j\ \right\rangle$.

$$\left(J^2\right)_{j'jm_{j'}m_j} = \left\langle j',m_{j'}\left|\hat{J}^2\right|j,m_j\right\rangle = j(j+1)\hbar^2\delta_{j'j}\delta_{m_{j'}m_j} \qquad (13.56)$$

The matrix elements of $\hat{J}_z$ are

$$(J_z)_{j'jm_{j'}m_j} = \left\langle j',m_{j'}\left|\hat{J}_z\right|j,m_j\right\rangle = m_j\hbar\delta_{j'j}\delta_{m_{j'}m_j}\ . \qquad (13.57)$$

So the matrices J^2 and J_z are diagonal in the $\left|j,m_j\right\rangle$ basis.

We next determine the matrix elements of $\hat{J}_x$ and $\hat{J}_y$. We rewrite Eqs. (13.9a and b) in terms of as $\hat{J}$ as

Raising Operator $\hat{J}_{+} = \hat{J}_x + i\hat{J}_y$ (13.58a)

Lowering Operator $\hat{J}_{-} = \hat{J}_x - i\hat{J}_y$ (13.58b)

So $$\hat{J}_x = \frac{1}{2}\left(\hat{J}_{+} + \hat{J}_{-}\right)\ ;\ \hat{J}_y = \frac{1}{2i}\left(\hat{J}_{+} - \hat{J}_{-}\right) \qquad (13.59)$$

The matrix elements of $\hat{J}_x$ in the basis $\left|j,m_j\right\rangle$ can be written as

$$(J_x)_{j'jm_{j'}m_j} = \left\langle j',m_{j'}\left|\hat{J}_x\right|j,m_j\right\rangle = \frac{1}{2}\left\langle j',m_{j'}\left|\hat{J}_{+} + \hat{J}_{-}\right|j,m_j\right\rangle \qquad (13.60)$$

Using the results of Eqs. (13.52) and (13.53) we get:

$$(J_x)_{j'jm_{j'}m_j} = \frac{1}{2}\left[(j - m_j)(j + m_j + 1)\right]^{1/2}\hbar\left\langle j',m_{j'}\ \middle|\ j,m_j + 1\right\rangle$$

$$+ \frac{1}{2}\left[(j + m_j)(j - m_j + 1)\right]^{1/2}\hbar\left\langle j',m_{j'}\ \middle|\ j,m_j - 1\right\rangle$$

$$= \frac{\hbar}{2}\left[(j - m_j)(j + m_j + 1)\right]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j+1}$$

$$+ \frac{\hbar}{2}\left[(j + m_j)(j - m_j + 1)\right]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j-1} \qquad (13.61)$$

You can also determine $(J_y)_{j'jm_{j'}m_j}$ in the same way.

SAQ 5

Determine $(J_y)_{j'jm_{j'}m_j}$.

So

$$(J_y)_{j'jm_{j'}m_j} = \frac{\hbar}{2i}[(j-m_j)(j+m_j+1)]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j+1}$$

$$-\frac{\hbar}{2i}[(j+m_j)(j-m_j+1)]^{1/2}\hbar\delta_{j'j}\delta_{m_{j'}m_j-1} \quad (13.62)$$

The matrices J_x and J_y are not diagonal.

Using the relations developed so far you can write down the matrix elements of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ and $\hat{J}^2$ for different values of *j*.

We now write down the matrices for $\hat{J}_x, \hat{J}_y, \hat{J}_z$ and $\hat{J}^2$ for $j = 1/2$ which is the lowest value the *j* can have. It is the lowest dimensional (2 dimensional) space in which the angular momentum can be realized. We will discuss the significance of this in the next unit.

13.5.1 Matrix Elements for *j* = 1/2

In SAQ 3 we wrote down the eigenkets for $j = 1/2$. There are two eigenkets as you know, which in the $|j, m_j\rangle$ basis are $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$. To make it simpler to write down the matrix elements we label the eigenkets as $|\phi_1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and $|\phi_2\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

Matrix elements of $\hat{J}^2$

From Eq. (13.56) you know that the matrix J^2 is diagonal in this basis and the only non-zero element are for $j = j'$ and $m_j = m_{j'}$. So with $j = j' = 1/2$ we can write:

$$J^2{}_{11} = \langle\phi_1|\hat{J}^2|\phi_1\rangle = \left\langle\frac{1}{2}, \frac{1}{2}\middle|\hat{J}^2\middle|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2$$

$$= \frac{3}{4}\hbar^2 \quad (13.63a)$$

$$J^2{}_{22} = \langle\phi_2|\hat{J}^2|\phi_2\rangle = \left\langle\frac{1}{2}, -\frac{1}{2}\middle|\hat{J}^2\middle|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2$$

$$= \frac{3}{4}\hbar^2 \quad (13.63b)$$

$$J^2{}_{12} = \langle\phi_1|\hat{J}^2|\phi_2\rangle = \left\langle\frac{1}{2}, \frac{1}{2}\middle|\hat{J}^2\middle|\frac{1}{2}, -\frac{1}{2}\right\rangle = 0 \quad (13.63c)$$

$$J^2{}_{21} = \langle\phi_2|\hat{J}^2|\phi_1\rangle = \left\langle\frac{1}{2}, -\frac{1}{2}\middle|\hat{J}^2\middle|\frac{1}{2}, \frac{1}{2}\right\rangle = 0 \quad (13.63d)$$

