Indira Gandhi National Open University
School of Sciences

MST-001
Foundation in Mathematics and Statistics

Block

3

MATRICES, DETERMINANTS AND COLLECTION OF DATA

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BLOCK 3 MATRICES, DETERMINANTS AND COLLECTION OF DATA

This is the third block of the course MST-001. This block is devoted to matrices, determinants, and introduction to statistics, collection and scrutiny of data. The aim of units 9 and 10 is just to give you an idea of matrices and determinants as the concepts related to these terms will be used in some sections of some courses. Last two units of this block and all the four units of the next block of this course put a foundation stone to the statistics.

Unit 9: Matrices and Determinants

This unit discusses what we mean by matrices and its different types, operation on matrices and trace of a square matrix. Concept of determinants and how we evaluate the determinants of square matrices of orders 1, 2, 3 and higher orders have been also discussed. This unit ends by giving some properties of determinants and how these properties are used, is explained with the help of some examples.

Unit 10: Applications of Matrices and Determinants

In this unit adjoint and inverse of a square matrix are discussed. Applications of matrices and determinants in solving a system of linear equations by matrix method and Cramer's rule have been also discussed.

Unit 11: Introduction to Statistics

This unit throws the light on origin and development, definition, scope and uses, and limitations of statistics. Different types of measurement of scale have been discussed in detail. Time series, cross section, discrete, continuous, frequency and non frequency data have been also discussed.

Unit 12: Collection and Scrutiny of Data

In this unit main methods of collection of primary data are discussed. Sources of collection of secondary data including some government publications, scrutiny of data and preparation of different kinds of questionnaires have been also discussed.

Notations and Symbols

$\left[a_{ij}\right]_{m\times n}$	:	matrix of order $m\times n$ with $(i, j)^{th}$ element as a_{ij}
A'	:	transpose of matrix A
tr(A)	:	trace of matrix A
$\left\|A\right\|$	:	determinant of the square matrix A
M_{ij}	:	minor of $(i, j)^{th}$ element of the matrix $\left[a_{ij}\right]$
A_{ij}	:	cofactor of $(i, j)^{th}$ element of the matrix $\left[a_{ij}\right]$
adjA	:	adjoint of A
A^{-1}	:	inverse of matrix A
Δ	:	capital delta

UNIT 9 MATRICES AND DETERMINANTS

Structure

9.1 INTRODUCTION

The knowledge of matrices has become necessary for the individuals working in different branches of science, technology, commerce, management and social sciences. In this unit, we introduce the concept of matrices and its elementary properties. The unit also discusses the determinant, which is a number associated with a square matrix and its properties. Trace of a matrix is also defined.

Objectives

After completing this unit, you should be able to:

- define a matrix and give examples of matrices;
- explain the types of matrices;
- know how operations on matrices are done;
- find multiplication of a matrix by a scalar;
- compute transpose of a matrix;
- find the trace of a square matrix;
- evaluate determinants find minors and cofactors of square matrices of different orders; and
- apply properties of determinants.

9.2 DEFINITION OF A MATRIX

Let us consider the following example to arrive at the definition of a matrix: Suppose there are three girls "Kavita, Preksha and Tanu" Kavita has 9 hundred rupees notes, 4 fifty rupees notes and 5 ten rupees notes. Preksha has 17 hundred rupees notes, 6 fifty rupees notes and one ten rupee note. Tanu has 8 hundred rupees notes, 3 fifty rupees notes and 2 ten rupees notes.
This information can be represented as:

$$
\begin{array}{lcc}
 & & \begin{array}{ccc} \text{Column 1} & \text{Column 2} & \text{Column 3} \\ \downarrow & \downarrow & \downarrow \\ \text{Rs.100} & \text{Rs.50} & \text{Rs.10} \\ \text{Notes} & \text{Notes} & \text{Notes} \end{array} \\
\begin{array}{l} \text{Row 1} \rightarrow \\ \text{Row 2} \rightarrow \\ \text{Row 3} \rightarrow \end{array} & \begin{array}{c} \text{Kavita} \\ \text{Preksha} \\ \text{Tanu} \end{array} & \begin{bmatrix} 9 & 4 & 5 \\ 17 & 6 & 1 \\ 8 & 3 & 2 \end{bmatrix}
\end{array}
$$

This is an arrangement of 9 (3×3) numbers in 3 rows and 3 columns. Such an arrangement is nothing but a matrix. Let us now define a matrix as follows:

Definition of a Matrix

An arrangement of $m \times n$ elements in m rows and n columns enclosed by the brackets ( ) or [ ] only, is called a matrix of order $m \times n$ and is generally denoted by

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ . & . & . & & . \\ . & . & . & \ldots & . \\ . & . & . & & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \text{ or } \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ . & . & . & & . \\ . & . & . & \ldots & . \\ . & . & . & & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}
$$

where a_{ij} denotes the $(i, j)^{th}$ element of the matrix, i.e. element of i^{th} row and j^{th} column is denoted by a_{ij}.

Remark 1:

(i) A matrix is denoted by capital letters A, B, C, etc. of the English alphabets.

(ii) First suffix of an element of the matrix indicates the position of row and second suffix of the element of the matrix indicates position of column. e.g. a_{23} means it is an element in the second row and the third column.

(iii) The order of a matrix is written as "number of rows $\times$ number of columns"

For example,

(i) $A = \begin{bmatrix} 2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ is a matrix of order 2×3

(ii) $B = \begin{bmatrix} 9 & 6 \\ 1 & 0 \\ 8 & 4 \end{bmatrix}$ is a matrix of order 3×2

Let us consider some examples:

Example 1: Write the order of the matrix

$$
A = \begin{bmatrix} 9 & 7 & 8 & -3 & -8 \\ 4 & 3 & 6 & 1 & -10 \\ 10 & 12 & 15 & 2 & 5 \end{bmatrix}
$$

Also write the elements $a_{23}, a_{14}, a_{35}, a_{22}, a_{31}, a_{32}$.

Solution: Order of the matrix A is 3×5 and the desired elements are:

$a_{23} = 6,\ a_{14} = -3,\ a_{35} = 5,\ a_{22} = 3,\ a_{31} = 10,\ a_{32} = 12$

Example 2: Write all the possible orders of the matrix having following elements. (i) 8 (ii) 13

Solution:

(i) All the 8 elements can be arranged in single row, i.e. 1 row and 8 columns.

Or

They can be arranged in two rows with 4 elements in each row, i.e. 2 rows and 4 columns.

Or

in four rows with 2 elements in each row, i.e. 4 rows and 2 columns.

Or

in eight rows with 1 element in each row, i.e. 8 rows and 1 column.

$\therefore$ the possible orders are $1\times 8,\ 2\times 4,\ 4\times 2,\ 8\times 1$.

(ii) All the 13 element can be arranged in single row, i.e. 1 row and 13 columns.

Or

in 13 rows with 1 element in each row, i.e. 13 rows and 1 column.

$\therefore$ the possible orders are $1\times 13,\ 13\times 1$.

Example 3: Construct the matrix A = $[a_{ij}]_{2\times 3}$, where $a_{ij} = \frac{(i-j)^2}{2}$

Solution: A = $[a_{ij}]_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$,where $a_{ij} = \frac{(i-j)^2}{2}$

$$a_{11} = \frac{(1-1)^2}{2} = \frac{0}{2} = 0,\ a_{12} = \frac{(1-2)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2},$$

$$a_{13} = \frac{(1-3)^2}{2} = \frac{(-2)^2}{2} = \frac{4}{2} = 2,\ a_{21} = \frac{(2-1)^2}{2} = \frac{(1)^2}{2} = \frac{1}{2},$$

$$a_{22} = \frac{(2-2)^2}{2} = \frac{0}{2} = 0,\ a_{23} = \frac{(2-3)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$\therefore A = \begin{bmatrix} 0 & 1/2 & 2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Here is an exercise for you.

E 1) Construct A = $[a_{ij}]_{3\times 2}$, where $a_{ij} = |i-j|$

9.3 TYPES OF MATRICES

On the basis of number of rows and number of columns and depending on the values of elements, the type of a matrix gets changed. Various types of matrix are explained as below:

Row Matrix

A matrix having only one row is called a row matrix.

For example, $\begin{bmatrix} 2 & 5 & 7 \end{bmatrix}$, $\begin{bmatrix} 8 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 3 & 2 \end{bmatrix}$ all are row matrices.

Column Matrix

A matrix having only one column is called a column matrix.

For example, $\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 9 \\ -3 \\ 2 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 5 \\ -11 \end{bmatrix}$ all are column matrices.

Remark 2: If a matrix has one element, e.g. $A = [6]$, then matrix A has only one row and only one column. So, it is both row matrix as well as column matrix.

Rectangular Matrix

A matrix having m rows and n columns is called a rectangular matrix if $m \neq n$.

For example, $\begin{bmatrix} 2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ is a rectangular matrix having 2 rows and 3 columns.

Square Matrix

A matrix having equal number of rows and columns is called a square matrix.

For example,

(i) $\begin{bmatrix} 4 & 6 \\ 5 & 3 \end{bmatrix}$ is a square matrix of order 2.

