UNIT 14

TIME SERIES MODELLING TECHNIQUES

Structure

14.1 INTRODUCTION

In the previous units (Units 12 and 13), you have learnt stationary and nonstationary time series, the concept of autocorrelation and partial autocorrelation in time series. When the data is autocorrelated, then most of the standard modelling methods may become misleading or sometimes even useless because they are based on the assumption of independent observations. Therefore, we need to consider alternative methods that take into account the autocorrelation in the data. Such types of models are known as time series models. In this unit, you will study time series models, such as autoregressive (AR), moving average (MA) autoregressive moving average (ARMA), autoregressive integrated moving average (ARIMA), etc.

In this unit, you will learn some time series models. We begin with a simple introduction of the necessity of time series models instead of ordinary regression models in Sec. 14.2. In Secs. 14.3 and 14.4, we discuss the autoregressive and moving average models with their types and properties, respectively. In Sec. 14.5, the autoregressive moving average models are

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explained. The AR, MA and ARMA models are used for stationary time series. If a time series is nonstationary then we use autoregressive integrated moving average (ARIMA), therefore, you will learn it in Sec. 14.6. When you deal with real-time series data, then the first question may arise in your mind how you know which time series model is most suitable for a particular time series data. For that, we discuss time series model selection in Sec. 14.7.

Expected Learning Outcomes

After studying this unit, you would be able to:

- ❖ explain the necessity of time series models;
- ❖ describe and use autoregressive models;
- ❖ explain and use moving average models;
- ❖ explore autoregressive moving average model;
- ❖ describe and use autoregressive integrated moving average models; and
- ❖ selection of a particular time series model for real-life time series data.

14.2 TIME SERIES MODELS

In the previous classes, you have learnt about linear regression, and it is one of the most common methods for identifying and quantifying the relationship between a dependent variable and a single (simple linear regression) or multiple (multivariate linear regression) independent variables. The dependent variable (Y) is also called the regress, explained or forecast variable whereas the independent variable(X) is also called the predictor, regressor or explanatory variable.

In the simplest case, the regression model allows for a linear relationship between the forecast variable Y and a single predictor variable X. In the case of a single independent variable, the regression equation is as follows:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

The coefficients β_0 and β_1 denote the intercept and the slope of the regression line, respectively. The intercept β_0 represents the predicted value of Y when X = 0 and the slope β_1 represents the average predicted change in Y resulting from a one-unit change in X. Also each observation Y is consisting of the systematic or explained part of the model, $\beta_0 + \beta_1 X$ and random error ε. The term "error" in this context refers to a departure from the underlying straight line model rather than a mistake and it includes everything affecting Y other than predictor variable X. The error term has the following assumptions:

- The mean of the error term should be zero, i.e., $E[\varepsilon] = 0$.
- The error term should have constant variance, i.e., $Var[\varepsilon] = \sigma^2 = \text{constant}$
- It should not correlate with the predictor variable, i.e.,

$$Cor[\varepsilon, X] = Cov[\varepsilon, X] = E[\varepsilon, X] = 0$$

- The error term should not correlate with its previous values, i.e.,

$$Cor\left[\varepsilon_i, \varepsilon_{i-1}\right] = Cov\left[\varepsilon_i, \varepsilon_{i-1}\right] = E\left[\varepsilon_i, \varepsilon_{i-1}\right] = 0$$

- It should be normally distributed with mean 0 and constant variance.

We may use the regression model in time series, but it has some issues which are as follows:

- The primary issue is that linear regression models assume that the data points of a variable are independent of each other, while time series models work with data points that have a temporal relationship with one another (autocorrelated). For example, there exists a strong relationship between today's temperature and the next day's temperature.
- A second issue is the error distribution. In the regression model, we assume that the errors are independent and identically distributed ("IID") and the error in the model at one time is uncorrelated with the error at other times, which is usually not fulfilled for time series data, which are generally autocorrelated.

However, time series data have their historical features of the trend seasonal, cyclical and irregular components, therefore, they might not fit well with traditional regression models. In such cases, we use **Autoregressive** (**AR**) models, **Moving Average** (**MA**) models, **Autoregressive Moving Average** (**ARMA**) models or **Autoregressive Integrated Moving Average** (**ARIMA**) models, etc. Out of these, AR, MA and ARMA models are used when the time series is stationary whereas the ARIMA model is used when the time series is nonstationary. We will discuss one at a time.

14.3 AUTOREGRESSIVE MODELS

In time series data, it is generally observed that the value of a variable observed in the current time period will be similar to its value in the previous period, or even the period before that, and so on. For example, if the temperature is quite high today, it is reasonable to assume that the temperature will also be high tomorrow. Therefore, the past temperature has an impact more on the future temperature. In other words, we can say that there exists a strong relationship between today's temperature and the next day's temperature. Therefore, to forecast the temperature, we can take previous days' temperatures as a predictor when fitting a regression model for such time series data. Consider one more example, suppose we want to predict today's behaviour of a child. Generally, we see the behaviour of a child matches with his/her father/mother or grandfather/grandmother. So if today's behaviour of a child depends on the behaviour of his/her father/mother then we can take the father/mother as a predictor/lag because the father/mother is the previous generation or grandfather/grandmother as lag two generations as previous to the previous generation. So we can say that there exists autocorrelation and we can predict today's behaviour of a child using the behaviour of his/her parents or grandparents. The regression models in which the predictors are the past values of the series instead of independent variables are called autoregressive (AR) models. Because such a regression model uses data from the same input variable at previous time steps, therefore, it is referred to as an autoregression (regression of self). We can define autoregressive models as

Autoregressive (AR) models are the models in which the value of a variable in the current period is regressed against its own lagged values, that is, the value in the previous period.

The AR models are very useful in situations where the next forecasted value is a function of the previous time period, such as, if it's rainy today, the data suggests that it's more likely to rain tomorrow than if it's clear today. The autoregressive models are used for stationary time series. On the basis of the correlation with the previous values, the autocorrelation model has different types which are discussed in the next sub-sections.

14.3.1 First-order Autoregressive Models

The autoregressive model is a model in which the value of a variable in the current period is regressed against its previous value then it is called first-order autoregression. The number of lags used as regressors is called the order of autoregression. So, the preceding model is a first-order autoregression and it is written as AR(1) where 1 represents the order.

Since in time series, we have measured values of a variable over time, therefore, we use **"t"** as a subscript in the variable of time series models. If y_t and y_{t-1} are the values of a variable at time t and $t-1$, respectively then we can express the first-order autoregressive model as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t$$

The model expresses the present value as a linear combination of the mean of the series δ (read as **delta)**, the previous value of the variable y_{t-1} and the error term ε_t (read as **epsilon)**. The magnitude of the impact of the previous value on the present value is quantified using a coefficient denoted with ϕ_1 (read as **phai)**. The "error term " is called white noise and it is normally distributed with mean zero and constant variance (σ^2).

We now study the properties of the model.

Mean and Variance

If X and Y are independent random variables and a, b & c are constant then $E(aX \pm bY \pm c) = aE(X) \pm bE(Y) \pm c$

We can find mean of an AR(1) model as follows:

$$E[y_t] = \mu = E[\delta] + \phi_1 E[y_{t-1}] + E[\varepsilon_t]$$

$$\text{Mean} = \delta + \phi_1\mu + 0 \left[\begin{array}{l}\text{Since time series is stationary, therefore,}\\ E[y_t] = E[y_{t-1}] = \mu \text{ and}\\ \varepsilon_t \sim N[0,\sigma^2], \text{ therefore, } E[\varepsilon_t] = 0\end{array}\right]$$

Thus,

$$\text{Mean} = \frac{\delta}{1-\phi_1}$$

Similarly,

If X and Y are independent random variables and a, b & c are constant then $Var(aX \pm bY \pm c) = a^2Var(X) \pm b^2Var(Y)$

We can find the variance of an AR(1) model as follows:

$$Var[y_t] = Var[\delta] + \phi_1^2 Var[y_{t-1}] + Var[\varepsilon_t]$$

Since time series is stationary, therefore, $Var[y_t] = Var[y_{t-1}]$

Also, δ is a constant, therefore, $Var[\delta] = 0$ and $\varepsilon_t \sim N[0, \sigma^2]$, therefore, $Var[\varepsilon_t] = \sigma^2$.