So the matrix J^2 is

$$J^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{13.64}$$

Matrix elements of $\hat{J}_z$

Using Eq. (13.57) you know that the matrix J_z (like J^2) is diagonal and the only non-zero elements are for $j = j'$ and $m_j = m_{j'}$. So with $j = j' = 1/2$ we can write:

$$(J_z)_{11} = \left\langle \phi_1 \middle| \hat{J}_z \middle| \phi_1 \right\rangle = \left\langle \frac{1}{2},\frac{1}{2} \middle| \hat{J}_z \middle| \frac{1}{2},\frac{1}{2} \right\rangle = \frac{1}{2}\hbar \tag{13.65a}$$

$$(J_z)_{22} = \left\langle \phi_2 \middle| \hat{J}_z \middle| \phi_2 \right\rangle = \left\langle \frac{1}{2},-\frac{1}{2} \middle| \hat{J}_z \middle| \frac{1}{2},-\frac{1}{2} \right\rangle = -\frac{1}{2}\hbar \tag{13.65b}$$

$$(J_z)_{12} = \left\langle \phi_1 \middle| \hat{J}_z \middle| \phi_2 \right\rangle = \left\langle \frac{1}{2},\frac{1}{2} \middle| \hat{J}_z \middle| \frac{1}{2},-\frac{1}{2} \right\rangle = 0 \tag{13.65c}$$

$$(J_z)_{21} = \left\langle \phi_2 \middle| \hat{J}_z \middle| \phi_1 \right\rangle = \left\langle \frac{1}{2},-\frac{1}{2} \middle| \hat{J}_z \middle| \frac{1}{2},\frac{1}{2} \right\rangle = 0 \tag{13.65d}$$

So the matrix J_z is

$$J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{13.66}$$

Matrix elements of $\hat{J}_x$

Using Eq. (13.61) you know that the matrix J_x is not diagonal. An element $\left\langle j', m_{j'} \middle| \hat{J}_x \middle| j, m_j \right\rangle$ is non-zero only if $j = j'$ **and** $m_j' = m_j - 1$ or $m_j' = m_j + 1$. So with $j = j' = 1/2$ we can write:

$$(J_x)_{11} = \left\langle \phi_1 \middle| \hat{J}_x \middle| \phi_1 \right\rangle = \left\langle \frac{1}{2},\frac{1}{2} \middle| \hat{J}_x \middle| \frac{1}{2},\frac{1}{2} \right\rangle = 0 \tag{13.67a}$$

$$(J_x)_{22} = \left\langle \phi_2 \middle| \hat{J}_x \middle| \phi_2 \right\rangle = \left\langle \frac{1}{2},-\frac{1}{2} \middle| \hat{J}_x \middle| \frac{1}{2},-\frac{1}{2} \right\rangle = 0 \tag{13.67b}$$

$$\begin{aligned}(J_x)_{12} &= \left\langle \phi_1 \middle| \hat{J}_x \middle| \phi_2 \right\rangle = \left\langle \frac{1}{2},\frac{1}{2} \middle| \hat{J}_z \middle| \frac{1}{2},-\frac{1}{2} \right\rangle \\ &= \frac{\hbar}{2}\left[\left\{\frac{1}{2}-\left(-\frac{1}{2}\right)\right\}\left\{\frac{1}{2}+\left(-\frac{1}{2}\right)\right\}+1)\right]^{1/2} = \frac{\hbar}{2}\end{aligned} \tag{13.67c}$$

To get this result we use $j = j' = 1/2$ and $m_{j'} = \frac{1}{2} = m_j + 1$ where $m_j = -\frac{1}{2}$ in Eq. (13.61).

$$\begin{aligned}(J_x)_{21} &= \left\langle \phi_2 \middle| \hat{J}_x \middle| \phi_1 \right\rangle = \left\langle \frac{1}{2},-\frac{1}{2} \middle| \hat{J}_x \middle| \frac{1}{2},\frac{1}{2} \right\rangle \\ &= \frac{\hbar}{2}\left[\left\{\frac{1}{2}+\left(\frac{1}{2}\right)\right\}\left\{\frac{1}{2}-\left(\frac{1}{2}\right)\right\}+1)\right]^{1/2} = \frac{\hbar}{2}\end{aligned} \tag{13.67d}$$

To get this result we use $j = j' = 1/2$ and $m_{j'} = -\frac{1}{2} = m_j - 1$ where $m_j = \frac{1}{2}$ in Eq. (13.61).