(ii) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 3 & -4 & 8 \end{bmatrix}$ is a square matrix of order 3.

Remark 3: For a square matrix, there is no need of mentioning the number of columns, e.g. in example (i) the order has been written as 2 and not 2×2.

Diagonal Matrix

Principal Diagonal of a Matrix

If $A = [a_{ij}]_{n \times n}$ be a square matrix of order n then the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called diagonal elements of the square matrix A, and the diagonal along which these elements lie is called principal diagonal or **main diagonal** or simply **diagonal** of the matrix A.

For example,

(i) Diagonal elements of the matrix $A = \begin{bmatrix} 8 & 9 \\ 5 & 6 \end{bmatrix}$ are 8, 6.

(ii) Write the diagonal elements (if possible) of the matrix $A = \begin{bmatrix} 8 & 9 & 7 \\ 6 & 5 & 2 \end{bmatrix}$

Here, A is not a square matrix, so writing diagonal elements of a rectangular matrix is impossible.

Diagonal Matrix

A square matrix A = $[a_{ij}]_{n\times n}$ is said to be diagonal matrix if

$a_{ij} = 0, \ \forall\, i \neq j$

For example,

(i) If A = $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ then it is a diagonal matrix because all its non-diagonal elements are zero. Sometimes, we denote it by writing diag. [4, 3, 6].

(ii) A = $\begin{bmatrix} 2 & 9 \\ 0 & 5 \end{bmatrix}$ is not a diagonal matrix because non-diagonal element $a_{12} \neq 0$.

Remark 4:

(i) For a diagonal matrix all non diagonal elements must be zero.

(ii) In a diagonal matrix some or all the diagonal elements may be zero.

Example 4: Write all the diagonal matrices of order 2×2 having its elements only 0 or 1.

Solution: For a diagonal matrix, all the non-diagonal elements are zero. Therefore, we are to write 0 and 1 in the diagonal elements in different ways, i.e. 0, 0; 0, 1; 1, 0; and 1, 1.

$\therefore$ possible diagonal matrices with elements only 0 and 1 are given below:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Scalar Matrix

A diagonal matrix is said to be scalar matrix if all its diagonal elements are same.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ all are scalar matrices.

Identity Matrix

A diagonal matrix is said to be Identity or **Unit matrix** if all the diagonal elements are equal to unity.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ all are identity (or Unit) matrices of order 2, 3, 4 respectively.

Upper Triangular Matrix

A square matrix A = $[a_{ij}]_{n\times n}$ is said to be upper triangular matrix if all the elements below the principal diagonal are zero.

For example, $\begin{bmatrix} 2 & 5 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 0 & 6 \\ 0 & 5 & 4 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 5 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ all are upper triangles matrices.

But $\begin{bmatrix} 2 & 9 & 7 \\ 0 & 5 & 8 \\ 2 & 0 & 9 \end{bmatrix}$ is not an upper triangular matrix because one element below the diagonal line, i.e. a_{31} is non zero, which is 2, in this case.

Lower Triangular Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be lower triangular matrix if all the elements above the principal diagonal are zero.

For example, $\begin{bmatrix} 5 & 0 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 9 & 7 \end{bmatrix}$, are lower triangular matrices of orders 2 and 3 respectively.

Null Matrix

A matrix $A = [a_{ij}]_{m \times n}$ is said to be null matrix if all its elements are equal to zero.

i.e. $a_{ij} = 0, \quad \forall$ i, j

$\therefore$ a null matrix is generally denoted by **O.**

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc. are null matrices.

Comparable Matrices

Two matrices are said to be comparable if they are of the same order.

For example,

if $A = \begin{bmatrix} 2 & 5 & 3 \\ 6 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ then A and B are comparable because both are of the same order, i.e. of order 2×3.

Equal Matrices

Two matrices are said to be equal if

(i) they are of same order, and

(ii) the corresponding elements of the matrices are equal.

For example, if $A = \begin{bmatrix} 2 & 8 \\ 3 & x \end{bmatrix}, B = \begin{bmatrix} a & 8 \\ 3 & 5 \end{bmatrix}$, then A = B, if a = 2, x = 5.

Example 5: Write orders and types of the following matrices:

(i) $\begin{bmatrix} 2 & 9 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 6 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 \\ 9 \\ 6 \end{bmatrix}$ (viii) $\begin{bmatrix} 8 & 9 & 1 & 5 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & 9 & 3 \\ 6 & 4 & 5 \end{bmatrix}$

Solution:

	Order	**Type**
(i)	2×2	Square matrix [$\because$ rows and columns are equal in number.]
(ii)	2×2	Diagonal matrix [$\because$ all the non-diagonal elements are zero.]
(iii)	2×2	Scalar matrix [$\because$ all the diagonal elements are equal and non diagonal element, are zero.]
(iv)	2×2	Identify matrix [$\because$ all the diagonal elements are unity and non diagonal element are zero.]
(v)	3×3	Upper triangular matrix [$\because$ all the elements below the principal diagonal are zero.]
(vi)	3×3	Lower triangular matrix [$\because$ all the elements above the principal diagonal are zero.]
(vii)	3×1	Column matrix [$\because$ it has only one column.]
(viii)	1×4	Row matrix [$\because$ it has only one row.]
(ix)	2×3	Rectangular matrix [$\because$ number of rows $\neq$ numbers of columns.]

Example 6:

(i) If $\begin{bmatrix} 3 & x+y \\ xy & 7+z \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 8 & 4 \end{bmatrix}$, find x, y, z.

(ii) If $\begin{bmatrix} a+5 & -2a & b+6 \\ 2c & 3b+2 & x \\ y+1 & z+3 & x+2 \end{bmatrix} = \begin{bmatrix} 2 & a+9 & 11 \\ c+4 & -b+22 & 3-x \\ 2 & 2z+3 & 5-x \end{bmatrix}$ find a, b, c, x, y, z.

Solution:

(i) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$3 = 3$

$x + y = 6$ … (1)

$xy = 8$ … (2)

$7 + z = 4 \Rightarrow z = -3$

From (1), $y = 6 - x$ … (3)

Putting y from (3) in (2), we get

$x(6-x) = 8$

$\Rightarrow 6x - x^2 - 8 = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x^2 - 4x - 2x + 8 = 0$

$\Rightarrow x(x-4) - 2(x-4) = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4, 2$

When x = 4, y = 6 – 4 = 2 and when x = 2, y = 6 – 2 = 4

$\therefore x = 4, y = 2, z = -3$ or $x = 2, y = 4, z = -3$.

(ii) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$a + 5 = 2 \Rightarrow a = -3$

$-2a = a + 9 \Rightarrow -3a = 9 \Rightarrow a = -3$

$b + 6 = 11 \Rightarrow b = 5$

$2c = c + 4 \Rightarrow c = 4$

$3b + 2 = -b + 22 \Rightarrow 4b = 20 \Rightarrow b = 5$

$x = 3 - x \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$

$y + 1 = 2 \Rightarrow y = 1$

$z + 3 = 2z + 3 \Rightarrow -z = 0 \Rightarrow z = 0$

$x + 2 = 5 - x \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$

$\therefore a = -3, b = 5, c = 4, x = \frac{3}{2}, y = 1, z = 0.$

Here is an exercise for you.

E 2) Find the values of x, y, z, w if $\begin{bmatrix} 3x - 2y & z + w \\ 3z - w & x + y \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 5 & 3 \end{bmatrix}$.

9.4 OPERATIONS ON MATRICES

In school times, a child first learns the natural numbers and then learns how these numbers are added, subtracted, multiplied and divided. Similarly, here also we now see as to how such operations (except division) are applied on matrices.

These operations are explained by first giving a general formula and then examples followed by some exercises.

Remark 5: Division of a matrix by another matrix is meaning less and hence it is not permitted in case of matrices.

9.4.1 Addition of Matrices

Addition of two matrices A and B make sense only if they are of the same order and obtained by adding their corresponding elements. It is denoted by A + B.

That is, if A= $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{m\times n}$ then A + B = $[a_{ij} + b_{ij}]_{m\times n}$

For example,

(i) If $A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 9 & 8 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2+1 & 3+5 & 4+6 \\ 7+2 & 5+9 & 1+8 \end{bmatrix} = \begin{bmatrix} 3 & 8 & 10 \\ 9 & 14 & 9 \end{bmatrix}.$$

(ii) If A = $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, B= $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 5 \end{bmatrix}$ then A + B does not make any sense because A and B are of different orders.

Properties of Addition of Matrices

If A, B, C are of the same orders over R, (i.e. elements of A, B, C are real numbers) then

(i) A + B = B + A (commutative law)

(ii) (A + B) + C = A + (B + C) (associative law)

(iii) A + **O** = **O** + A = A, where **O** is a null matrix. (existence of additive identity)

(iv) For a given matrix A, there exists a matrix B of the same order such that A + B **=** **O** = B + A.
Here B is called additive inverse of A. (existence of additive inverse)

9.4.2 Scalar Multiplication

Let A = $[a_{ij}]_{m \times n}$ and k is any scalar then scalar multiplication of A by k is denoted by kA and obtained by multiplying each element of A by k.

i.e. kA= $[ka_{ij}]_{m \times n}$

For example,

If A = $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and k = 7, then kA = 7A = $\begin{bmatrix} 7\times 3 & 7\times 4 \\ 7\times 5 & 7\times 6 \end{bmatrix} = \begin{bmatrix} 21 & 28 \\ 35 & 42 \end{bmatrix}$.