Therefore,

$$Var[y_t] = \phi_1^2 Var[y_t] + \sigma^2$$

$$Var[y_t] = \frac{\sigma^2}{1-\phi_1^2} \geq 0 \text{ when } \phi_1^2 < 1$$

Autocovariance and Autocorrelation Functions

We now find the autocovariance of a stationary AR(1) model as follows:

$$\gamma_k = Cov[y_t, y_{t+k}] = Cov[\delta + \phi_1 y_{t-1} + \varepsilon_t, y_{t+k}]$$

$$= Cov[\delta, y_{t+k}] + +\phi_1 Cov[y_{t-1}, y_{t+k}] + Cov[\varepsilon_t, y_{t+k}]$$

Since the covariance between a constant and a variable is zero, therefore,

$$Cov[\delta, y_{t+k}] = 0,$$

Also, $Cov[y_{t-1}, y_{t+k}] = \gamma_{k-1}$ and by the property of ε_t

$$Cov[\varepsilon_t, y_{t+k}] = 0$$

Therefore,

$$\gamma_k = \phi_1 \gamma_{k-1}$$

Hence,

$$\gamma_1 = \phi_1 \gamma_0 = \phi_1 Var(y_t) = \phi_1 \frac{\sigma^2}{1-\phi_1^2} \left[\text{Since } Var[y_t] = \frac{\sigma^2}{1-\varphi_1^2}\right]$$

$$\gamma_2 = \phi_1 \gamma_1 = \phi_1^2 \frac{\sigma^2}{1-\phi_1^2} \qquad \left[\text{Since } \gamma_1 = \phi_1 \frac{\sigma^2}{1-\phi_1^2}\right]$$

$$\gamma_k = \phi_1 \gamma_{k-1} = \phi_1^k \frac{\sigma^2}{1-\phi_1^2}$$

The autocorrelation function(ACF) for an AR(1) model is as follows:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k \text{ for } k = 0, 1, 2, \ldots$$

Since ϕ_1 lies between -1 to 1, therefore, for a positive value of ϕ_1, the ACF (ϕ_1^k) exponentially decreases to 0 as the lag k increases. For the negative value of ϕ_1, the ACF also exponentially decays to 0 as the lag increases, but the algebraic signs for the autocorrelations alternate between positive and negative.

Conditions for Stationarity

An autoregressive model can be used on time series data if and only if the time series is stationary. Therefore, some constraints on the values of the parameters are required for stationarity which are as follows:

$$|\phi_1| < 1 \Rightarrow -1 < \phi_1 < 1$$

Since stationarity is necessary for applying autoregressive models, therefore, before applying it to time series data, you will have to check whether the time series is stationary or not. If it is nonstationary, then you will have to apply some transformation methods (such as differencing, log transformation, etc. as discussed in Unit 13) to transform the series into a stationary and we use the ARIMA model. We will discuss it later in this unit.

14.3.2 Second-order Autoregressive Models

The autoregressive model in which the value of a variable in the current period is regressed against its two previous values then the autoregressive model is called the second-order autoregression model.

The second-order autoregression model is written as AR(2) where 2 represents the order.

If y_t, y_{t-1} and y_{t-2} are the values of a variable at time $t, t-1$ and $t-2$, respectively then the second-order autoregressive model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

We now study the properties of the AR(2) model as we have studied for AR(1).

Mean and Variance

We can find the mean and variance of an AR(2) model as we have obtained for AR(1). Here, we just write these as follows:

$$\text{Mean} = \frac{\delta}{1-\phi_1-\phi_2}$$

Similarly,

$$\text{Var}[y_t] = \frac{\sigma^2}{1-\phi_1^2-\phi_2^2} \geq 0 \text{ when } \phi_1^2+\phi_2^2 < 1$$

Autocovariance and Autocorrelation Functions

We can find the autocovariance of an AR(2) model as we have obtained for AR(1) model. Here, we just write these as follows:

$$\gamma_k = \text{Cov}[y_t, y_{t+k}] = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \begin{cases} \sigma^2 & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

Therefore,

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2$$

and

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \sigma^2; k = 1, 2, \ldots$$

The autocorrelation function for an AR(2) model can be obtained by dividing γ_k by γ_0 as follows:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \quad \text{for } k = 1, 2, \ldots$$

Therefore,

$$\rho_1 = \phi_1\rho_0 + \phi_2\rho_{-1}$$

$$\rho_1 - \phi_2\rho_1 = \phi_1 \quad (\text{Since } \rho_0 = 1, \rho_{-1} = \rho_1)$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0 = \phi_1\rho_1 + \phi_2$$

$$\rho_3 = \phi_1\rho_2 + \phi_2\rho_1$$

Conditions for Stationarity

Since an autoregressive model can be used on time series data if and only if the time series is stationary. Therefore, some constraints on the values of the parameters are required for stationarity which are as follows:

- $|\phi_2| < 1 \Rightarrow -1 < \phi_2 < 1$
- $\phi_1 + \phi_2 < 1$
- $\phi_2 - \phi_1 < 1$

After understanding the first and second-order autoregressive models, you are interested to know the general form of the autoregressive model. Let's discuss the general form of the autoregressive model.

14.3.3 pth-order Autoregressive Models

More generally, a pth-order autoregression, written as AR(p), is a multiple linear regression in which the value of the series at any time t is a linear function of the values at times t−1,t−2,…,t−p.

If $y_t, y_{t-1}, ..., y_{t-p}$ are the values of a variable at time t, t−1,t−2,…,t−p, respectively, then the pth-order autoregressive model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t$$

We now study the properties of the AR(p) model as we have studied for AR(1) and AR(2) models.

Mean and Variance

We can find the mean and variance of an AR(p) model as we have obtained for AR(1). Here, we just write these as follows:

$$\text{Mean} = \frac{\delta}{1-\phi_1-\phi_2-...-\phi_p}$$

and

$$\text{Var}[y_t] = \frac{\sigma^2}{1-\phi_1^2-\phi_2^2-...-\phi_p^2} \geq 0 \text{ when } \phi_1^2+\phi_2^2+...+\phi_p^2 < 1$$

Autocovariance and Autocorrelation Functions

We can find the autocovariance of an AR(2) model as we have obtained for AR(1). Here, we just write these as follows:

$$\gamma_k = \text{Cov}[y_t, y_{t+k}] = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + ... + \phi_p\gamma_{k-p} + \begin{cases} \sigma^2 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

Therefore,

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + ... + \phi_p\gamma_p$$

and

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + ... + \phi_p\gamma_{k-p} + \sigma^2;\ k = 1,2,...$$

The autocorrelation function for an AR(2) model can be obtained by dividing γ_k by γ_0 as follows:

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \ldots + \phi_p\rho_{k-p} \quad \text{for } k = 1, 2, \ldots$$

Conditions for Stationarity

When p ≥ 3, the restrictions for stationarity are much more complicated, therefore, we have not discussed them.

After understanding the autoregressive models, let's look at an example which helps you to understand how to check stationarity and calculate mean, variance, autocovariance and autocorrelation functions of an autoregressive model.

Example 1: Consider a time series model

$$y_t = 10 + 0.2y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim N[0,1]$

(i) Is this a stationary time series?

(ii) What are the mean and variance of the time series?

(iii) Calculate the autocorrelation function.

(iv) If the current observation is y_{100} = 7.5, would you expect the next observation to be above or below the mean?

Solution:

For checking the stationarity of the time series, first of all, we find the parameters of the time series model. Since the variable in the current period is regressed against its previous value, therefore, it is the first-order autoregressive model. We now compare it with its standard form, that is, $y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t$ and we obtain

$$\delta = 10 \text{ and } \phi_1 = 0.2$$

The condition for stationarity of the first-order autoregressive model is $-1 < \phi_1 < 1$. Since $-1 < \phi_1 = 0.2 < 1$, therefore, the series is stationary.

We now calculate the mean and variance of the series as

$$\text{Mean} = \frac{\delta}{1-\phi_1} = \frac{10}{1-0.2} = 12.5$$

$$\text{Var}[y_t] = \frac{\sigma^2}{1-\phi_1^2} = \frac{1}{1-0.04} = \frac{1}{0.96} = 1.04$$

We now calculate the autocorrelation function as

$$\gamma_1 = \phi_1\gamma_0 = \phi_1\frac{\sigma^2}{1-\phi_1^2} = 0.2 \times 1.04 = 0.21$$

$$\gamma_2 = \phi_1\gamma_1 = 0.2 \times 0.21 = 0.04$$

$$\gamma_3 = \phi_1\gamma_2 = 0.2 \times 0.04 = 0.008$$

We can forecast, the next value of y_{100} using the prediction model as

$$\hat{y}_t = 10 + 0.2y_{t-1}$$

$$\hat{y}_{101} = 10 + 0.2y_{100} = 10 + 0.2 \times 7.5 = 10 + 0.15 = 11.15$$

Therefore, if the current observation is y_{100} = 7.5, then the next observation y_{101} = 11.15 will be above the mean (12.5).