So the matrix J_x is

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13.68}$$

Matrix elements of $\hat{J}_y$

We use Eq. (13.62) to obtain the matrix J_y, which like J_x, is not diagonal. An element $\langle j', m_{j'} | \hat{J}_x | j, m_j \rangle$ is non-zero only if $j = j'$ **and** $m_j{}' = m_j - 1$ or $m_j{}' = m_j + 1$. So with $j = j' = 1/2$ we get:

$$(J_y)_{11} = \langle \phi_1 | \hat{J}_y | \phi_1 \rangle = \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{J}_y \middle| \frac{1}{2}, \frac{1}{2} \right\rangle = 0 \tag{13.69a}$$

$$(J_y)_{22} = \langle \phi_2 | \hat{J}_y | \phi_2 \rangle = \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{J}_y \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle = 0 \tag{13.69b}$$

$$\begin{aligned} (J_y)_{12} &= \langle \phi_1 | \hat{J}_y | \phi_2 \rangle = \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{J}_y \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ &= \frac{\hbar}{2i}\left[\left\{ \frac{1}{2} - \left(-\frac{1}{2} \right) \right\} \left\{ \frac{1}{2} + \left(-\frac{1}{2} \right) \right\} + 1) \right]^{1/2} = -i\frac{\hbar}{2} \end{aligned} \tag{13.69c}$$

To get this result we use $j = j' = 1/2$ and $m_{j'} = \frac{1}{2} = m_j + 1$ where $m_j = -\frac{1}{2}$ in Eq. (13.62).

$$\begin{aligned} (J_y)_{21} &= \langle \phi_2 | \hat{J}_y | \phi_1 \rangle = \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{J}_y \middle| \frac{1}{2}, \frac{1}{2} \right\rangle \\ &= -\frac{\hbar}{2i}\left[\left\{ \frac{1}{2} + \left(\frac{1}{2} \right) \right\} \left\{ \frac{1}{2} - \left(\frac{1}{2} \right) \right\} + 1) \right]^{1/2} = i\frac{\hbar}{2} \end{aligned} \tag{13.69d}$$

To get this result we use $j = j' = 1/2$ and $m_{j'} = -\frac{1}{2} = m_j - 1$ where $m_j = \frac{1}{2}$ in Eq. (13.62).

So the matrix J_y is :

$$J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{13.70}$$

Since we started with a recap of the spherical harmonics in the coordinate basis in Sec 13.2, it seems only fair that we see how we can get back the spherical harmonics in the coordinate basis from the eigenket notation.

13.6 EIGENFUNCTIONS OF THE ANGULAR MOMENTUM OPERATORS

We start with the definition $\hat{J} \equiv \hat{L} = \hat{r} \times \hat{p}$, so the angular momentum has only the component $\hat{L}$. We write down the operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in the coordinate representation as:

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \quad (13.71a)$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \quad (13.71b)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \quad (13.71c)$$

Using Eqs. (13.71a and b) in the definition of $\hat{L}_+$ and $\hat{L}_-$ from Eqs. (13.9a and b) we get

$$\hat{L}_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \quad (13.72a)$$

$$\hat{L}_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \quad (13.72b)$$

Consider the eigenket $|l, m_l\rangle$ which is a simultaneous eigenket of $\hat{L}^2$ and $\hat{L}_z$.

We know that

$$\hat{L}^2|l, m_l\rangle = l(l+1)\hbar^2|l, m_l\rangle \quad (13.73a)$$

$$\hat{L}_z|l, m_l\rangle = m_l\hbar|l, m_l\rangle \quad (13.73b)$$

The coordinate representation of $|l, m_l\rangle$ is $Y_{l,m_l}(\theta,\phi)$:

$$|l, m_l\rangle \xrightarrow{\text{coordinate representation}} Y_{l,m_l}(\theta,\phi) \quad (13.74)$$

$Y_{l,m_l}(\theta,\phi) = P_l(\theta)\Phi_{m_l}(\phi)$ where $\Phi_{m_l}(\phi)$ is the eigenfunction of $\hat{L}_z$.

$$\hat{L}_z|l, m_l\rangle = m_l\hbar|l, m_l\rangle \longrightarrow -i\hbar\frac{\partial}{\partial\phi}[P_l(\theta)\Phi_{m_l}(\phi)]$$

$$= m_l\hbar[P_l(\theta)\Phi_{m_l}(\phi)] \quad (13.75)$$

Solving the ODE of Eq. (13.75) we get $\Phi_{m_l}(\phi) = e^{im_l\phi}$ and hence $Y_{l,m_l}(\theta,\phi) = P_l(\theta)e^{im_l\phi}$.

We now try to determine the eigenfunctions using the raising operator.