Properties of Scalar Multiplication

If A and B are two matrices of the same order and α, β are scalars (real numbers), then

(i) $\alpha(A + B) = \alpha A + \alpha B$

(ii) $\alpha(\beta A) = (\alpha\beta)A$

(iii) $(\alpha + \beta)A = \alpha A + \beta A$

(iv) 1A = A

9.4.3 Subtraction of Matrices

Subtraction of two matrices A and B make sense only if they are of the same order, and is given by

A – B = A + (– B) = A+ (–1) B, i.e. A – B means addition of two matrices A and – B. So, if A = $[a_{ij}]_{m \times n}$, B = $[b_{ij}]_{m \times n}$, then

A – B = $[a_{ij} + (-1)b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$

For example,

(i) If A = $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, B = $\begin{bmatrix} 6 & 2 \\ 1 & 10 \end{bmatrix}$, then A – B = $\begin{bmatrix} 2-6 & 4-2 \\ 6-1 & 8-10 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 5 & -2 \end{bmatrix}$.

(ii) If A = $\begin{bmatrix} 2 & 9 & 3 \\ 8 & 4 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix}$, then A – B does not make any sense because A and B are of different orders.

9.4.4 Matrix Multiplication

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices, then product of A and B is denoted by AB and is defined only if number of columns in A = number of rows in B and is given by

$AB = C = [c_{ij}]_{m\times p}$

where c_{ij} = (i, j)th element of C and is equal to (i^{th} row of A) (j^{th} column of B)

$$= [a_{i1} \quad a_{i2} \quad \ldots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ . \\ . \\ . \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

$= \sum_{k=1}^{n} a_{ik}b_{kj}$, i.e. sum of product of first, second, third, … elements of i^{th} row of A with first, second, third, … , elements of j^{th} column of B respectively.

You may notice that the number of rows in AB = number of rows in A, and number of columns in AB = number of columns in B.

Let us make the above concept more clear by taking the following matrices, in particular let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Here A is a matrix of order 3×2 and B be a matrix of order 2×2. As number of columns of A = 2 = number of rows of B.

$\therefore$ AB is defined and is given by

$$AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}_{3\times 2}$$

where c_{11} = Product of first row of A and first column of B
= Sum of product of first, second elements of first row of A with first, second elements of first column of B respectively.
= $a_{11}b_{11} + a_{12}b_{21}$,

c_{12} = Product of first row of A and second column of B
= Sum of product of first, second elements of first row of A with first, second elements of second column of B respectively.
= $a_{11}b_{12} + a_{12}b_{22}$,

c_{21} = …, etc.

Properties of Matrix Multiplication

If A, B, C are three matrices such that corresponding multiplications hold then

(1) A(BC) = (AB)C (associative law)

(2) (i) A(B + C) = AB + AC (left distributive law)

(ii) $(A + B)C = AC + BC$ (right distributive law)

(3) If A is a square matrix of order n, then
$I_n A = AI_n = A$, where I_n is the identity matrix of order n.

Remark 6: Commutative law does not hold, in general,

i.e. $AB \neq BA$, in general. But for some cases AB may be equal to BA. This has been explained below:

(i) AB may be defined but BA may not be defined and hence $AB \neq BA$ in this case.

For example, let A be a matrix of order 3×2 and B be a matrix of order 2×4.
Here AB is defined and is of order 3×4.
But BA is not defined ($\because$ number of columns of $B \neq$ number of rows of A).

(ii) AB and BA both may defined but may not be of same order and hence $AB \neq BA$.
For example, let A be a matrix of order be 3×2 and B be a matrix of order 2×3.

Here as number of columns of A = number of rows of B.
$\therefore$ AB is defined and is of order 3×3.
Also, number of columns of B = number of rows of A.
Hence BA is defined but of order 2×2.
$\therefore AB \neq BA$.

(iii) AB and BA both may be defined and of same order but even then they may not be equal.

Let $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix}$

Here, AB and BA both are defined and are of same order.

But $AB = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 30+8 & 40+36 \\ 12+14 & 16+63 \end{bmatrix} = \begin{bmatrix} 38 & 76 \\ 26 & 79 \end{bmatrix}$ and

$BA = \begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 30+16 & 24+56 \\ 10+18 & 8+63 \end{bmatrix} = \begin{bmatrix} 46 & 80 \\ 28 & 71 \end{bmatrix}$.

So, $AB \neq BA$.

However, sometimes, we may observe that AB = BA.

For example, Let $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$.

Here $AB = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 10+12 & 6-6 \\ 20-20 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$ and

$BA = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10+12 & -15+15 \\ -8+8 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$.

Here, AB = BA.

Example 7: If $A = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$, then evaluate the following

(i) 3A + 2B (ii) 2A – 3B (iii) AB (iv) BA

Solution:

(i) $3A + 2B = 3\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} + 2\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 15 \\ -9 & 18 & 21 \\ 3 & 24 & 27 \end{bmatrix} + \begin{bmatrix} 6 & 12 & 4 \\ 2 & 8 & 10 \\ 16 & 14 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 6+6 & 12+12 & 15+4 \\ -9+2 & 18+8 & 21+10 \\ 3+16 & 24+14 & 27-2 \end{bmatrix} = \begin{bmatrix} 12 & 24 & 19 \\ -7 & 26 & 31 \\ 19 & 38 & 25 \end{bmatrix}$$

(ii) $2A - 3B = 2\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ -6 & 12 & 14 \\ 2 & 16 & 18 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 6 \\ 3 & 12 & 15 \\ 24 & 21 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4-9 & 8-18 & 10-6 \\ -6-3 & 12-12 & 14-15 \\ 2-24 & 16-21 & 18+3 \end{bmatrix} = \begin{bmatrix} -5 & -10 & 4 \\ -9 & 0 & -1 \\ -22 & -5 & 21 \end{bmatrix}$$

(iii) $AB = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 6+4+40 & 12+16+35 & 4+20-5 \\ -9+6+56 & -18+24+49 & -6+30-7 \\ 3+8+72 & 6+32+63 & 2+40-9 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 63 & 19 \\ 53 & 55 & 17 \\ 83 & 101 & 33 \end{bmatrix} \quad \ldots (1)$$

(iv) $BA = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 6-18+2 & 12+36+16 & 15+42+18 \\ 2-12+5 & 4+24+40 & 5+28+45 \\ 16-21-1 & 32+42-8 & 40+49-9 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 64 & 75 \\ -5 & 68 & 78 \\ -6 & 66 & 80 \end{bmatrix} \quad \ldots (2)$$

9.4.5 Integral Powers of a Square Matrix

Here, we will learn how higher powers of A are evaluated. We define

$A^2 = A.A$

$A^3 = A^2A$ or $A^3 = AA^2$

$A^4 = A^3A$ or $A^4 = AA^3$ or $A^4 = A^2A^2$

and so on

in general $A^{p+q} = A^pA^q = A^qA^p$.

Remark 7:

(i) We define $A^0 = I$, where I is the identity matrix of the same order as A.

(ii) $(A+B)^2 = A^2 + AB + BA + B^2$.

(iii) $(A+B)^2 = A^2 + 2AB + B^2$ if and only if AB = BA.

Example 8: If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then find A^4.

Solution: $A^2 = AA = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}$

$A^4 = A^2A^2 = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}\begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} = \begin{bmatrix} 81+128 & 72+136 \\ 144+272 & 128+289 \end{bmatrix} = \begin{bmatrix} 209 & 208 \\ 416 & 417 \end{bmatrix}$

Now, you can try the following exercises.

E 3) If $3X + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$ and $2X - 3Y = \begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$, then find matrices X and Y.

E 4) Find AB, if defined, in each of the following cases:

(i) $A = [5 \quad 4]$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $B = [5 \quad 6]$

(iii) $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

E 5) Evaluate the product $[2 \quad 3 \quad 5]\begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$.

E 6) If $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, then find A^8.

9.5 TRANSPOSE OF A MATRIX

Transpose of a matrix A is denoted by A' or A^T and is obtained by interchanging rows and columns of A.

For example, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$.

Properties of Transpose

(i) $(A')' = A$

(ii) $(kA)' = kA'$, where k is a scalar

(iii) $(A+B)' = A' + B'$

(iv) $(A-B)' = A' - B'$

(v) $(AB)' = B'A'$

Symmetric Matrix

A square matrix A is said to be symmetric matrix if $A' = A$.

For example, let $A = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 4 & 3 \\ 6 & 3 & 8 \end{bmatrix}$, then $A' = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 4 & 3 \\ 6 & 3 & 8 \end{bmatrix} = A$.

$\therefore$ A is symmetric.

Skew-Symmetric Matrix

A square matrix A is said to be skew-symmetric matrix if $A' = -A$.

For example, let $A = \begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$ then

$$A' = \begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} = -A.$$

$\therefore$ A is skew-symmetric.