You may like to try the following Self Assessment Question before studying further.

SAQ 1

Consider the time series model

$$y_t = 5 + 0.8y_{t-1} - 0.5y_{t-2} + \varepsilon_t$$

where $\varepsilon_t \sim N[0,2]$

(i) Is this a stationary time series?

(ii) What are the mean and variance of the time series?

(iii) Calculate the autocorrelation function.

(iv) Plot the correlogram.

(v) Suppose the observations for time periods 50 and 51 are 38 and 40 then forecast the next observation.

14.4 MOVING AVERAGE MODELS

In the previous section, you learned the autoregression models in which the value of a variable in the current period is regressed against its own lagged values, that is, the value in the previous period. In some cases, the forecasting model is unable to capture all series patterns, and therefore some information is left over in model residuals which are called forecasting errors. For example, in our example of today's behaviour of a child, it may be possible that the behaviour of a child does not depend on the behaviour of his/her family, and it may be possible that it depends on the behaviour of his/her friends, therefore, whatever his/her friend perform in the past the child performs today. It means that the behaviour may also depend on other unknown factors. In other words, we can say that we are not aware of that factors and they are not considered in the model. As you know, the factors which are not considered in the model come under the residual. It means that the current value of a variable may also depend on the past residuals. The goal of the moving average models is to capture patterns in the residuals, if they exist, by modelling the relationship between the current value of a variable, the error term(ϵ_t), and the past error (residuals) terms of the models. We can define moving average models as

Moving average (MA) models are the models in which the value of a variable in the current period is regressed against the residuals in the previous period.

A moving average model, states that the current value is linearly dependent on the past error terms.

A Moving Average model is similar to an autoregressive model, except that instead of being a linear combination of past time series values, it is a linear combination of the past error/residual/white noise terms.

Concept of Invertibility

An autoregressive model can be used on time series data if and only if the time series is stationary. The moving average models are always stationary. But some restrictions also are imposed on the parameters of the moving average model as in the case of the autoregressive model.

The MA model is defined in many textbooks and computer software with minus sign before the θ terms. Although this switches the algebraic signs of estimated coefficient values and (unsquared) theta terms in formulas for ACFs and variances. It has no effect on the model's overall theoretical features. In order to accurately construct the estimated model, you must examine your software to make sure that either to make sure that either negative or positive signs are used. R software uses positive signs in its underlying model, as we take here.

As by the definition of the moving average model, the value of a variable in the current period is regressed against the residuals in the previous period and if y_t is the value of a variable at time t and ε_{t-1} is the residuals at time t –1 then we can express moving average model of first-order as

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

We can write the above expression as

$$\varepsilon_t = y_t - \mu - \theta_1 \varepsilon_{t-1}$$

$$\varepsilon_t = y_t - \mu - \theta_1 \left(y_{t-1} - \mu - \theta_1 \varepsilon_{t-2} \right)$$

$$= \mu\left(\theta_1 - 1\right) + y_t + \theta_1 y_{t-1} + \theta_1^2 \varepsilon_{t-2}$$

$$\vdots$$

$$\varepsilon_t = c + \sum_{i=0}^{\infty} \theta_1^i y_t^i$$

It means that we can convert/invert the past residuals/errors//noises into past observations. In other words, we can convert/invert a moving average model into the autoregressive model. This property is called **invertibility**. This notion is very important if one wants to forecast the future values of the dependent variable, otherwise, the forecasting task will be impossible (i.e., the residuals in the past cannot be estimated, as it cannot be observed). Actually, when the model is not invertible, the innovations can still be represented by observations of the future, this is not helpful at all for forecasting purposes.

Any autoregressive process is necessarily invertible but a stationarity condition must be imposed to ensure the uniqueness of the model for a particular autocorrelation structure. A moving average process on the other hand is stationary but in order for there to be a unique model for a particular autocorrelation structure an invertbility condition must be imposed.

We now discuss different types of moving average models on the basis of the correlation with the residuals of the previous periods in the following sub-sections.

14.4.1 First-order Moving Average Models

The moving average model in which the value of a variable in the current period is regressed against its previous residual is called the first-order moving average. For example, if today's price of a share depends on whatever has happened in the other factor on the previous day except for the price of the share on the previous day then we use the first-order moving average.

If y_t is the value of a variable at time t and ε_{t-1} is the residuals at time t –1 then the moving average model of first-order is given as follows:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

A first-order moving average model expresses the present value of a variable as a linear combination of the mean of the series μ, the present error term ε_t and the past error term ε_{t-1}. Of course, we do not observe the values of ε_t.

The magnitude of the impact of past residual on the present value is quantified using a coefficient denoted with θ_1. We use θ_1 for that instead of ϕ_1 like in the autoregressive model, to avoid confusion.

The number of residuals used as regressors is called the order of the moving average model. So, the first-order moving average is written as MA(1) where 1 represents the order.

After understanding the expression of MA(1) model, we now study its properties.

Mean and Variance

We can find the mean of a MA(1) model as follows:

$$E[y_t] = E[\mu] + E[\varepsilon_t] + \theta_1 E[\varepsilon_{t-1}]$$

$$= \mu + 0 + \theta_1 \times 0 \quad \begin{bmatrix} E[\varepsilon_t] = E[\varepsilon_{t-1}] = 0 \text{ because } \varepsilon_t \sim N[0, \sigma^2] \\ \text{and } \mu \text{ is a constant so } E(\mu) = \mu \end{bmatrix}$$

Therefore,

Mean of MA(1) $= \mu$

Similarly,

We can find the variance of a MA(1) model as follows:

$$\text{Var}[y_t] = \text{Var}[\mu] + \theta_1^2 \text{Var}[\varepsilon_{t-1}] + \text{Var}[\varepsilon_t]$$

Since

$$\text{Var}[\mu] = 0 \text{ because } \mu \text{ is a constant}$$

$$\text{Var}[\varepsilon_t] = \text{Var}[\varepsilon_{t-1}] = \sigma^2 \text{ because } \varepsilon_t \sim N[0, \sigma^2]$$

Therefore,

$$\text{Var}[y_t] = \theta_1^2 \sigma^2 + \sigma^2$$

$$\text{Var}[y_t] = (1 + \theta_1^2)\sigma^2 \geq 0$$

Autocovariance and Autocorrelation Functions

The autocovariance function of an MA(1) model is as follows:

$$\gamma_0 = \text{Var}(y_t) = \frac{\sigma^2}{1 + \theta_1^2}$$

$$\gamma_1 = \theta_1 \sigma^2$$

$\gamma_k = 0;\ k > 1$

Similarly, we have the autocorrelation function of MA(1) model as

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1}{1+\theta_1^2}$$

$$\rho_k = 0;\ k > 1$$

It indicates that the autocorrelation function for MA(1) model becomes zero after lag 1.

Conditions for Invertibility

The moving average models are always stationary. However, some restrictions are also imposed on the parameters of the moving average models otherwise the model can not converge. Therefore, some constraints on the values of the parameters are required for the invertibility of the MA (1) model which is as follows:

$$|\theta_1| < 1 \Rightarrow -1 < \theta_1 < 1$$

14.4.2 Second-order Moving Average Models

The moving average model in which the value of a variable in the current period is regressed against its two previous residuals then it is called the second-order moving average. It is represented by MA(2). For example, if today's price of a share depends on whatever has happened in the other factor in the previous day and the day of the previous day then we use a second-order moving average.

If y_t is the value of a variable at time t and ε_t, ε_{t-1} and ε_{t-2} are the residuals at time t, t – 1 and t – 2, respectively, then the moving average model of second order is expressed as follows:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$$

As for MA (1) model, the coefficients θ_1 & θ_2 represent the magnitude of the impact of past residuals on the present value.

Let us study the properties of the MA(2) model.