For any value of *l* the maximum possible value of m_l is *l*. Hence using Eq. (13.72) and (13.52) we get:

$$\hat{L}_+|l, l\rangle = 0 \xrightarrow{\text{Coordinate Representation}} \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_{l,l}(\theta,\phi) = 0 \quad (13.76)$$

Writing $Y_{l,l}(\theta,\phi) = P_l(\theta)e^{il\phi}$, we can solve the ODE of Eq. (13.76) to determine $P_l(\theta)$ (TQ 3). The eigenfunction is:

$$Y_{l,l}(\theta,\phi) = Ce^{il\phi}\sin^l\theta \quad (13.77)$$

where *C* is the normalization constant. Using m_l, l in Eq. (13.53) we can write:

$$\hat{L}_-|l, m_l\rangle = [(l+m_l)(l-m_l+1)]^{1/2}\hbar|l, m_l-1\rangle \quad (13.78)$$

which gives us:

$$|l, m_l - 1\rangle = \frac{1}{[(l + m_l)(l - m_l + 1)]^{1/2}\hbar}\left[\hat{L}_-|l, m_l\rangle\right] \quad (13.79)$$

With $m_l = l$, Eq. (13.79) is

$$|l, l - 1\rangle = \frac{1}{\hbar(2l)^{1/2}}\left[\hat{L}_-|l, l\rangle\right] \quad (13.80)$$

From Eq. (13.79) using the coordinate representation of $\hat{L}_-$ we get:

$$Y_{l,l-1}(\theta,\phi) = \frac{1}{\hbar(2l)^{1/2}}\left[\hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_{l,l}(\theta,\phi)\right] \quad (13.81)$$

Using Eq. (13.81), if $Y_{l,l}(\theta,\phi)$ is known, we can derive $Y_{l,l-1}(\theta,\phi)$. This process can be repeated to get all the functions Y_{l,m_l} upto $Y_{l,0}$. For $m_l < 0$, we can use the condition

$$Y_{l,-m_l}(\theta,\phi) = (-1)^{m_l}\left[Y_{l,m_l}(\theta,\phi)\right]^* \quad (13.82)$$

to derive the eigenfunctions.

13.7 SUMMARY

- In the Dirac notation the eigenfunctions of $\hat{L}_z$, $\hat{L}^2$ and $\hat{H}$ (for a spherically symmetric potential), $\Phi_{m_l}(\phi)$, $Y_{l,m_l}(\theta,\phi)$ and $\psi_{nlm_l}(r,\theta,\phi)$, respectively, are represented by:

  $$\Phi_{m_l}(\phi) \to |\beta\rangle;\ Y_{l,m_l}(\theta,\varphi) \to |\alpha,\beta\rangle;\ \psi_{nlm}(r,\theta,\varphi) \to |E,\alpha,\beta\rangle$$

- We define the angular momentum raising and lowering operators as

  Raising Operator $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$

  Lowering Operator $\hat{L}_- = \hat{L}_x - i\hat{L}_y$

  $\hat{L}_+$ and $\hat{L}_-$ are the adjoints of each other. The commutation relations for the raising and lowering operators are

  $$\left[\hat{L}_z,\hat{L}_+\right] = \hbar\hat{L}_+\ ;\quad \left[\hat{L}_z,\hat{L}_-\right] = -\hbar\hat{L}_-;\ \left[\hat{L}_+,\hat{L}_-\right] = 2\hbar\hat{L}_z;$$
  $$\left[\hat{L}^2,\hat{L}_+\right] = 0;\ \left[\hat{L}^2,\hat{L}_-\right] = 0$$

  And

  $$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z \text{ and } \hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z$$

- We determine the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ using the raising and lowering operators, assuming the commutation rules for the angular momentum operators as proved in Units 4 and 8.

- The eigenstates of $\hat{L}^2$ are found to correspond to the eigenvalues $\hbar^2 j(j+1)$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$

There are $2j+1$ eigenvalues of $\hat{L}_z$ corresponding to every eigenvalue of $\hat{L}^2$. So for any value of j, the eigenvalues of $\hat{L}_z$ are $m_j\hbar$ with

$$m_j = -j, -j+1, -j+2, ..., j-2, j-1, j .$$

Each eigenstate of $\hat{L}^2$ is also an eigenstate of $\hat{L}_z$ and has a $(2j+1)$-fold degeneracy, corresponding to the $2j+1$ eigenvalues of $\hat{L}_z$.

The eigenstates are labelled by the numbers j and m_j as $|j, m_j\rangle$

The eigenkets $|j, m_j\rangle$ form a basis and are assumed to satisfy the following orthonormality condition: $\langle j, m_j | j', m_{j'}\rangle = \delta_{jj'}\delta_{m_j m_{j'}}$

- We introduce the total angular momentum $\hat{J}$ as the sum of the orbital angular momentum $\hat{L}$ and the spin angular momentum $\hat{S}$ (or just spin): $\hat{J} = \hat{L} + \hat{S}$. The components of $\hat{J}$ satisfy the same commutation rules as $\hat{L}$ and for the eigenstates $|j, m_j\rangle$ we can write:

$$\hat{J}^2|j, m_j\rangle = \hbar^2 j(j+1)|j, m_j\rangle; \quad \hat{J}_z|j, m_j\rangle = m_j\hbar|j, m_j\rangle$$

The **orbital angular momentum** $\hat{L}$ has the classical analogue $\vec{L} = \vec{r} \times \vec{p}$ and the eigenvalues of $\hat{L}_z$ are $0, \pm\hbar, \pm 2\hbar, ...$. The eigenvalues of $\hat{S}_z$ are $0, \pm\frac{\hbar}{2}, \pm\hbar, \pm\frac{3\hbar}{2}$. **$\hat{S}$ does not have any classical analogue**