Remark 8: A square matrix $A = [a_{ij}]_{m\times n}$ will be symmetric if $a_{ij} = a_{ji}, \forall$ i, j and will be skew-symmetric if $a_{ij} = -a_{ji}, \forall i, j$ and hence for a skew-symmetric matrix

$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$$

That is, all the diagonal elements of a skew-symmetric matrix are zero.

Example 9: If $A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ then show that

(i) $\frac{1}{2}(A + A')$ is symmetric, and (ii) $\frac{1}{2}(A - A')$ is skew-symmetric.

Solution:

(i) Let $P = \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}'\right)$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 & 3 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix} \quad \ldots (1)$$

$$P' = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix}' = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2)

$P' = P \Rightarrow P$ is symmetric, i.e. $\frac{1}{2}(A + A')$ is symmetric.

(ii) Let $Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}'\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}\right)$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 5+2 \\ -2-5 & 4-4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 7 \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -7/2 \\ 7/2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix} = -Q$$

$\Rightarrow$ Q is skew symmetric, i.e. $\frac{1}{2}(A - A')$ is skew symmetric.

Remark 9:

$$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}A + \frac{1}{2}A' + \frac{1}{2}A - \frac{1}{2}A' = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$$

i.e. A = P + Q, where P is symmetric and Q is skew symmetric.
i.e. every square matrix can be expressed as a sum of a symmetric and a skew-symmetric matrix.

9.6 TRACE OF A MATRIX

In this section we will define trace of a matrix.

Trace of a square matrix $A = [a_{ij}]_{n\times n}$ is denoted by tr (A) and is defined as
tr (A) = sum of diagonal elements of the matrix.
i.e. $tr(A) = a_{11} + a_{22} + a_{33} + \ldots + a_{nn}$

For example, if $A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 8 & 4 \\ 9 & 1 & -3 \end{bmatrix}$ then $tr(A) = 2 + 8 + (-3) = 7.$

Properties of Trace of a Matrix

If $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ then

(i) tr (A + B) = tr (A) + tr (B)

(ii) tr (kA) = k tr (A) , where k is a scalar

(iii) tr (AB) = tr (BA)

Remark 10: tr (AB) $\neq$ tr (A) tr (B)

Here is an exercise for you.

E 7) (i) Find trace of the matrix A, where $A = \begin{bmatrix} 8 & 7 \\ 5 & 6 \end{bmatrix}$

(ii) Find trace of the matrices I_2, I_3, I_n.

9.7 DETERMINANT OF SQUARE MATRICES

Determinant is a number associated with each square matrix. In this section, we will deal with determinant of square matrices of order 1, 2, 3 and 4. Determinants of square matrices of order greater than 4 can be evaluated in a similar fashion.

9.7.1 Determinant of a Square Matrix of Order 1

If $A = [a_{11}]$ be a square matrix of order 1 then determinant of A is given by $|A| = |a_{11}| = a_{11}$.

For example,

(i) If $A = [5]$ then $|A| = |5| = 5$.

(ii) If $A = [-3]$ then $|A| = |-3| = -3$.

Remark 11:

(i) $|A|$ is read as determinant of A, do not read it modulus of A, i.e.

if $A = [-8]$ then $|A| = |-8| = -8$.

But in case of modulus $|-8| = -(-8) = 8$.

(ii) The context in which we are using $|\ |$ will clear whether it represents modulus or determinant.

9.7.2 Determinant of a Square Matrix of Order 2×2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Let us take an example:

Example 10: Evaluate the following determinants:

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 5 \\ 8 & 9 \end{vmatrix}$ (iii) $\begin{vmatrix} x^2 & x^2+1 \\ x & x+1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(ii) $\begin{vmatrix} 3 & 5 \\ 8 & 9 \end{vmatrix} = 27 - 40 = -13$

(iii) $\begin{vmatrix} x^2 & x^2+1 \\ x & x+1 \end{vmatrix} = x^3 + x^2 - (x^3 + x) = x^2 - x$

Now, you can try the following exercise.

E 8) Find x in each of the following cases:

(i) $\begin{vmatrix} x & 7 \\ 9 & x+2 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x & x^2 \\ 15 & 5 \end{vmatrix} = 0$

9.7.3 Determinant of a Square Matrix of Order 3×3

Before evaluating, the determinant of order 3×3, let us define the minors and cofactors of a square matrix as follows:

Minors and Cofactors

Minor

If $A = [a_{ij}]_{n \times n}$ be a square matrix of order n then minor of $(i, j)^{th}$ element a_{ij} is denoted by M_{ij} and is defined as

M_{ij} = determinant of sub matrix of order n – 1 obtained after deleting i^{th} row and j^{th} column from A.

Example 11: Find the minor of each element of the following matrices:

(i) $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Solution:

(i) Let A = $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Let M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A, i, j = 1, 2.

$\therefore M_{11} = |-7| = -7$ $\quad \begin{bmatrix} \text{Determinant obtained after deleting first} \\ \text{row and first column of matrix A } = |-7| \end{bmatrix}$

Similarly, $M_{12} = |4| = 4, \quad M_{21} = |5| = 5, \quad M_{22} = |2| = 2$

(ii) Let A = $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Let M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A, where i, j = 1, 2, 3.

$\therefore M_{11} = \begin{vmatrix} 5 & 7 \\ 9 & 1 \end{vmatrix} = 5 - 63 = -58$ $\quad \begin{bmatrix} \text{After deleting the first row and} \\ \text{first column from A.} \end{bmatrix}$

$M_{12} = \begin{vmatrix} 6 & 7 \\ -8 & 1 \end{vmatrix} = 6 + 56 = 62$ $\quad \begin{bmatrix} \text{After deleting the first row and} \\ \text{second column from A.} \end{bmatrix}$

Similarly,

$M_{13} = \begin{vmatrix} 6 & 5 \\ -8 & 9 \end{vmatrix} = 54 + 40 = 94$

$M_{21} = \begin{vmatrix} 4 & -2 \\ 9 & 1 \end{vmatrix} = 4 + 18 = 22$

$M_{22} = \begin{vmatrix} -3 & -2 \\ -8 & 1 \end{vmatrix} = -3 - 16 = -19$

$M_{23} = \begin{vmatrix} -3 & 4 \\ -8 & 9 \end{vmatrix} = -27 + 32 = 5$

$M_{31} = \begin{vmatrix} 4 & -2 \\ 5 & 7 \end{vmatrix} = 28 + 10 = 38$

$M_{32} = \begin{vmatrix} -3 & -2 \\ 6 & 7 \end{vmatrix} = -21 + 12 = -9$

$M_{33} = \begin{vmatrix} -3 & 4 \\ 6 & 5 \end{vmatrix} = -15 - 24 = -39$

Cofactor

If $A = [a_{ij}]_{n\times n}$ be a square matrix of order n then cofactor of $(i, j)^{th}$ element a_{ij} of matrix A is denoted by C_{ij} and is defined by

$C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A.

Example 12: Find the cofactor of each element of the following matrices:

(i) $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Solution:

(i) Let $A = \begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Let C_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A, i, j = 1, 2.

$\therefore C_{11} = (-1)^{1+1} M_{11} = (-1)^2(-7) = -7$ [Using Example 11 (i)]

Similarly,

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 (5) = -5$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) Let $A = \begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Let C_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A, then

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-58) = -58$ [Using Example 11 (ii)]

Similarly,

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (62) = -62$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 (94) = 94$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 (22) = -22$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 (-19) = -19$

$C_{23} = (-1)^{2+3} M_{23} = (-1)^5 (5) = -5$

$C_{31} = (-1)^{3+1} M_{31} = (-1)^4 (38) = 38$

$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-9) = 9$

$C_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-39) = -39$

Here is an exercise for you.

E 9) Find minor and cofactor of the elements $a_{12}, a_{23}, a_{31}, a_{13}$ where

$$A = [a_{ij}]_{3\times 3} = \begin{bmatrix} 5 & 6 & -2 \\ -3 & 8 & 9 \\ 7 & 10 & -4 \end{bmatrix}$$

Now, we discuss the determinant of a square matrix of order 3×3.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ = Sum of products of the elements of any line (row or column) with their corresponding co-factors.

Let us expand along first row (R_1), we have

$|A| = a_{11}$ (co-factor of a_{11}) $+ a_{12}$ (co-factor of a_{12}) $+ a_{13}$(co-factor of a_{13})

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Remark 12:

(i) We can expand the determinant along any row or column, we will get the same value.

(ii) When we expand a determinant along any row or column we attach + or – sign with each term containing the product of elements of a row (or column) and its corresponding minor. Pattern of +, – signs is shown as under.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

We put + at (1, 1) position and then alternatively– and + are placed, provided either we can move along row or column (we cannot walk diagonally).

(iii) There is no hard and fast rule, to choose a row or column to expand a determinant. But if we choose that row or column which contains maximum number of zero, it will reduce a lot of our calculation work.