Mean and Variance

The mean and variance of MA(2) model are given as follows:

$$\text{Mean} = \mu + 0 + \theta_1 \times 0 + \theta_2 \times 0 \begin{bmatrix} E[\varepsilon_t] = E[\varepsilon_{t-1}] = E[\varepsilon_{t-2}] = 0 \\ \text{because } \varepsilon_t \sim N[0, \sigma^2] \end{bmatrix}$$

Mean of MA(2) $= \mu$

Similarly, We can find the variance of a MA(2) model as follows:

$$\text{Var}[y_t] = \text{Var}[\mu] + \theta_1^2\text{Var}[\varepsilon_{t-1}] + \theta_2^2\text{Var}[\varepsilon_{t-2}] + \text{Var}[\varepsilon_t]$$

Since

$$\text{Var}[\mu] = 0 \text{ because } \mu \text{ is constant}$$

$$\text{Var}[\varepsilon_t] = \text{Var}[\varepsilon_{t-1}] = \text{Var}[\varepsilon_{t-2}] = \sigma^2 \text{ because } \varepsilon_t \sim N[0, \sigma^2]$$

Therefore,

$$\text{Var}[y_t] = \theta_1^2\sigma^2 + \theta_2^2\sigma^2 + \sigma^2$$

$$\text{Var}[y_t] = (1 + \theta_1^2 + \theta_2^2)\sigma^2 \geq 0$$

Autocovariance and Autocorrelation Functions

The autocovariance function of an MA(2) model is given as follows:

$$\gamma_0 = \text{Var}(y_t) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

$$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma^2$$

$$\gamma_2 = \theta_2\sigma^2$$

$$\gamma_k = 0; \ k > 2$$

Similarly, we have the autocorrelation function of MA(2) model as

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_k = 0; \ k > 2$$

It indicates that the autocorrelation function of MA(2) model becomes zero after lag 2.

Conditions for Invertibility

The constraints on the values of the parameters for the invertibility of the MA (2) model are as follows:

- $|\theta_2| < 1 \Rightarrow -1 < \theta_2 < 1$
- $\theta_1 + \theta_2 < 1$
- $\theta_2 - \theta_1 < 1$

After understanding the first and second-order moving average models, you are interested to know the general form of the moving average model. Let's discuss the general form of the moving average model.

14.4.3 qth-order Moving Average Models

A moving average model states that the current value is linearly dependent on the current and past error terms.

If y_t is the value of a variable at time t and ε_t, ε_{t-1}, $\varepsilon_{t-2}, \ldots, \varepsilon_{t-q}$ are the residuals at time $t, t-1, t-2, \ldots, t-q$, respectively, then the moving average model of the qth order is expressed as the present value (y_t) as a linear combination of the mean of the series (μ), the present error term (ε_t), and past error terms (ε_{t-1}, $\varepsilon_{t-2}, \ldots, \varepsilon_{t-q}$). Mathematically, we express a general moving average model as follows:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + ... + \theta_q\varepsilon_{t-q}$$

where $\theta_1, \theta_2, ..., \theta_q$ represent the magnitude of the impact of past errors on the present value.

After understanding the form of the general moving average model, we now study the properties of the model.

Mean and Variance

The mean and variance of MA(q) model are given as follows:

$$\text{Mean} = \mu$$

$$\text{Var}[y_t] = \left(1 + \theta_1^2 + \theta_2^2 + ... + \theta_q^2\right)\sigma^2 \geq 0$$

Autocovariance and Autocorrelation Functions

The autocovariance function of a MA(q) model is given below:

$$\gamma_0 = \left(1 + \theta_1^2 + \theta_2^2 + ... + \theta_q^2\right)\sigma^2$$

$$\gamma_k = \begin{cases} \left(\theta_k + \theta_1\theta_{k+1} + ... + \theta_{q-k}\theta_q\right)\sigma^2; k = 1, 2, ..., q \\ 0; k > q \end{cases}$$

Similarly, we have the autocorrelation function of MA(q) model as

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\theta_k + \theta_1\theta_{k+1} + ... + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + ... + \theta_q^2}; k = 1, 2, ..., q \\ 0; k > q \end{cases}$$

It indicates that the autocorrelation function of MA (q) model becomes zero after lag q.

Conditions for Invertibility

The constraints on the values of the parameters for invertibility of the MA (q) when q ≥ 3, are much more complicated, therefore, we are not discussing them.

After understanding the moving average models, let's look at an example which helps you to understand how to check invertibility and calculate mean, variance, autocovariance and autocorrelation functions for the moving average models.

Example 2: Consider the time series model

$$y_t = 2 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

where $\varepsilon_t \sim N[0, 1]$

(i) Identify the model.

(ii) Is this a moving average model? If yes check whether it is invertible.

(iii) What are the mean and variance of the time series?

(iv) Calculate autocovariance and autocorrelation functions.

(v) Plot the correlogram.

(vi) If the residual at t = 100 is 2.3, then would you expect the next observation to be above or below the mean?

Solution: Since the variable in the current period is regressed against its previous residual, therefore, it is the first-order moving average model. To check whether it is invertible, first of all, we find the parameters of the time series model. We now compare it with its standard form, that is,

$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$

We obtain

$\mu = 10,\ \theta_1 = 0.8$

The invertibility constraints for MA(1) is $-1 < \theta_1 < 1$. Since θ_1 lies between –1 and 1, therefore, the time series model MA(1) is invertible.

We now calculate the mean and variance of the series as

$\text{Mean} = 2$

$\text{Var}[y_t] = (1+\theta_1^2)\sigma^2 = (1+0.64)\times 1 = 1.64$

We now calculate the autocorrelation function as

$\gamma_0 = \text{Var}(y_t) = 1.64$

$\gamma_1 = \theta_1\sigma^2 = 0.8 \times 1 = 0.8$

$\gamma_k = 0;\ k > 1$

Similarly, we calculate the autocovariance function of MA(1) as

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.8}{1.64} = 0.49$$

$\rho_k = 0;\ k > 1$

We can forecast, the next value of y_{100} using the prediction model as

$\hat{y}_t = 2 + 0.8\varepsilon_{t-1}$

$\hat{y}_{101} = 2 + 0.8\varepsilon_{100} = 2 + 0.8 \times 2.3 = 2 + 0.18 = 2.18$

Therefore, if the residual at t = 100 is 2.3, then the next observation y_{101} = 2.18 will be above the mean (2).

You may try the following Self Assessment Question before studying further.

SAQ 2

Consider the time series model

$y_t = 42 + \varepsilon_t + 0.7\varepsilon_{t-1} - 0.2\varepsilon_{t-2}$

where $\varepsilon_t \sim N[0,2]$

(i) Identify the model.

(ii) Is this a moving average model? If yes check whether it is invertible.

(iii) What are the mean and variance of the time series?

(iv) Calculate the autocorrelation function.

(v) Suppose the residual errors for time periods 20 and 21 are 0.23 and 0.54 then forecast the next observation.

14.5 AUTOREGRESSIVE MOVING AVERAGE MODELS

In the previous sections, you have learnt autoregressive and moving average models which are used to model time series data. The autoregressive(AR) models are used when the current value of the time series variable depends on the past values of the series whereas the moving average models are used when the current value of the time series variable depends on the unpredictable shocks (residuals) in the previous periods. But in real-life data, we also observe that the current value of the time series variable depends not only on its past values but also on the residuals in the previous periods. For example, the sale of a product of a company at a current time depends on the prior sales happening in the past time which plays a role of the AR component and it also depends on the time-based campaigns launched by the company, such as distribution of coupons, by one get one free, etc. will increase sales temporarily and such change in sales is captured by the moving average component. Therefore, we need models that simultaneously use past data as a foundation for estimates, and can also quickly adjust to unpredictable shocks (residuals). In this section, we are going to talk about one such model, called autoregressive moving average (ARMA), which takes into account past values as well as past errors when constructing future estimates.

Autoregressive moving average (ARMA) models play a key role in the modelling of time series. An ARMA process consists of two models: an autoregressive (AR) model and a moving average (MA) model. In analysis, we tend to put the residuals at the end of the model equation, so that’s why the “MA” part comes second. As compared with the pure AR and MA models, ARMA models provide the most effective linear model of stationary time series since they are capable of modelling the unknown process with the minimum number of parameters. In this model, the impact of previous lags along with the residuals is considered for forecasting the future values of the time series. We can define ARMA model as follows:

Autoregressive moving average models are simply a combination of an AR model and an MA model.

Autoregressive Moving Average (ARMA) models are models in which the value of a variable in the current period is related to its own values in the previous period as well as values of the residual in the previous period.

Since ARMA is a combination of both autoregressive terms(p) and moving average(q) terms, therefore, we represent it as ARMA (p,q). It is also used for stationary time series.