- For the raising and lowering operators $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ and $\hat{J}_- = \hat{J}_x - i\hat{J}_y$, respectively we have

$$\hat{J}_+|j, m_j\rangle = C_+(j, m_j)|j, m_j + 1\rangle$$
$$= [(j - m_j)(j + m_j + 1)]^{1/2}\hbar|j, m_j + 1\rangle$$
$$J_-|j, m_j\rangle = C_-(j, m_j)|j, m_j - 1\rangle$$
$$= [(j + m_j)(j - m_j + 1)]^{1/2}\hbar|j, m_j - 1\rangle$$

- We can determine the matrix element of the angular momentum operators $\hat{J}^2, \hat{J}_x, \hat{J}_y, \hat{J}_z$ using the relations:

$$(J^2)_{j'jm_{j'}m_j} = \langle j', m_{j'}|\hat{J}^2|j, m_j\rangle = j(j+1)\hbar^2\delta_{j'j}\delta_{m_{j'}m_j}$$

$$(J_z)_{j'jm_{j'}m_j} = \langle j', m_{j'}|\hat{J}_z|j, m_j\rangle = m_j\hbar\delta_{j'j}\delta_{m_{j'}m_j}$$

$$(J_x)_{j'jm_{j'}m_j} = \frac{\hbar}{2}[(j - m_j)(j + m_j + 1)]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j+1}$$
$$+ \frac{\hbar}{2}[(j + m_j)(j - m_j + 1)]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j-1}$$

$$(J_y)_{j'jm_{j'}m_j} = \frac{\hbar}{2i}[(j - m_j)(j + m_j + 1)]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j+1}$$
$$- \frac{\hbar}{2i}[(j + m_j)(j - m_j + 1)]^{1/2}\hbar\delta_{j'j}\delta_{m_{j'}m_j-1}$$

- The matrix elements for j = 1/2 are

$$J^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$J_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ J_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};\ J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

13.8 TERMINAL QUESTIONS

1. Determine:
   i) $\langle j,m_j|\hat{J}_x|j,m_j\rangle$ ii) $\langle j,m_j|\hat{J}_y|j,m_j\rangle$ iii) $\langle j,m_j|\hat{J}_x^2|j,m_j\rangle$
   iv) $\langle j,m_j|\hat{J}_y^2|j,m_j\rangle$ v) $\langle j,m_j|\hat{J}_z^2|j,m_j\rangle$
2. Using the results of TQ 1 show that $\left\langle\left(\Delta\hat{J}_x\right)^2\right\rangle = \left\langle\left(\Delta\hat{J}_y\right)^2\right\rangle$ and that $\left\langle\left(\Delta\hat{J}_x\right)^2\right\rangle\left\langle\left(\Delta\hat{J}_y\right)^2\right\rangle$ satisfies the generalized uncertainty principle.
3. For a system in which $\hat{J}\equiv\hat{L}=\hat{r}\times\hat{p}$, using the condition $\hat{L}^+|l,l\rangle = 0$ show that $Y_{l,l} = Ce^{il\phi}\sin^l\theta$.
4. Determine the angular momentum matrices for $j = 1$.
5. Verify that the matrices J_x, J_y, J_z satisfy the commutation rules (Eq. 13.41b): $[J_x,J_y]=i\hbar J_z;[J_y,J_z]=i\hbar J_x;[J_z,J_x]=i\hbar J_y$ for $j=\frac{1}{2}$.

13.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. i) $$\begin{aligned}[\hat{L}_z,\hat{L}_+] &= [\hat{L}_z,\hat{L}_x + i\hat{L}_y] = [\hat{L}_z,\hat{L}_x] + i[\hat{L}_z,\hat{L}_y]\\ &= i\hbar\hat{L}_y + i(-i\hbar\hat{L}_x) = \hbar[\hat{L}_x + i\hat{L}_y]\\ &= \hbar\hat{L}_+\end{aligned}$$

   ii) $$\begin{aligned}[\hat{L}_z,\hat{L}_-] &= [\hat{L}_z,\hat{L}_x - i\hat{L}_y] = [\hat{L}_z,\hat{L}_x] - i[\hat{L}_z,\hat{L}_y]\\ &= i\hbar\hat{L}_y - \hbar\hat{L}_x = -\hbar[\hat{L}_x - i\hat{L}_y]\\ &= -\hbar\hat{L}_-\end{aligned}$$

   iii) $$\begin{aligned}[\hat{L}_+,\hat{L}_-] &= [\hat{L}_x + i\hat{L}_y,\hat{L}_x - i\hat{L}_y]\\ &= [\hat{L}_x,\hat{L}_x] + i[\hat{L}_y,\hat{L}_x] + [L_y,L_y] - i[\hat{L}_x,\hat{L}_y]\\ &= 0 + i(-i\hbar\hat{L}_z) + 0 - i(i\hbar\hat{L}_z) = 2\hbar\hat{L}_z\end{aligned}$$