Example 13: Evaluate the following determinants:

(i) $\begin{vmatrix} 3 & 2 & -1 \\ 5 & 4 & 6 \\ -3 & 1 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 9 & 6 \\ 0 & 5 & 4 \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} 3 & 2 & -1 \\ 5 & 4 & 6 \\ -3 & 1 & 7 \end{vmatrix}$

Expanding along R_1 (first row)

$\Delta = 3(28 - 6) - 2(35 + 18) - 1(5 + 12) = 66 - 106 - 17 = -57$

(ii) Let $\Delta = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

Expanding along R_1 (first row)

$\Delta = 2(2-4) - 1(1-4) + 2(2-4) = -4 + 3 - 4 = -5$

(iii) Let $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 0 & 9 & 6 \\ 0 & 5 & 4 \end{vmatrix}$

Expanding along C_1 (first column) $\left[\begin{array}{l} \because \text{it contains maximum} \\ \text{number of zeros.} \end{array}\right]$

$\Delta = 3(36-30) - 0 + 0 = 18$

Here is an exercise for you.

E 10) If $A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 5 & 1 \\ 2 & 4 & 8 \end{bmatrix}$ then show that $|A| = 0$.

9.7.4 Determinant of Square Matrices of Order 4×4 and of Higher Order

The procedure of expanding the determinant of order 4 or more is the same as we discussed in case of order 3×3.

Example 14: Evaluate $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 3 & 5 \\ 6 & 4 & -2 & 7 \\ -4 & 8 & 9 & -5 \end{vmatrix}$

Solution: Expanding along R_1, we get

$$\Delta = 1\begin{vmatrix} -1 & 3 & 5 \\ 4 & -2 & 7 \\ 8 & 9 & -5 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 5 \\ 6 & -2 & 7 \\ -4 & 9 & -5 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 & 5 \\ 6 & 4 & 7 \\ -4 & 8 & -5 \end{vmatrix} - 4\begin{vmatrix} 2 & -1 & 3 \\ 6 & 4 & -2 \\ -4 & 8 & 9 \end{vmatrix}$$

Expanding each determinant of order 3×3 along R_1, we get

$$\begin{aligned}\Delta &= 1[-1(10-63) - 3(-20-56) + 5(36+16)] - 2[2(10-63) \\ &\quad - 3(-30+28) + 5(54-8)] + 3[2(-20-56) - (-1)(-30+28) + 5(48+16)] \\ &\quad - 4[2(36+16) - (-1)(54-8) + 3(48+16)] \\ &= (53+228+260) - 2(-106+6+230) + 3(-152-2+320) - 4(104+46+192) \\ &= 541 - 260 + 498 - 1368 \\ &= -589\end{aligned}$$

Remark 13:

(i) If A is square matrix then determinant of A is unique.

(ii) If A is not a square matrix then determent of A does not exist.

9.8 PROPERTIES OF DETERMINANTS

In Sec. 9.7 of this unit you have become familiar about how to expand the determinants of orders 1, 2, 3, or of higher order. But as you have seen that it requires lot of calculations and is a time consuming process. To avoid such calculations and to reduce the time of evaluation, we will use properties of determinants.

In this section, we will discuss some properties of the determinants. We shall give the proofs of these properties only for determinants of order 3×3. But remember that these properties hold good for all orders of the determinants. Let us discuss these one by one. Our way to move further is that, first we list all the properties and then some examples will be solved to get the idea how these properties are used and useful.

P 1 $|A'|=|A|$, i.e. determinants of a matrix and its transpose are equal.

Proof: Let $A=\begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$ … (1)

$$|A|=\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$$

Expanding along R_1

$|A|=a(ny-mz)-b(nx-lz)+c(mx-ly)$ … (2)

From (1), we get

$$A'=\begin{bmatrix} a & x & l \\ b & y & m \\ c & z & n \end{bmatrix}$$

$$\therefore\ |A'|=\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix}$$

Expanding along R_1

$$\begin{aligned} |A'| &= a(ny-mz)-x(bn-cm)+l(bz-cy) \\ &= a(ny-mz)-bnx+cmx+lbz-cly \\ &= a(ny-mz)-b(nx-lz)+c(mx-ly) \quad \text{… (3)} \end{aligned}$$

From (2) and (3), we get

$|A'|=|A|$

P 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant is multiplied by (–1).

Proof: Let $\Delta=\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$ … (1)

Expanding along R_1

$\Delta=a(ny-mz)-b(nx-lz)+c(mx-ly)$ … (2)

Let us interchange the first and second rows of the given determinant we have a new determinant Δ_1 (say) as

$$\Delta_1 = \begin{vmatrix} x & y & z \\ a & b & c \\ l & m & n \end{vmatrix}$$

Expanding along R_1

$$\begin{aligned}\Delta_1 &= x(bn - cm) - y(an - cl) + z(am - bl)\\ &= bnx - cmx - any + cly + amz - blz\\ &= -a(ny - mz) + b(nx - lz) - c(mx - ly)\\ &= -[a(ny - mz) - b(nx - lz) + c(mx - ly)]\\ &= -\Delta \qquad\qquad [\text{Using (2)}]\end{aligned}$$

Remark 14: Here we interchanged R_1 and R_2. In fact we can interchange any two rows or any two columns, result remains the same in each case.

P 3 If any two rows or columns of a determinant are identical then value of the determinant vanishes.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix}$, where R_1 and R_2 are identical

Expanding along R_1, we get

$$\Delta = a(bz - cy) - b(az - cx) + c(ay - bx) = abz - acy - abz + bcx + acy - bcx = 0$$

P 4 If each element of a row (or a column) of a determinant is multiplied by a scalar k (say), then value of the new determinant is k times the original given determinant.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$

Expanding along R_1

$$\Delta = a(ny - mz) - b(nx - lz) + c(mx - ly) \qquad \ldots (1)$$

Let $\Delta_1 = \begin{vmatrix} ka & b & c \\ kx & y & z \\ kl & m & n \end{vmatrix}$ $\left[\begin{array}{l}\text{Here, the elements of first column of}\\ \Delta \text{ have been multiplied with k.}\end{array}\right]$

Expanding along R_1

$$\begin{aligned}\Delta_1 &= ka(ny - mz) - b(knx - klz) + c(kmx - kly)\\ &= k[a(ny - mz) - b(nx - lz) + c(mx - ly)] = k\Delta \qquad \ldots (2)\ [\text{Using (1)}]\end{aligned}$$

From (1) and (2)

$$\Delta_1 = k\Delta$$

Hence proved

Remark 15: This property implies that if there is some factor common in all elements of any line then we can write it as the factor of the whole determinant.

For example, $\begin{vmatrix} 5a & b & c \\ 5x & y & z \\ 5l & m & n \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$

P 5 If each element of a row (or column) of a determinant is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

Proof: Let $\Delta = \begin{vmatrix} a+\lambda & b+\mu & c+\nu \\ x & y & z \\ l & m & n \end{vmatrix}$, then expanding along R_1, we get

$$\Delta = (a+\lambda)\begin{vmatrix} y & z \\ m & n \end{vmatrix} - (b+\mu)\begin{vmatrix} x & z \\ l & n \end{vmatrix} + (c+\nu)\begin{vmatrix} x & y \\ l & m \end{vmatrix}$$

$$= \left[a\begin{vmatrix} y & z \\ m & n \end{vmatrix} - b\begin{vmatrix} x & z \\ l & n \end{vmatrix} + c\begin{vmatrix} x & y \\ l & m \end{vmatrix} \right] + \left[\lambda\begin{vmatrix} y & z \\ m & n \end{vmatrix} - \mu\begin{vmatrix} x & z \\ l & n \end{vmatrix} + \nu\begin{vmatrix} x & y \\ l & m \end{vmatrix} \right]$$

$$= \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} + \begin{vmatrix} \lambda & \mu & \nu \\ x & y & z \\ l & m & n \end{vmatrix}$$

P 6 If to each element of any row (or column), we add some scalar multiple of another row (or column) and some other scalar multiple of some other row (or column), the value of determinant remains unaltered.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$ … (1)

and $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ l+ka & m+kb & n+kc \end{vmatrix}$

where Δ_1 is obtained from Δ by operating $R_3 \to R_3 + kR_1$ i.e. k times R_1 has been added to R_3.

$$\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ ka & kb & kc \end{vmatrix} \quad \text{[Using property 5]}$$

$$= \Delta + k\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} \quad \left[\begin{array}{l} \text{Using (1) and taking k} \\ \text{common from third} \\ \text{rows of second determanant} \end{array}\right]$$

$$= \Delta + k(0) = \Delta + 0 = \Delta \quad \left[\begin{array}{l} \because R_1 \text{ and } R_2 \text{ are identical and} \\ \text{so using property 3} \end{array}\right]$$

Hence proved

Remark 16: If operations of the type $R_i \to R_i + kR_j$ are used more than one in a single step then keep it always in mind that row which has been affected in one operation cannot be used in other operation.

For example,

(i) $R_1 \to R_1 + 2R_3,\ R_2 \to R_2 + 5R_1$ is not allowed because R_1 has been affected by first operation, so it cannot be used in second operation in the same step.

(ii) $R_1 \to R_1 + 3R_3,\ R_2 \to R_2 + 2R_3,$ etc. are allowed.

P 7 If all the elements of any line (row or column) are zero then value of the determinant vanishes.