On the basis of different values of p and q, the ARMA model has different types which are discussed as follows:

14.5.1 Various Forms of ARMA Models

The ARMA model has various forms for different values of the parameters p and q of the model. We discuss some standard forms as follows:

ARMA(1,1) Models

ARMA(1,1) models are the models in which the value of a variable in the current period is related to its own value in the previous period as well as values of the residual in the previous period. It is a mixture of AR(1) and MA(1).

If y_t and y_{t-1} are the values of a variable at time t and t –1, respectively and if ε_t and ε_{t-1} are the residuals at time t and t –1, respectively then the ARMA (1,1) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

As usual, the coefficients δ and ε_t denote the intercept/constant factor and error term at time t, respectively whereas the coefficients ϕ_1 and θ_1 represent AR and MA coefficients and represent the magnitude of the impact of past values and past error on the present value, respectively.

After understanding the form of the ARMA(1,1) model, we now study the properties of the model.

Mean and Variance

We can find the mean and variance of an ARMA(1, 1) model as we have found in AR and MR models which are given as follows:

$$\text{Mean} = \frac{\delta}{1-\phi_1}$$

$$\text{Var}\left[y_t\right] = \frac{\left(1+2\phi_1\theta_1+\theta_1^2\right)\sigma^2}{1-\phi_1^2} \geq 0 \text{ when } \phi_1^2 < 1$$

Autocovariance and Autocorrelation Functions

The autocovariance function of an ARMA(1, 1) model is given as follows:

$$\gamma_0 = \text{Var}\left[y_t\right] = \frac{\left(1+2\phi_1\theta_1+\theta_1^2\right)\sigma^2}{1-\phi_1^2}$$

$$\gamma_1 = \frac{\left(\phi_1+\theta_1\right)\left(1+\phi_1\theta_1\right)\sigma^2}{1-\phi_1^2}$$

$$\gamma_k = \phi_1 \gamma_{k-1} \text{ for } k = 2,3,\ldots$$

Similarly, the autocorrelation function of an ARMA(1,1) model is given as follows:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\left(\phi_1+\theta_1\right)\left(1+\phi_1\theta_1\right)}{1+2\phi_1\theta_1+\theta_1^2} & \text{for } k = 1 \\ \phi_1^k \rho_{k-1} \text{ for } k = 2,3,\ldots & \end{cases}$$

The autocorrelation function of an ARMA(1, 1) model exhibits exponential decay and/or sinusoid pattern towards zero. It does not cut off but gradually decreases as lag k increases. Also the autocorrelation function of an ARMA(1, 1) model displays the shape of an AR(1) process. The partial autocorrelation function of an ARMA(1, 1) model also gradually dies out (the same property as a moving average model) as k increases. It is relatively difficult to the identification of the order of the ARMA model.

Conditions for Stationarity and Invertibility

The stationarity of the ARMA (1, 1) is related to the AR component in the ARMA (1, 1) model. Therefore, stationarity conditions which are discussed for AR(1) are also for ARMA (1,1) model which is as follows:

$$|\phi_1| < 1 \Rightarrow -1 < \phi_1 < 1$$

Similarly to the stationarity conditions, the invertibility of an ARMA(1,1) model is related to the MA(1) component. Therefore, the invertibility conditions which are discussed for MA(1) are also for ARMA (1,1) model which is as follows:

$$|\theta_1| < 1 \Rightarrow -1 < \theta_1 < 1$$

ARMA (1, 2) Models

ARMA(1,2) models are the models in which the value of a variable in the current period is related to its own value in the previous period as well as the residuals of two previous periods. It is a mixture of AR(1) and MA(2) models.

If y_t and y_{t-1} are the values of a variable at time t and t –1, respectively, and if $\varepsilon_t, \varepsilon_{t-1}$ and ε_{t-2} are the residuals at time t, t –1 and t – 2, respectively, then the ARMA (1,2) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t$$

The expression for mean, variance, autocovariance and autocorrelation functions of ARMA(1, 2) model are more complicated, therefore, we are not giving the expression of these.

Conditions for Stationarity and Invertibility

The stationarity of the ARMA (1, 2) model is related to the AR component in the ARMA (1, 1) model. Therefore, stationarity conditions which are discussed for AR(1) are also for ARMA (1,2) model which is as follows:

$$|\phi_1| < 1 \Rightarrow -1 < \phi_1 < 1$$

Similarly, the invertibility conditions, of an ARMA(1,1) model is related to the MA(2) component which are as follows:

- $|\theta_2| < 1 \Rightarrow -1 < \theta_2 < 1$
- $\theta_1 + \theta_2 < 1$
- $\theta_2 - \theta_1 < 1$

ARMA (p, q) Models

More generally, ARMA(p,q) models are the models in which the value of a variable in the current period is related to its own p values in the previous periods as well as q values of the residual in the previous periods. It is a mixture of AR(p) and MA(q) models.

If $y_t, y_{t-1}, ..., y_{t-p}$ are the value of a variable at time $t, t-1, ..., t-p$, respectively and if $\varepsilon_t, \varepsilon_{t-1}, ..., \varepsilon_{t-q}$ are the residuals at time $t, t-1, ..., t-q$, respectively, then the ARMA (p, q) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \theta_1 \varepsilon_1 + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

The expression for mean, variance, autocovariance and autocorrelation functions of ARMA (p, q) models are more complicated, therefore, we are not giving the expression of these.

Autocorrelation and Partial Autocorrelation Functions

The autocorrelation function of an ARMA(p, q) model exhibits exponential decay and/or sinusoid pattern towards zero. It does not cut off but gradually decreases as lag k increases. Also the autocorrelation function of an ARMA(p, q) model displays the shape of an AR(p) process. The partial autocorrelation function of an ARMA(p, q) model also gradually dies out (the same property as a moving average model) as k increases. It is relatively difficult to the identification of the order of the ARMA model.

Conditions for Stationarity and Invertibility of ARMA (p, q) model

When p ≥ 3, the restrictions for stationarity are much more complicated. Similarly, when q ≥ 3, the restrictions for invertibility become more complicated, therefore, we are not discussing them here.

After understanding the ARMA models, let's look at an example which helps you to understand how to identify the order, check stationary and invertible and calculate mean, variance, autocovariance and autocorrelation functions for the ARMA models.

Example 3: Consider the time series model

$$y_t = 20 + \varepsilon_t - 0.5 y_{t-1} + 0.7 \varepsilon_{t-1}$$

Assuming that the variance of the white noise is 2.

(i) Identify the model.

(ii) Check whether the model is stationary and invertible.

(iii) Calculate the autocorrelation function ρ_1, ρ_2 and ρ_3.

Solution:

Since the variable in the current period is regressed against its previous value as well as previous residual, therefore, it is the ARMA model of order (1, 1). To check whether it is stationary and invertible, first of all, we find the parameters of the time series model. We now compare it with its standard form, that is, $y_t = \delta + \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

We obtain

$$\delta = 20, \; \phi_1 = -0.5, \; \theta_1 = 0.7$$

The stationary constraints for ARMA(1, 1) is $-1 < \phi_1 < 1$. Since ϕ_1 lies between –1 and 1, therefore, the time series model ARMA(1, 1) is stationary.

Similarly, the invertibility constraints for ARMA(1,1) is $-1 < \theta_1 < 1$. Since θ_1 lies between –1 and 1, therefore, the time series model ARMA(1,1) is invertible.

We now calculate the autocovariance function of ARMA(1, 1) as

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 + 2\phi_1\theta_1 + \theta_1^2} & \text{for } k = 1 \\ \phi_1^k \rho_{k-1} & \text{for } k = 2, 3, \ldots \end{cases}$$

$$\rho_1 = \frac{(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 + 2\phi_1\theta_1 + \theta_1^2} = \frac{(-0.5 + 0.7)(1 - 0.5 \times 0.7)}{1 + 2 \times -0.5 \times 0.7 + (0.7)^2} = \frac{0.13}{0.79} = 0.165$$

$$\rho_2 = \phi_1^2 \rho_1 = -0.5 \times -0.5 \times 0.165 = 0.041$$

$$\rho_3 = \phi_1^3 \rho_2 = (-0.5)^3 \times 0.041 = -0.005$$

Now, Let us try some more Self Assessment Questions.

SAQ 3

Consider the ARMA time series model

$$y_t = 27 + 0.8y_{t-1} + 0.3\varepsilon_{t-1} + \varepsilon_t$$

Assuming that the variance of the white noise is 1.5.