   iv) $$\begin{aligned}\hat{L}_-\hat{L}_+ &= (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y)\\ &= \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) = \hat{L}^2 - \hat{L}_z^2 + i(i\hbar\hat{L}_z)\\ &= \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z\end{aligned}$$

   v) $$[\hat{L}^2,\hat{L}_+] = [\hat{L}^2,\hat{L}_x + i\hat{L}_y] = [\hat{L}^2,\hat{L}_x] + i[\hat{L}^2,\hat{L}_y] = 0$$
   $$[\hat{L}^2,\hat{L}_-] = [\hat{L}^2,\hat{L}_x - i\hat{L}_y] = [\hat{L}^2,\hat{L}_x] - i[\hat{L}^2,\hat{L}_y] = 0$$

2. i) From Eq. (13.10b) $\hat{L}_z\hat{L}_- = \hat{L}_-\hat{L}_z - \hbar\hat{L}_-$

Therefore $\hat{L}_z\hat{L}_-|\alpha,\beta\rangle = \left(\hat{L}_-\hat{L}_z - \hbar\hat{L}_-\right)|\alpha,\beta\rangle$

Using Eq. (13.14b) we get:

$$\hat{L}_z\left(\hat{L}_-|\alpha,\beta\rangle\right) = (\beta-\hbar)\left(\hat{L}_-|\alpha,\beta\rangle\right)$$

ii) Using Eq. (13.10e) we get:

$$\hat{L}^2\left(\hat{L}_-|\alpha,\beta\rangle\right) = \hat{L}_-\hat{L}^2|\alpha,\beta\rangle = \alpha\left(\hat{L}_-|\alpha,\beta\rangle\right)$$

3.

j	$\alpha = \hbar^2 j(j+1)$	m_j $(-j \text{ to } j)$	$\beta_{max} = j\hbar$	**States** $\lvert j,m_j\rangle$	**Degeneracy** $(2j+1)$
0	0	0	0	$\lvert 0,0\rangle$	1
$\frac{1}{2}$	$\alpha = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}\hbar^2$	$-\frac{1}{2}$	$-\frac{1}{2}\hbar$	$\left\lvert\frac{1}{2},-\frac{1}{2}\right\rangle$	2
		$\frac{1}{2}$	$\frac{1}{2}\hbar$	$\left\lvert\frac{1}{2},\frac{1}{2}\right\rangle$	
1	$\alpha = \hbar^2 1(1+1) = 2\hbar^2$	-1	$-\hbar$	$\lvert 1,-1\rangle$	3
		0	0	$\lvert 1,0\rangle$	
		1	$\hbar$	$\lvert 1,1\rangle$	
$\frac{3}{2}$	$\alpha = \hbar^2\frac{3}{2}\left(\frac{3}{2}+1\right) = \frac{15}{4}\hbar^2$	$-\frac{3}{2}$	$-\frac{3}{2}\hbar$	$\left\lvert\frac{1}{2},-\frac{3}{2}\right\rangle$	4
		$-\frac{1}{2}$	$-\frac{1}{2}\hbar$	$\left\lvert\frac{3}{2},-\frac{1}{2}\right\rangle$	
		$\frac{1}{2}$	$\frac{1}{2}\hbar$	$\left\lvert\frac{3}{2},\frac{1}{2}\right\rangle$	
		$\frac{3}{2}$	$\frac{3}{2}\hbar$	$\left\lvert\frac{1}{2},\frac{3}{2}\right\rangle$	

4. We define a state $|Q\rangle$ as follows:

$$|Q\rangle = \hat{J}_-|j,m_j\rangle = C_-|j,m_j-1\rangle$$

The adjoint of $|Q\rangle$ is

$$\langle Q| = \langle j,m_j|\hat{J}_-{}^\dagger = \langle j,m_j|\hat{J}_+ \Rightarrow \langle Q| = C_-^*\langle j,m_j-1| \text{ (since } \hat{J}_-^\dagger = \hat{J}_+)$$

The inner product of the states $|Q\rangle$ and $\langle Q|$ is:

$$\langle Q|Q\rangle = \langle j,m_j \mid \hat{J}+\hat{J}_- \mid j,m_j\rangle = C_-C_-{}^*\langle j,m_j-1|\, j,m_j-1\rangle$$

$$\Rightarrow \qquad \langle j,m_j \mid \hat{J}_+\hat{J}_- \mid j,m_j\rangle = |C_-|^2 \qquad \text{(i)}$$

Eq. (13.11a) can be rewritten for $\hat{J}$ as : $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$

Eq. (i) reduces to:

$$\left\langle j,m_j\left|\hat{J}^2-\hat{J}_z^2+\hbar\hat{J}_z\right|j,m_j\right\rangle=|C_-|^2$$

$$\Rightarrow \quad |C_-|^2=j(j+1)\hbar^2-m_j{}^2\hbar^2+m_j\hbar^2=(j+m_j)(j-m_j+1)\hbar^2$$

$$\Rightarrow \quad C_-=\left[(j+m_j)(j-m_j+1)\right]^{1/2}\hbar \quad \text{(but for a phase factor)}$$