Proof: Let $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x & y & z \\ l & m & n \end{vmatrix}$, then evaluating along R_1, we get

$$\Delta = (0)(ny - mz) - (0)(nx - lz) + (0)(mx - ly) = 0 - 0 + 0 = 0$$

Example 15: Evaluate the following determinants:

(i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ (ii) $\begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

(iv) $\begin{vmatrix} 3 & x & xyz \\ 3 & y & xyz \\ 3 & z & xyz \end{vmatrix}$ (v) $\begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

(vii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ (ix) $\begin{vmatrix} 2x+3 & x & x \\ x & 2x+3 & x \\ x & x & 2x+3 \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$ [$\because$ all elements of C_1 are zero, so using P7.]

(ii) Let $\Delta = \begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} -3+5-2 & 5 & -2 \\ 8+9-17 & 9 & -17 \\ 3-6+3 & -6 & 3 \end{vmatrix}$

$= \begin{vmatrix} 0 & 5 & -2 \\ 0 & 9 & -17 \\ 0 & -6 & 3 \end{vmatrix} = 0$ [$\because$ all the element of C_1 are zero, so using P7.]

(iii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

Operating $R_3 \to R_3 + R_2$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

Taking (a + b + c) common from R_3

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a+b+c)(0) = 0 [\because R_1 \text{ and } R_3 \text{ are identical.}]$$

(iv) Let $\Delta = \begin{vmatrix} 3 & x & xyz \\ 3 & y & xyz \\ 3 & z & xyz \end{vmatrix}$

Taking 3, xyz common from C_1 and C_3 respectively

$$\Delta = 3xyz\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} = 3xyz\,(0) = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical.}]$$

(v) Let $\Delta \begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$

Taking 6 common from C_3

$$\Delta = 6\begin{vmatrix} 2 & 3 & 5 \\ 5 & 4 & 9 \\ 6 & 1 & 7 \end{vmatrix}$$

Operating $C_3 \to C_3 - C_1 - C_2$

$$\Delta = 6\begin{vmatrix} 2 & 3 & 0 \\ 5 & 4 & 0 \\ 6 & 1 & 0 \end{vmatrix} = 6\,(0) = 0 [\because \text{all the elements of } C_3 \text{ are zero, so using P7.}]$$

(vi) Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Operating $R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0.(y-x) & (y-x).1 & (y-x)(y+x) \\ 0.(z-x) & (z-x).1 & (z-x)(z+x) \end{vmatrix}$$

Taking y – x, z – x common from R_2, R_3 respectively

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0.(z-y) & 0.(z-y) & 1.(z-y) \end{vmatrix}$$

Taking (z – y) common from R_3

$$\Delta = (y-x)(z-x)(z-y)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (y-x)(z-x)(z-y)[1(1-0)-0+0]$$
$$= (x-y)(y-z)(z-x)$$

(vii) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

Taking (a + b+ c) common from C_1

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

Expanding along C_1

$$\Delta = (a+b+c)[1\{(c-b)(b-c)-(a-b)(a-c)\}-0+0]$$
$$= (a+b+c)[bc-c^2-b^2+bc-(a^2-ac-ab+bc)]$$
$$= (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

(viii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^2-x^3 \end{vmatrix}$$

Taking y – x, z – x common from C_2, C_3 respectively

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^3 & y^2+x^2+xy & z^2+x^2+zx \end{vmatrix}$$

Operating $C_3 \to C_3 - C_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & z^2-y^2+x(z-y) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & (z-y)(z+y+x) \end{vmatrix}$$

Taking (z – y) (x + y + z) common from C_3

$$\Delta = (y-x)(z-x)(z-y)(x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & 1 \end{vmatrix}$$

Expanding along R_1

$$\Delta = (y-x)(z-x)(z-y)(x+y+z)[1(1-0)-0+0]$$

$$= (x-y)(y-z)(z-x)(x+y+z)$$

(ix) Let $\Delta = \begin{vmatrix} 2x+3 & x & x \\ x & 2x+3 & x \\ x & x & 2x+3 \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 4x+3 & x & x \\ 4x+3 & 2x+3 & x \\ 4x+3 & x & 2x+3 \end{vmatrix}$$

Taking 4x + 3 common from C_1

$$\Delta = (4x+3)\begin{vmatrix} 1 & x & x \\ 1 & 2x+3 & x \\ 1 & x & 2x+3 \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Delta = (4x+3)\begin{vmatrix} 1 & x & x \\ 0 & x+3 & 0 \\ 0 & 0 & x+3 \end{vmatrix}$$

Expanding along C_1

$$\Delta = (4x+3)[1\{(x+3)^2-0\}-0+0] = (4x+3)(x+3)^2$$

Now, you can try the following exercise.

E 11) Prove the following

(i) $\begin{vmatrix} ab & 1 & c(a+b) \\ bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \end{vmatrix} = 0$ [Without expanding]

(ii) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1+x^3 & 1+y^3 & 1+z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(1+xyz)$

[Using properties]

(iii) $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$ [Using properties]

9.9 SUMMARY

In this unit we have covered following topics:

1) Definition with examples of a matrix.
2) Types of matrices with examples.
3) Operations on matrices.
4) Integral powers of a square matrix.
5) Trace of a matrix.
6) Determinant and its properties.

9.10 SOLUTIONS/ANSWERS

E 1) $A = [a_{ij}]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, where $a_{ij} = |i - j|$

$a_{11} = |1-1| = |0| = 0,\quad a_{12} = |1-2| = |-1| = -(-1) = 1,\ a_{21} = |2-1| = |1| = 1,$

$a_{22} = |2-2| = |0| = 0,\ a_{31} = |3-1| = |2| = 2,\ a_{32} = |3-2| = |1| = 1$

$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

E 2) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$3x - 2y = -1$ … (1)

$z + w = 7$ … (2)

$3z - w = 5$ … (3)

$x + y = 3$ … (4)

Equation (1) + 2× equation (4) gives

$$\begin{aligned} 3x - 2y &= -1 \\ 2x + 2y &= 6 \\ \hline 5x \quad &= 5 \end{aligned}$$

$\Rightarrow x = 1$

Putting x = 1 in (4), we get

$1 + y = 3 \Rightarrow y = 2$

(2) + (3) gives.

$4z = 12 \Rightarrow z = 3$

Putting z = 3 in (2), we get

$3 + w = 7 \Rightarrow w = 4$

$\therefore x = 1, y = 2, z = 3, w = 4.$

E 3) $3X + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$ … (1)

$2X - 3Y = \begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$ … (2)

Equation (1) $\times 3 + 2 \times$ equation (2) gives

$$9X + 6Y + 4X - 6Y = 3\begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$$

$$13X = \begin{bmatrix} 12 & 39 \\ 54 & 39 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ -2 & -26 \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 52 & 13 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{13}\begin{bmatrix} 26 & 39 \\ 52 & 13 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$ [By scalar multiplication property]

Putting this value of X in (1), we get

$$3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$$

$$\Rightarrow 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} + \begin{bmatrix} -6 & -9 \\ -12 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 6 & 10 \end{bmatrix}$$

$$\Rightarrow Y = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$$ [By scalar multiplication property]

$$\therefore X = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}.$$

E 4) (i) Order of A is 1×2 and order of B is 3×1.

$\therefore$ number of columns in A $\neq$ number of rows in B.

$\Rightarrow$ AB is not defined.

(ii) Number of columns in A = number of rows in B = 1.

$\Rightarrow$ AB is defined and is given by

$$AB = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \end{bmatrix} = \begin{bmatrix} 3\times 5 & 3\times 6 \\ 4\times 5 & 4\times 6 \end{bmatrix} = \begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix}$$

(iii) AB is defined and is given by

$$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3+12+5 & 6+16+6 \\ 2+15+30 & 4+20+36 \end{bmatrix} = \begin{bmatrix} 20 & 28 \\ 47 & 60 \end{bmatrix}$$

(iv) AB is defined and is given by

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10+9 & 8+6 \\ 5+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 19 & 14 \\ 5 & 4 \end{bmatrix}$$

E 5) $$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 8+0+10 & 10+3+30 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 43 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 36+215 & 72+258 & 18+344 \end{bmatrix}$$

$$= \begin{bmatrix} 251 & 330 & 362 \end{bmatrix}$$

E 6) $$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 2+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$A^4 = A^2A^2 = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 8+72 & 0+81 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}$$

$$A^8 = A^4A^4 = \begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 80+6480 & 0+6561 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6560 & 6561 \end{bmatrix}$$

E 7) (i) tr(A) = sum of diagonal elements = 8 + 6 = 14

(ii) We know that in an identity matrix, all the diagonal elements are unity.

$\therefore$ tr (I_2) = 1+ 1 = 2 [$\because I_2$ is identity matrix of order 2×2].

Similarly, $tr(I_3) = 1+1+1 = 3$, $tr(I_n) = \underbrace{1+1+1+\ldots+1}_{\text{n times}} = n$.