(i) Is the process stationary and invertible?

(ii) Find ρ_1, ρ_2 and ρ_3 for the process.

14.6 AUTOREGRESSIVE INTEGRATED MOVING AVERAGE MODELS

In the previous sections of this unit, you have learnt different time series modes such as autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA). These models are based on the assumption that the time series is stationary. But in the real world, most of the time series variables are nonstationary. In general, trends, and periodicity exist in many time series data. Hence, the AR, MA, and ARMA models do not apply to nonstationary time series so there is a need to remove these effects before applying such models. Therefore, if the input time series is nonstationary, then first we have to transform the series from a nonstationary into a stationary and after that, we shall apply models such as the AR, MA, and ARMA. For transforming a nonstationary time series to stationary, we may use differencing, as discussed in Unit 13, once, twice or three times, and so on until the series is at least approximately stationary. As AR and MA processes are described by the order, in a similar way, the differencing process is also described by the order of differencing, as 1, 2, 3.... Therefore, to describe a model for nonstationary time series, the elements make up a triple (p,d,q) instead of two (p, q) that defines the type of model applied where the degree of the differencing is represented by the d parameter. Combining the differencing of a nonstationary time series with the ARMA model provides a powerful family of models that can be applied in a wide range of situations. The model is described as an autoregressive moving average (ARMA) model. In this form, the letter **"I"** in

ARIMA refers to the fact that the time series data has been initially differenced and when the modelling is completed the results then have to be **summed or integrated** to produce the final estimations and forecasts. Box and Jenkins played a significant role in the development of this extended variant of the model, therefore, ARIMA models are also referred to as Box-Jenkins models. The ARIMA model is discussed below:

Autoregressive Integrated Moving Average (ARIMA) model is a combination of differencing with autoregressive and moving average models.

We can express the ARIMA model as follows:

$$y'_t = \delta + \phi_1 y'_{t-1} + \phi_2 y'_{t-2} + \ldots + \phi_p y'_{t-q} - \theta_1\varepsilon_1 - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \varepsilon_t$$

where y'_t is the differenced series which may have been differenced more than once and p and q are the orders of autoregressive and moving average parts.

For the first difference, we can write the ARIMA model as

$$y_t - y_{t-1} = \delta + \phi_1(y_{t-1} - y_{t-2}) + \phi_2(y_{t-2} - y_{t-3}) + \ldots + \phi_p(y_{t-q} - y_{t-q-1}) - \theta_1\varepsilon_1 - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \varepsilon_t$$

On the basis of different values of p, q and d, the ARIMA model has different types which are discussed as follows:

14.6.1 Various Forms of ARIMA Models

The ARIMA model has various forms for different values of the parameters p, d and q of the model. We discuss some standard forms as follows:

Model	Name	Form	Use
ARIMA(0,0,0)	White noise	$y_t = \varepsilon_t$	The errors are uncorrelated across time.
ARIMA(1,0,0)	first-order autoregressive model	$y_t = \mu + \phi_1 y_{t-1} + \varepsilon_t$	The series is stationary and autocorrelated with its previous value.
ARIMA(0,1,0)	Random walk	$y_t - y_{t-1} = \mu + \varepsilon_t$ $y_t = \mu + y_{t-1} + \varepsilon_t$	The series is not stationary.
ARIMA(1,1,0)	Differenced first-order autoregressive model	$y_t - y_{t-1} = \mu + \phi_1(y_{t-} - y_{t-2}) + \varepsilon_t$ $y_t = \mu + y_{t-1} + \phi_1(y_{t-} - y_{t-2}) + \varepsilon_t$	The time series is not stationary and the autocorrelated with its previous values.
ARIMA(0,1,1) with constant	Simple exponential smoothing model	$y_t - y_{t-1} = \theta_1\varepsilon_{t-1} + \varepsilon_t$ $y_t = y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t$	The time series is not stationary and the errors are correlated across time.
ARIMA(0,1,1)	Simple exponential smoothing with growth	$y_t - y_{t-1} = \mu + \theta_1\varepsilon_{t-1} + \varepsilon_t$ $y_t = \mu + y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t$	The time series is not stationary and the errors are correlated across time.
ARIMA(1,1,1)	Damped-trend linear exponential smoothing model	$y_t - y_{t-1} = \mu + \phi_1 y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t$ $y_t = \mu + (1+\phi_1)y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t$	The series is not stationary and the series has an upward trend

After understanding various time series models, we now discuss an important topic how to select a suitable time series model for real-life time series data in the next section.

14.7 TIME SERIES MODEL SELECTION

I hope you understand the various time series models. Broadly, you can divide all time series models into two categories: the models which are used for stationary time series such as AR, MA, ARMA and the models which are used for nonstationary time series such as ARIMA. When you deal with real-time series data, the first question that may arise in your mind is how you know which time series model is most suitable for a particular time series data. Don't worry about that here we describe the methodology for the same in steps so that you can easily identify/select the model and its order for the given time series data. It has the following steps:

It is important that the choice of order makes sense. For example, suppose you have had blood pressure readings for every day over the past two years. You may find that an AR(1) or AR(2) model is appropriate for modeling blood pressure. However, the PACF may indicate a large partial autocorrelation value at a lag of 17, but such a large order for an autoregressive model likely does not make much sense.

Step 1: Since there are two types of models which are used for stationary and nonstationary time series, therefore, first of all, we plot the time series data and check whether the time series data is stationary or nonstationary as you have learned in Unit 13.

Step 2: **If the time series is stationary**, we have to decide which model out of AR, MA and ARMA is suitable for our time series data. To distinguish among them, we calculate the autocorrelation function (ACF) and the partial autocorrelation function (PACF) as discussed in Unit 13. After that, we plot the ACF and PACF versus the lag, that is, correlogram as discussed in Unit 13 and try to identify the pattern of both. The ACF plot is most useful for identifying the AR model and PACF plot for the order of the AR model whereas the PACF plot is most useful for identifying the MA model and ACF plot for the order of the MA model. We now try to understand how to distinguish between AR, MA and ARMA models as follows:

Case I (AR model): In the plot of ACF versus the lag (correlogram), if you see a gradual diminish in amount or exponential decay then this indicates that the values of the time series are serially correlated, and the series can be modelled through an AR mode. For determining the order of an AR model, we use a plot of PACF versus the lag. If the PACF output cuts off, which means the PACF is almost zero at lag p+1, then it indicates that the AR model of order p. We can also calculate PACF by increasing the order one by one and as soon as this lies within the range of $\pm 2/\sqrt{n}$ (where n is the size of the time series) we should stop and take the order as the last significant PACF as the order of the AR model (see SAQ 4).

Case II (MA model): In the plot of PACF versus the lag, if you see a gradual diminish in amount or exponential decay then this indicates that the series can be modelled through an MA model and if the ACF output cuts off, means the ACF almost zero, at lag q+1, then it indicates that the MA model of order q.

Case III (ARMA model): If the autocorrelation function (ACF) as well as the partial autocorrelation function (PACF) plots show a gradual diminish in amount (exponential decay) or damped sinusoid pattern then this indicates that the series can be modelled through an ARMA model but it makes the identification of the order of the ARMA (p, q) model relatively more difficult. For that extended ACF, generalised sample PACS, etc. are used which are beyond the scope of this course. For more detail, you can consult **Time Series Analysis Forecasting and Control**, 4th Edition, written by Box, Jenkins and Reinsel.

Step 3: **If the time series is nonstationary**, we obtain the first, second, etc. differences of the time series as discussed in Unit 13 until it becomes stationary and ensure that trend and seasonal components are removed and find **d**. Suppose after the second difference the series becomes stationary then d is 2. Generally, one or two-stage differencing is sufficient. The differenced series will be shorter (as you have observed in Unit 13) than the source series. An ARMA model is then fitted to the resulting time series. Since ARIMA models have three parameters, therefore, there are many variations to the possible models that could be fitted. We should choose the ARIMA models as simple as possible, i.e. contain as few terms as possible (small values of p and q). For more detail, you can consult **Time Series Analysis Forecasting and Control**, 4th Edition, written by Box, Jenkins and Reinsel.

Step 4: After identifying the model, we estimate the parameters of the model using the method of moments, maximum likelihood estimation, least squares methods, etc. The method of moments is the simplest of these. In this method, we equate the sample autocorrelation functions to the corresponding population autocorrelation functions which are the function of the parameters of the model and solve these equations for the parameters of the model. However, this method is not a very efficient method of estimation of parameters. For moving average processes usually the maximum likelihood method is used which gives more efficient estimates when n is large. We shall not discuss this anymore here and if someone is interested in this, may refer to **Time Series Analysis Forecasting and Control**, 4th Edition, written by Box, Jenkins and Reinsel.