And

$$J_-\left|j,m_j\right\rangle=C_-(j,m_j)\left|j,m_j-1\right\rangle$$

$$=\left[(j+m_j)(j-m_j+1)\right]^{1/2}\hbar\left|j,m_j-1\right\rangle$$

5. $$(J_y)_{j'jm_{j'}m_j}=\left\langle j',m_{j'}\left|\hat{J}_y\right|j,m_j\right\rangle=\frac{1}{2i}\left\langle j',m_{j'}\left|\hat{J}_+-\hat{J}_-\right|j,m_j\right\rangle$$

$$(J_y)_{j'jm_{j'}m_j}=\frac{1}{2i}\left[(j-m_j)(j+m_j+1)\right]^{1/2}\hbar\left\langle j',m_{j'}\,|\,j,m_j+1\right\rangle$$

$$-\frac{1}{2i}\left[(j+m_j)(j-m_j+1)\right]^{1/2}\hbar\left\langle j',m_{j'}\,|\,j,m_j-1\right\rangle$$

$$=\frac{\hbar}{2i}\left[(j-m_j)(j+m_j+1)\right]^{1/2}\delta_{j'j}\delta_{m_{j'}m_j+1}$$

$$-\frac{\hbar}{2i}\left[(j+m_j)(j-m_j+1)\right]^{1/2}\hbar\delta_{j'j}\delta_{m_{j'}m_j-1}$$

Terminal Questions

1. i) $\left\langle j,m_j\left|\hat{J}_x\right|j,m_j\right\rangle=0$ from Eq. (13.61)

ii) $\left\langle j,m_j\left|\hat{J}_y\right|j,m_j\right\rangle=0$ From Eq. (13.62)

iii) $\hat{J}_x=\frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)\Rightarrow\hat{J}_x^2=\frac{1}{4}\left(\hat{J}_+^2+\hat{J}_-^2+\hat{J}_+\hat{J}_-+\hat{J}_-\hat{J}_+\right)$

Now

$$\hat{J}_+\left|j,m_j\right\rangle=C_+(j,m_j)\left|j,m_j+1\right\rangle$$

$$\Rightarrow \hat{J}_+{}^2\left|j,m_j\right\rangle=C_+(j,m_j)C_+(j,m_j+1)\left|j,m_j+2\right\rangle$$

$$\left\langle j,m_j\left|\hat{J}_+^2\right|j,m_j\right\rangle=C_+(j,m_j)\,C_+(j,m_j+1)\left\langle j,m_j\,|\,j,m_j+2\right\rangle$$

$$=0$$

Similarly $\left\langle j,m_j\left|\hat{J}_-^2\right|j,m_j\right\rangle=0$. Therefore

$$\therefore \quad \left\langle J_x^2\right\rangle=\left\langle j,m_j\left|\hat{J}_x^2\right|j,m_j\right\rangle=\frac{1}{4}\left\langle j,m_j\left|\hat{J}_+\hat{J}_-+\hat{J}_-\hat{J}_+\right|j,m\right\rangle$$

$$=\frac{1}{2}\left\langle j,m_j\left|\hat{J}^2-\hat{J}_z^2\right|j,m_j\right\rangle=\frac{1}{2}\left\langle j,m_j\left|\hat{J}^2\right|j,m_j\right\rangle-\frac{1}{2}\left\langle j,m_j\left|\hat{J}_z^2\right|j,m_j\right\rangle$$

$$=\frac{1}{2}\left[j(j+1)\hbar^2-m_j^2\,\hbar^2\right]=\frac{1}{2}\hbar^2\left[j\,(j+1)-m_j^2\right]$$

iv) $\hat{J}_y = \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right) \Rightarrow \hat{J}_y^2 = -\frac{1}{4}\left(\hat{J}_+^2 + \hat{J}_-^2 - \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+\right)$

$$\left\langle J_y^2\right\rangle = \left\langle j, m_j\middle|\hat{J}_y^2\middle|j, m_j\right\rangle = \frac{1}{4}\left\langle j, m_j\middle|\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\middle|j, m_j\right\rangle$$

$$= \frac{1}{2}\left\langle j, m_j\middle|\hat{J}^2 - \hat{J}_z^2\middle|j, m_j\right\rangle = \frac{\hbar^2}{2}\left[j(j+1) - m_j^2\right]$$

v) $\left\langle J_z^2\right\rangle = \left\langle j, m_j\middle|\hat{J}_z^2\middle|j, m_j\right\rangle$

$$\hat{J}_z\hat{J}_z|\, j, m_j\rangle = m_j\hbar\,\hat{J}_z|\, j, m_j\rangle$$

$$= (m_j\,\hbar)^2|\, j, m_j\rangle \Rightarrow \left\langle j, m_j\middle|J_z^2\middle|j, m_j\right\rangle = m_j^2\hbar^2$$