E 8) (i) $\begin{vmatrix} x & 7 \\ 9 & x+2 \end{vmatrix} = 0 \Rightarrow x(x+2) - 63 = 0$

$\Rightarrow x^2 + 2x - 63 = 0 \Rightarrow x^2 + 9x - 7x - 63 = 0$

$\Rightarrow x(x+9) - 7(x+9) = 0$

$\Rightarrow (x+9)(x-7) = 0 \Rightarrow x = 7, -9$

(ii) $\begin{vmatrix} x & x^2 \\ 15 & 5 \end{vmatrix} = 0 \Rightarrow 5x - 15x^2 = 0$

$\Rightarrow 15x^2 - 5x = 0 \Rightarrow 5x(3x-1) = 0 \Rightarrow x = 0, 1/3$

E 9) Let M_{ij} and C_{ij} denote the minor and cofactor of $(i, j)^{th}$ element in the matrix A respectively then

$$M_{12} = \begin{vmatrix} -3 & 9 \\ 7 & -4 \end{vmatrix} = 12 - 63 = -51,\ M_{23} = \begin{vmatrix} 5 & 6 \\ 7 & 10 \end{vmatrix} = 50 - 42 = 8$$

$$M_{31} = \begin{vmatrix} 6 & -2 \\ 8 & 9 \end{vmatrix} = 54 - (-16) = 54 + 16 = 70$$

$$M_{13} = \begin{vmatrix} -3 & 8 \\ 7 & 10 \end{vmatrix} = -30 - 56 = -86$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3(-51) = 51,\ C_{23} = (-1)^{2+3} M_{23} = (-1)^5(8) = -8$$

$$C_{31} = (-1)^{3+1} M_{31} = (-1)^4(70) = 70$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4(-86) = -86$$

E 10) $|A| = \begin{vmatrix} 1 & 2 & 4 \\ -3 & 5 & 1 \\ 2 & 4 & 8 \end{vmatrix}$

Expanding along R_1

$|A| = 1(40-4) - 2(-24-2) + 4(-12-10) = 36 + 52 - 88 = 0$

E 11) (i) L.H.S. $= \begin{vmatrix} ab & 1 & c(a+b) \\ bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \end{vmatrix} = \begin{vmatrix} ab & 1 & ac+bc \\ bc & 1 & ab+ac \\ ca & 1 & bc+ab \end{vmatrix}$

Operating $C_3 \to C_3 + C_1$

$$\text{L.H.S.} = \begin{vmatrix} ab & 1 & ab+bc+ca \\ bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \end{vmatrix}$$

Taking ab + bc + ca common from C_3

$$\text{L.H.S.} = (ab+bc+ca)\begin{vmatrix} ab & 1 & 1 \\ bc & 1 & 1 \\ ca & 1 & 1 \end{vmatrix}$$

$= (ab+bc+ca)(0) = 0 =$ R.H.S. [$\because C_2$ and C_3 are identical.]

(ii) Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1+x^3 & 1+y^3 & 1+z^3 \end{vmatrix}$

$= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ [Using property 5]

Taking x, y, z common from C_1, C_2, C_3 of the second determinant respectively.

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $R_1 \leftrightarrow R_3$ on first determinant

$$\Delta = (-1)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $R_2 \leftrightarrow R_3$ on first determinant

$$\Delta = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Taking determinant common from both terms

$$\Delta = (1+xyz)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1$

$$\Delta = (1+xyz)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

Taking $y-x$, $z-x$ common from C_2, C_3 respectively

$$\Delta = (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & x+y & z+x \end{vmatrix}$$

Operating $C_3 \to C_3 - C_2$

$$\Delta = (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & x+y & z-y \end{vmatrix}$$

Expanding along R_1

$$\Delta = (1+xyz)(y-x)(z-x)[1\{(z-y)-0\}-0+0]$$

$$= (1+xyz)(y-x)(z-x)(z-y) = (x-y)(y-z)(z-x)(1+xyz) = \text{R.H.S.}$$

(iii) $\text{L.H.S} = \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix}$

Operating $R_1 \to R_1 - R_2 - R_3$

$$\text{L.H.S.} = \begin{vmatrix} 0 & -2b & -2a \\ a & b+c & a \\ b & b & c+a \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\text{L.H.S.} = \begin{vmatrix} 0 & -2b & -2a \\ a & b+c-a & 0 \\ b & 0 & c+a-b \end{vmatrix}$$

Expanding along R_1

$$\text{L.H.S.} = 0-(-2b)[a(c+a-b)-0]+(-2a)[0-b(b+c-a)]$$

$$= 2ab(c+a-b)+2ab(b+c-a) = 2ab(c+a-b+b+c-a)$$

$$= 2ab(2c) = 4abc = \text{R.H.S.}$$

UNIT 10 APPLICATIONS OF MATRICES AND DETERMINANTS

Structure

10.1 INTRODUCTION

In the previous unit, we have discussed matrices, types of matrices and determinants of square matrices of orders 1, 2, 3 and higher orders. We have also discussed minors, cofactors of square matrices and properties of determinants.

In this unit, we will learn some applications of matrices and determinants such as solutions of simultaneous linear equations by using matrix method and Cramer's rule.

Objectives

After completing unit, you should be able to:

- find the adjoint of a square matrix;
- find the inverse of a square matrix;
- solve the simultaneous linear equations with the help of matrix methods; and
- solve the simultaneous linear equations with the help of Cramer's rule.

10.2 ADJOINT OF A MATRIX

In next section, i.e. Sec.10.3, we will discuss inverse of a square matrix. But in order define inverse of a square matrix we use the concept of adjoint of the square matrix, so in this section we are going to discuss adjoint of the square matrix.

Adjoint of a Matrix: Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order $n \times n$, then adjoint of A is denoted by adjA and is defined as

$$\text{adjA} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ . & . & . & & . \\ . & . & . & \dots & . \\ . & . & . & & . \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix}'$$

where A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

It may be verified that A(adjA) = (adjA) A = $|A|$ I ,
where I is an identity matrix of order n.

Example 1: Find the adjA, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$.

Also verify that A(adjA) = (adjA)A= $|A|$I.

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 0 \\ -2 & 1 \end{vmatrix} = (-1)^2(5-0) = 5$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -4 & 0 \\ -1 & 1 \end{vmatrix} = (-1)^3(-4-0) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -4 & 5 \\ -1 & -2 \end{vmatrix} = (-1)^4(8+5) = 13$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = (-1)^3(2+6) = -8$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = (-1)^4(1+3) = 4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = (-1)^5(-2+2) = 0$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = (-1)^4(0-15) = -15$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix} = (-1)^5(0+12) = -12$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -4 & 5 \end{vmatrix} = (-1)^6(5+8) = 13$$

$$\therefore \text{adj}A = \begin{bmatrix} 5 & 4 & 13 \\ -8 & 4 & 0 \\ -15 & -12 & 13 \end{bmatrix}' = \begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}$$

To verify A(adjA) = (adjA)A= $|A|$I, we proceed as follows

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{vmatrix}$$

Expanding along C_3

$$|A| = 3(8+5) - 0 + 1(5+8) = 39 + 13 = 52 \quad \ldots (1)$$

$$A(\text{adj}A) = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 5+8+39 & -8+8+0 & -15-24+39 \\ -20+20+0 & 32+20+0 & 60-60+0 \\ -5-8+13 & 8-8+0 & 15+24+13 \end{bmatrix} = \begin{bmatrix} 52 & 0 & 0 \\ 0 & 52 & 0 \\ 0 & 0 & 52 \end{bmatrix}$$

$$= 52\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 52I = |A|I \qquad \ldots (2) \quad \text{[Using (1)]}$$

$$(\text{adj}A)A = \begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+32+15 & 10-40+30 & 15-0-15 \\ 4-16+12 & 8+20+24 & 12+0-12 \\ 13+0-13 & 26+0-26 & 39+0+13 \end{bmatrix} = \begin{bmatrix} 52 & 0 & 0 \\ 0 & 52 & 0 \\ 0 & 0 & 52 \end{bmatrix}$$

$$= 52\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 52I = |A|I \qquad \ldots (3)$$

From (2) and (3), we have A(adjA) = (adjA)A= $|A|$I

Now, you can try the following exercise.

E 1) If A = $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ then verify that A (adjA) = (adjA) A = $|A|I_2$.

10.3 INVERSE OF A MATRIX

You are familiar with the concept of the multiplicative inverse of a real number.

e.g. you know the multiplicative inverse of 5, $\frac{2}{3}$, etc.

Multiplicative inverse of 5 is $\frac{1}{5}$ and of $\frac{2}{3}$ is $\frac{3}{2}$, etc. $\left[\because 5 \times \frac{1}{5} = 1, \frac{2}{3} \times \frac{3}{2} = 1\right]$

i.e. when a number is multiplied with its multiplicative inverse, it gives 1. Similarly, in case of matrices, when a square matrix is multiplied with its multiplicative inverse, it gives identity matrix (I). That is, in case of matrices, identity, matrix plays the role as 1 plays in case of real numbers.

Inverse of a Matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that

AB = BA = I_n, then B is known as inverse of A and is denoted by A^{-1}.