Step 5: After fitting the best model, we give a diagnostic check to the residuals to examine whether the fitted model is adequate or not. It helps us to ensure no more information is left for extraction and check the goodness of fit. For the residual analysis, we plot the ACF and PACF of the residual and check whether there is a pattern or not. For the adequate model there should be no structure in ACF and PACF of the residual and should not differ significantly from zero for all lags greater than one. For the goodness of fit, we use Akaike's information criterion (AIC) and Bayesian information criterion (BIC). We have not discussed all the above aspects in detail here but interested one should consult **Time Series Analysis Forecasting and Control**, 4th Edition, written by Box, Jenkins and Reinsel.

After understanding the procedure of selection of a time series model, let us take an example.

Example 4: The temperature (in °C) in a particular area on different days collected by the meteorological department is as given below:

Day	Temperature	Day	Temperature
1	27	9	28
2	29	10	30
3	31	11	30
4	27	12	26
5	28	13	30
6	30	14	31
7	32	15	27
8	29		

(i) Examine which model (AR or MA) is suitable of this data

(ii) Find the order and estimate the parameter of the selected model.

(iii) Write the model.

Solution: First of all, we check whether the given time series is stationary or nonstationary. For that, we plot the time series data by taking days on the X-axis and temperature on the Y-axis. We get the time series plot as shown in Fig. 14.1.

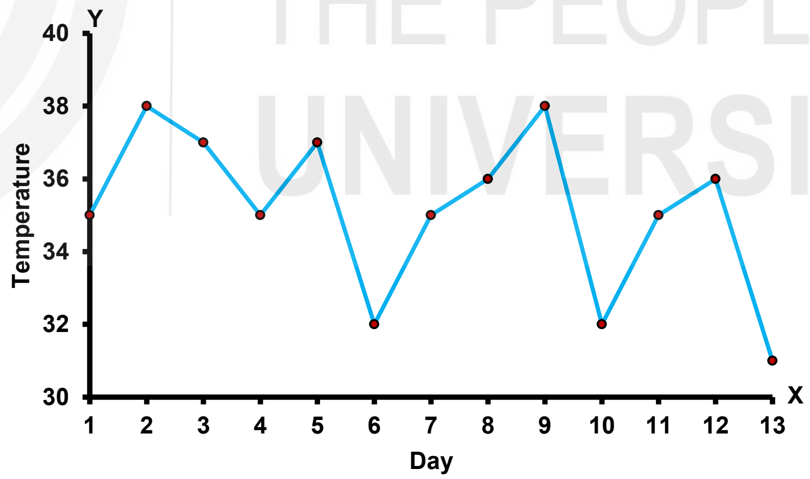

Fig. 14.1: Time series plots of the temperature data.

Fig. 14.1 shows that there is no consistent trend (upward or downward) over the entire period. The series appears to slowly wander up and down. Also, the variance is constant. Almost by definition, there is no seasonality or trend. So we can say that this time series is stationary.

To examine the model and its order, we have to compute the sample autocorrelation (ACF) and partial autocorrelation (PACF) as we have discussed in Unit 13. Therefore, for the sack of time, we just write them here

$$r_1 = 0.835,\ r_2 = 0.676,\ r_3 = 0.469,\ r_4 = 0.280$$

$\hat{\phi}_{11} = r_1 = 0.835$, $\hat{\phi}_{22} = 0.088$

Since the autocorrelation function (ACF) gradually diminishes (decreases) in amount, it indicates that the series can be modelled through an AR model and the PACF output is almost zero, at lag 2, it indicates that the AR model is of order 1. Hence, we may conclude that the AR (1) model is suitable of this data. Therefore, the model is

$$y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t$$

We now estimate the parameters $(\delta \text{ and } \phi_1)$ of the model using the method of moments. In this method, we equate the sample autocorrelation functions to the population autocorrelation functions which are the function of the parameters of the model and solve these as below:

$$r_1 = \rho_1$$

$$0.835 = \phi_1^1 \Rightarrow \phi_1 = 0.835$$

For estimating the parameter δ , first, we find the mean of the given data and then we use the relationship $\text{Mean} = \frac{\delta}{1-\phi_1}$

$$\text{Mean} = \frac{1}{15}\sum_{i=1}^{15} y_i = \frac{435}{15} = 29$$

Therefore,

$$\text{Mean} = 29 = \frac{\delta}{1-0.835} \Rightarrow \delta = 29 \times 0.165 = 4.785$$

Therefore the suitable model for the temperature data is

$$y_t = 4.785 + 0.835 y_{t-1} + \varepsilon_t$$

Before going to the next session, you may like to do some exercise yourself. Let us try Self Assessment Question.

SAQ 4

A researcher wants to develop an autoregressive model for the data on COVID-19 patients in a particular city. For that, he collected the data 100 days and calculated the autocorrelation function which are given as follow:

$$r_1 = 0.73,\ r_2 = 0.39,\ r_3 = 0.07$$

By calculating sample PACF, estimate the order of the autoregressive model to be fitted.

We end this unit by giving a summary of what we have covered in it.

14.8 SUMMARY

In this unit, we have discussed

- The necessity of time series models in comparison of regression models.
- Describe various time series modes such as AR, MA, ARMA, ARIMA.
- Explain the various properties of the models.

- Selection of a particular time series model for real-life time series data.
- The stationarity and invertibility conditions of the model also considered the role of autocorrelations and partial autocorrelations in the identification of the models.

14.9 TERMINAL QUESTIONS

1. Define various components of the ARIMA model.

2. Define the parameters of the ARIMA model.

3. For time series data, a researcher obtained the following information:

n = 100, Mean = 26, Error variance = 2.2

r_1 = 0.69, r_2 = 0.54, r_3 =0.38, r_4 = 0.38, r_5 = 0.29, r_6 = 0.0.5

(i) Plot the correlogram

(ii) Which one of the AR and MR models will be more suitable?

(iii) Fit the suitable model.

14.10 SOLUTIONS/ANSWERS

Self Assessment Questions (SAQs)

1. For checking the stationarity of the time series, first of all, we find the parameters of the time series model. Since the variable in the current period is regressed against its previous and previous to previous values, therefore, it is the second-order autoregressive model. We now compare it with its standard form, that is, $y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$. We obtain

$$\delta = 5, \phi_1 = 0.8, \phi_2 = -0.5$$

We now check the stationarity conditions for the second-order autoregressive model as

$$-1 < |\phi_2| < 1 \Rightarrow 0.5 < 1$$

$$\phi_1 + \phi_2 < 1 \Rightarrow 0.8 - 0.5 = 0.3 < 1$$

$$\phi_2 - \phi_1 < 1 \Rightarrow -0.5 - 0.8 = -1.3 < 1$$

Since all three conditions for stationarity are satisfied, hence, the time series is stationary.

We now calculate the mean and variance of the series as

$$\text{Mean} = \frac{\delta}{1-\phi_1-\phi_2} = \frac{5}{1-0.8+0.5} = \frac{5}{0.7} = 7.14$$

$$\text{Var}[y_t] = \frac{\sigma^2}{1-\phi_1^2-\phi_2^2} = \frac{2}{1-0.64-0.25} = \frac{2}{0.11} = 18.18$$

We now calculate the autocorrelation function as

$$\rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{0.8}{1+0.5} = 0.53$$

$\rho_2 = \phi_1\rho_1 + \phi_2 = 0.8 \times 0.53 - 0.5 = -0.076$

$= -0.061 - 0.265 = -0.326$

We can forecast, the next value of y_{52} using the prediction model as

$\hat{y}_t = 5 + 0.8y_{t-1} - 0.5y_{t-2}$

$\hat{y}_{52} = 5 + 0.8y_{51} - 0.5y_{50}$

$\hat{y}_{52} = 5 + 0.8 \times 0.40 - 0.5 \times 0.38 = 5.13$

2. Since the variable in the current period is regressed against its previous and previous to previous residuals, therefore, it is a second-order moving average model. For checking the invertibility of the MA(2) model, first of all, we find the parameters of the time series model. We now compare it with its standard form, that is, $y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$. We have

$\mu = 42$

$\theta_1 = 0.7$

$\theta_2 = -0.2$

The conditions for the invertibility of the MA(2) model are

- $|\theta_2| < 1 \Rightarrow -1 < \theta_2 < 1$
- $\theta_1 + \theta_2 < 1$
- $\theta_2 - \theta_1 < 1$

Since $\theta_2 = -0.2$, therefore, it lies between –1 and 1

$\theta_1 + \theta_2 = 0.7 - 0.2 = 0.5 < 1$

$\theta_2 - \theta_1 = -0.2 - 0.7 = -0.9 < 1$

Since all three conditions for the invertibility of MA(2) model are satisfied, hence the time series is invertible.