2. $$\left\langle\left(\Delta\hat{J}_x\right)^2\right\rangle = \left\langle\hat{J}_x^2\right\rangle - \left\langle\hat{J}_x\right\rangle^2 = \frac{\hbar^2}{2}\left[j(j+1) - m_j^2\right]$$

$$\left\langle\left(\Delta\hat{J}_y\right)^2\right\rangle = \frac{\hbar^2}{2}\left[j(j+1) - m_j^2\right] = \left\langle\left(\Delta\hat{J}_x\right)\right\rangle^2$$

$$\left\langle(\Delta J_x)^2\right\rangle\left\langle(\Delta J_y)^2\right\rangle = \frac{\hbar^4}{2}\left[j(j+1) - m_j^2\right]^2 \qquad \text{(i)}$$

Also, $\left[J_x, J_y\right] = i\,\hbar\,J_z$, so

$\therefore$ $$\left\langle j, m_j\middle|\left[\hat{J}_x, \hat{J}_y\right]\middle|\, j, m_j\right\rangle = i\,\hbar\left\langle j, m_j\middle|\hat{J}_z\middle|j, m_j\right\rangle = i\,\hbar^2\, m_j$$

$\Rightarrow$ $$\left|\left\langle j, m_j\middle|\left[\hat{J}_x, \hat{J}_y\right]\middle|j, m_j\right\rangle\right|^2 = \hbar^4\, m_j^2 \qquad \text{(ii)}$$

From Eqs. (i) and (ii) we see that $\left\langle(\Delta J_x)^2\right\rangle\left\langle(\Delta J_y)^2\right\rangle$ is always greater than $\frac{1}{4}\left|\langle[J_x, J_y]\rangle\right|^2$ and they will be equal only when $j = 0$. So the condition of the generalized uncertainty principle is satisfied.

3. Using Eqs. (13.76) and (13.77), we can write

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)P_l(\theta)\,e^{il\phi} = 0 \Rightarrow \frac{dP_l(\theta)}{d\theta}e^{il\phi}$$

$$+ i\cot\theta(il)P_l(\theta)e^{il\phi} = 0$$

$\Rightarrow$ $$\frac{dP_l(\theta)}{d\theta} - \cot\theta P_l(\theta) = 0 \qquad \text{(i)}$$

On solving Eq. (i) we get $P_l(\theta) = C\sin^l\theta$. Therefore $Y_{l,l}(\theta,\phi) = P_l(\theta)e^{il\phi} = Ce^{il\phi}\sin^l\theta$.

4. We define

$$|\phi_1\rangle = \left|\, j = 1, m_j = 1\right\rangle = |1,1\rangle\,;\ |\phi_2\rangle = \left|\, j = 1, m_j = 0\right\rangle = |1,0\rangle\,;$$

$$|\phi_3\rangle = \left|\, j = 1, m_j = -1\right\rangle = |1,-1\rangle$$

Using Eqs. (13.56 and 13.57) with $j = 1, m_j = 1, 0, -1$ we get

$$\hat{J}_z|\phi_1\rangle = \hbar|\phi_1\rangle\ ;\ \hat{J}_z|\phi_2\rangle = 0\ ;\ \hat{J}_z|\phi_3\rangle = -\hbar|\phi_3\rangle$$

$$\hat{J}^2|\phi_1\rangle = 2\hbar^2|\phi_1\rangle\ ;\ \hat{J}^2|\phi_2\rangle = 2\hbar^2|\phi_2\rangle\ ;\ \hat{J}^2|\phi_3\rangle = 2\hbar^2|\phi_2\rangle$$

So: $J_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ and $J^2 = 2\hbar^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

J_x and J_y are not diagonal matrices:

$$(J_x)_{11} = (J_x)_{22} = (J_x)_{33} = 0$$

$$(J_x)_{12} = \langle\phi_1|\hat{J}_x|\phi_2\rangle = \langle 1,1|\hat{J}_x|1,0\rangle = \frac{\hbar}{\sqrt{2}}$$

$$(J_x)_{21} = \langle\phi_2|\hat{J}_x|\phi_1\rangle = \langle 1,0|\hat{J}_x|1,1\rangle = \frac{\hbar}{\sqrt{2}}$$

$$(J_x)_{13} = \langle\phi_1|\hat{J}_x|\phi_3\rangle = \langle 1,1|\hat{J}_x|1,-1\rangle = 0 = (J_x)_{31}$$

$$(J_x)_{23} = \langle\phi_2|\hat{J}_x|\phi_3\rangle = \langle 1,0|\hat{J}_x|1,-1\rangle = \frac{\hbar}{\sqrt{2}} = (J_x)_{32}$$

So: $J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$

J_y can be found similarly using Eq. (13.62): $J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$

5. $$[J_x, J_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}2i&0\\0&-2i\end{bmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= i\hbar J_z$$

$$[J_y, J_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right] = \frac{\hbar^2}{4}\begin{pmatrix}0&2i\\2i&0\end{pmatrix}$$

$$= i\hbar\left[\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right] = i\hbar J_x$$

$$[J_z, J_x] = \frac{\hbar^2}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right]$$

$$= i\hbar\left[\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = i\hbar J_y\,.$$