We now calculate the mean and variance of the series as

$\text{Mean} = 42$

$$\text{Var}[y_t] = \frac{\sigma^2}{1 + \theta_1^2 + \theta_2^2} = \frac{2}{1 + 0.49 + 0.04} = \frac{2}{1.53} = 1.307$$

We now calculate the autocorrelation function as

$$\rho_1 = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{0.7 + 0.7(-0.2)}{1.53} = \frac{0.56}{1.53} = 0.366$$

$$\rho_2 = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-0.2}{1.53} = -0.131$$

$\rho_k = 0;\ k > 2$

We can forecast, the next value of y_{52} using the prediction model as

$\hat{y}_t = 42 + 0.7y_{t-1} - 0.2y_{t-2}$

$\hat{y}_{22} = 42 + 0.7\varepsilon_{21} - 0.2\varepsilon_{20}$

$$\hat{y}_{22} = 42 + 0.7 \times 0.54 - 0.2 \times 0.23 = 42.332$$

3. For checking whether it is stationary or invertible, first of all, we find the parameters of the time series model. We now compare it with its standard form, that is $y_t = \delta + \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

We obtain

$$\delta = 27, \ \phi_1 = 0.8, \ \theta_1 = 0.3$$

The stationary constraints for ARMA(1, 1) is $-1 < \phi_1 < 1$. Since $\phi_1 = 0.8$ lies between –1 and 1, therefore, the time series model ARMA(1, 1) is stationary.

Similarly, the invertibility constraints for ARMA(1) is $-1 < \theta_1 < 1$. Since $\theta_1 = 0.3$ lies between –1 and 1, therefore, the time series model ARMA(1) is invertible.

We now calculate the autocovariance function of ARMA(1, 1) as

$$\rho_1 = \frac{(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 + 2\phi_1\theta_1 + \theta_1^2} = \frac{(0.8 + 0.3)(1 - 0.8 \times 0.3)}{1 + 2 \times 0.8 \times 0.3 + (0.3)^2} = \frac{0.836}{1.57} = 0.532$$

$$\rho_2 = \phi_1^2 \rho_1 = (0.8)^2 \times 0.532 = 0.340$$

$$\rho_3 = \phi_1^3 \rho_2 = (0.8)^3 \times 0.340 = 0.174$$

4. As we know from Unit 13, the 1st-order partial autocorrelation equals the 1st-order autocorrelation, that is,

$$\phi_{11} = r_1 = 0.73$$

The range of the PACF function is $\pm 2/\sqrt{n} = \pm 2/10 = \pm 0.2$

Since $\phi_{11} = 0.73$ lies outside of the range of PACF so we have to calculate the next order PACF.

The 2nd order (lag) sample partial autocorrelation is

$$\hat{\phi}_{22} = \frac{(r_2 - r_1^2)}{(1 - r_1^2)} = \frac{0.39 - (0.73)^2}{1 - (0.73)^2} = -0.31$$

Since $\hat{\phi}_{22} = -0.31$ lies outside the range ±0.2, therefore, the autoregressive model AR(1) is not possible and we have to calculate next PACF.

Similarly, We now compute the 3rd order sample partial autocorrelation function as

$$\phi_{33} = \frac{\begin{vmatrix} 1 & r_1 & r_1 \\ r_1 & 1 & r_2 \\ r_2 & r_1 & r_3 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_3 \\ r_2 & r_1 & 1 \end{vmatrix}}$$

$$\begin{vmatrix} 1 & r_1 & r_1 \\ r_1 & 1 & r_2 \\ r_2 & r_1 & r_3 \end{vmatrix} = \begin{vmatrix} 1 & 0.73 & 0.73 \\ 0.73 & 1 & 0.39 \\ 0.39 & 0.73 & 0.07 \end{vmatrix}$$

$$= 1\times(0.07 - 0.73\times 0.39) - 0.73\times(0.73\times 0.07 - 0.39\times 0.39)$$
$$+ 0.73\times(0.73\times 0.73 - 0.39\times 1)$$

$$= -0.214 + 0.073 + 0.104 = -0.037$$

Similarly,

$$\begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_3 \\ r_2 & r_1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0.73 & 0.39 \\ 0.73 & 1 & 0.07 \\ 0.39 & 0.73 & 1 \end{vmatrix}$$

$$= 1\times(1 - 0.73\times 0.07) - 0.73\times(0.73\times 1 - 0.39\times 0.07)$$
$$+ 0.39\times(0.73\times 0.73 - 0.39\times 1)$$

$$= 0.949 - 0.513 + 0.056 = 0.492$$

Therefore,

$$\hat{\phi}_{33} = \frac{-0.037}{0.492} = -0.075$$

Since $\hat{\phi}_{22} = -0.075$ lies inside the range ±0.2, herefore, AR model of order 2 will be suitable for this time series. .

Terminal Questions (TQs)

1. The components of the ARIMA model are as follows:

- **Autoregression (AR)**: refers to a model that shows a changing variable that regresses on its own lagged, or prior, values.
- **Integrated (I)**: represents the differencing of raw observations to allow the time series to become stationary (i.e., data values are replaced by the difference between the data values and the previous values).
- **Moving Average (MA)**: incorporates the dependency between an observation and a residual error.

2. The ARIMA(p, d, q) model has the parameters p, d, q which are integers and different values of these indicate the type of ARIMA model used. The parameters can be defined as:

- **p:** the number of lag observations in the model, also known as the lag order.
- **d:** the number of times the raw observations are differenced; also known as the degree of differencing.
- **q:** the size of the moving average window, also known as the order of the moving average.

3. For the correlogram, we take lags on the X-axis and sample autocorrelation coefficients on the Y-axis. At each lag, we draw a line, which represents the level of correlation between the series and its lags as shown in the following Fig. 14.2.

In the plot of ACF versus the lag (correlogram), we see a gradual diminish in amount or exponential decay which indicates that the values of the time series are serially correlated, and the series can be modelled through an AR mode. For determining the order of an AR model, we use PACF. We calculate PACF by increasing the order one by one and as soon as this lies within the range of ± 2/√n (where n is the size of the time series) we should stop and take the order as the last significant PACF as the order of the AR model.

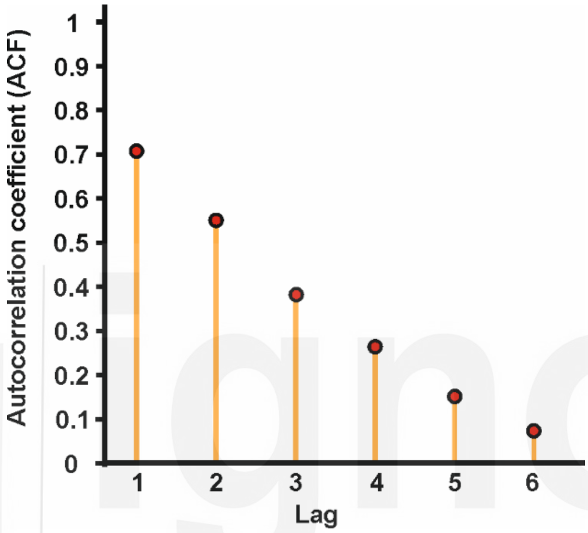

Fig. 14.2: The correlogram of the time series.

The 1st-order partial autocorrelation function equals the 1st-order autocorrelation function, that is,

$$\phi_{11} = r_1 = 0.691$$

Since PACF of first-order lies outside the range ± 2/√n = ± 2/√100 = 0.2, therefore, we calculate second-order PACF as

$$\hat{\phi}_{22} = \frac{\left(r_2 - r_1^2\right)}{\left(1 - r_1^2\right)} = \frac{0.54 - (0.69)^2}{1 - (0.69)^2} = \frac{0.064}{0.524} = 0.049$$

Since PACF (2) lies within the range of ± 2/√n = 0.22, therefore, AR(1) will be suitable for this time series.