As verified above that A(adjA) = (adjA)A = $|A|$I,

$\therefore A(\text{adj}A) = |A|I$

$$\Rightarrow A\left[\frac{1}{|A|}(\text{adj}A)\right] = I, \text{ provided } |A| \neq 0$$

Now, by definition of inverse of a matrix, $\frac{1}{|A|}(\text{adj}A)$ is inverse of A.

i.e. $A^{-1} = \frac{1}{|A|}(\text{adj}A)$

Properties of Inverse of a Matrix

(i) The inverse of a square matrix, if exists is unique.

(ii) A^{-1} exists if and only if $|A| \neq 0$.

(iii) $(A^{-1})' = (A')^{-1}$

(iv) $(AB)^{-1} = B^{-1}A^{-1}$

To find inverse of a square matrix use following steps

I Find $|A|$

II If $|A| = 0$ then A^{-1} does not exists.

III If $|A| \neq 0$ then find adjA and $A^{-1} = \frac{1}{|A|}(\text{adj}A)$.

Example 2: Find A^{-1}, where $A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & -1 & -2 \\ 5 & -3 & 6 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & -1 & -2 \\ 5 & -3 & 6 \end{bmatrix}$

$|A| = 4(-6-6) - 2(0+10) + 3(0+5) = -48 - 20 + 15 = -53 \neq 0$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of (i, j) th element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(-6-6) = -12,\quad A_{12} = (-1)^{1+2}(0+10) = -10$

$A_{13} = (-1)^{1+3}(0+5) = 5,\quad A_{21} = (-1)^{2+1}(12+9) = -21$

$A_{22} = (-1)^{2+2}(24-15) = 9,\quad A_{23} = (-1)^{2+3}(-12-10) = 22$

$A_{31} = (-1)^{3+1}(-4+3) = -1,\quad A_{32} = (-1)^{3+2}(-8-0) = 8$

$A_{33} = (-1)^{3+3}(-4-0) = -4$

$$\therefore \text{adj}A = \begin{bmatrix} -12 & -10 & 5 \\ -21 & 9 & 22 \\ -1 & 8 & -4 \end{bmatrix}' = \begin{bmatrix} -12 & -21 & -1 \\ -10 & 9 & 8 \\ 5 & 22 & -4 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{-53}\begin{bmatrix} -12 & -21 & -1 \\ -10 & 9 & 8 \\ 5 & 22 & -4 \end{bmatrix}$$

Here is an exercise for you.

E 2) For $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$, verify the result $(AB)^{-1} = B^{-1}A^{-1}$.

10.4 APPLICATION OF MATRICES

Matrix Method

Inverse of a matrix can be used to solve a system of linear simultaneous equations. Consider a system of n equations in n unknowns $x_1, x_2, x_3, ..., x_n$ given below.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$.$$
$$.$$
$$.$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix} \quad ... (1)$$

Or $AX = B$

$$\text{,where } A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

If $n = 3, x_1 = x, x_2 = y, x_3 = z$, then (1) reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or $AX = B$ ... (2)

Or $X = A^{-1}B$ ... (3)

Or $X = \frac{1}{|A|}(\text{adj}A)B$ $\left[\because A^{-1} = \frac{1}{|A|}(\text{adj}A)\right]$

Or $|A|X = (\text{adj}A)B$ ... (4)

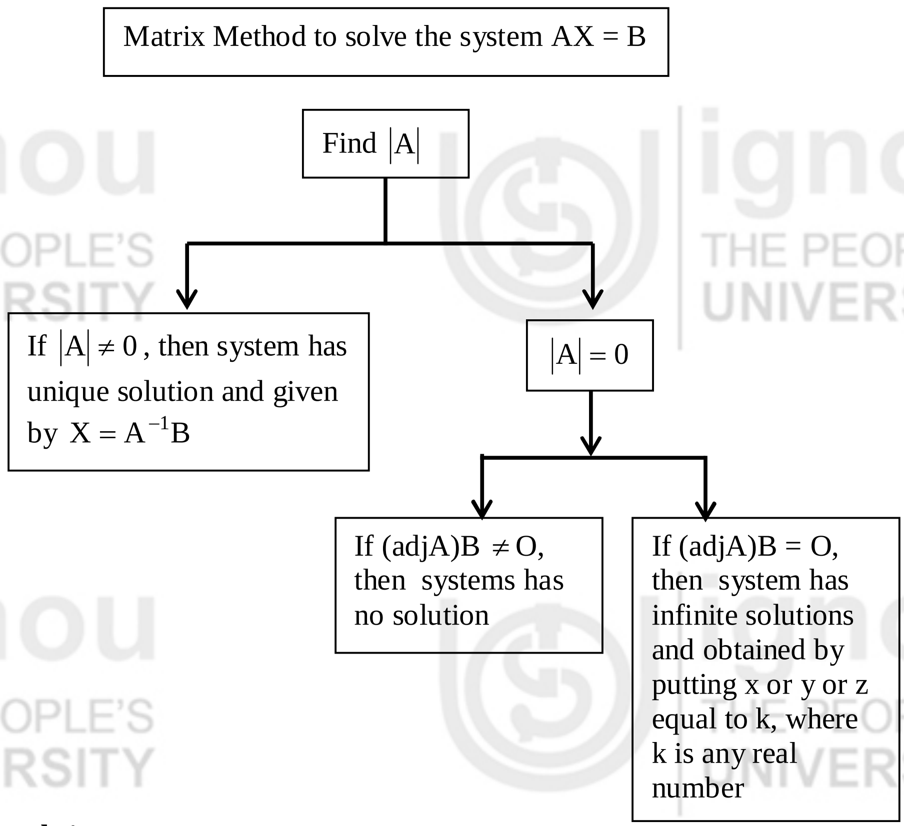
If $|A| = 0$, then A^{-1} does not exist. In this case, the given system of equations either has no solution or infinitely many solutions. In this case, we find (adjA)B. If (adjA)B = O, then system has infinitely many solutions

$$\left[\begin{array}{l}\because \text{in this case, (4)} \Rightarrow 0X = (\text{adj}A)B \\ \text{which holds for all matrices } X, \text{ i.e.} \\ \text{for all real values of } x, y, z.\end{array}\right]$$

and if $(\text{adj}A)B \neq O$, then system has no solution.

$$\left[\begin{array}{l}\because \text{in this case, L.H.S. of (4)} = O = \text{null matrix} \\ \text{but R.H.S. of (4)} = \text{non zero matrix.}\end{array}\right]$$

Above discussion can be summarised in the following diagram.



Remark 1:

(i) If B = O, i.e. if B is null matrix, then system given by (2) reduces to AX = O and is known as linear homogeneous system of equations.

(ii) Homogeneous system is always consistent.

Consistent system: A system of equations is said to be consistent if there is either unique solution or infinite number of solutions.
Inconsistent system: If system has no solution then it is known as inconsistent system.

Let us explain this method with the help of following example.

Example 3: The cost of 2 pens, 3 note-books, and 1 book is Rs 90. The cost of 1 pen, 4 note-books and 2 books is Rs 120. The cost of 2 pens, 4 note-books and 5 books is Rs 205. Find the cost of 1 pen, 1 note-book and 1 book by matrix method.

Solution: Let Rs x, y, z be the cost of 1 pen, 1 note-book and 1 book respectively, then according to given

$2x + 3y + z = 90$

$x + 4y + 2z = 120$

$2x + 4y + 5z = 205$

In matrix form this system can be written as

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 90 \\ 120 \\ 205 \end{bmatrix}$$

Or $AX = B$ … (1)

$$|A| = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 2 & 4 & 5 \end{bmatrix}$$

Expanding along R_1

$|A| = 2(20 - 8) - 3(5 - 4) + 1(4 - 8)$

$= 24 - 3 - 4 = 17 \neq 0$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(20-8) = 12$

$A_{12} = (-1)^{1+2}(5-4) = -1$

$A_{13} = (-1)^{1+3}(4-8) = -4$

$A_{21} = (-1)^{2+1}(15-4) = -11$

$A_{22} = (-1)^{2+2}(10-2) = 8$

$A_{23} = (-1)^{2+3}(8-6) = -2$

$A_{31} = (-1)^{3+1}(6-4) = 2$

$A_{32} = (-1)^{3+2}(4-1) = -3$

$A_{33} = (-1)^{3+3}(8-3) = 5$

$$\text{adj}A = \begin{bmatrix} 12 & -1 & -4 \\ -11 & 8 & -2 \\ 2 & -3 & 5 \end{bmatrix}' = \begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{17}\begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}$$

$$\text{Equation (1)} \Rightarrow X = A^{-1}B = \frac{1}{17}\begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}\begin{bmatrix} 90 \\ 120 \\ 205 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 1080 - 1320 + 410 \\ -90 + 960 - 615 \\ -360 - 240 + 1025 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 170 \\ 255 \\ 425 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$$

By definition of equality of two matrices, we have

$x = 10$, $y = 15$, $z = 25$

$\therefore$ costs of 1 pen, 1 note-book and one book are Rs 10, Rs 15, Rs 25, respectively.

Example 4: Solve the following system of equations:

(i) $4x + 2y = 6$
$6x + 3y = 8$

(ii) $3x + 6y - 4z = 3$, $3x - z = 0$,
$12x - 6y - z = -3$

Solution:

(i) Given system of equations is

$4x + 2y = 6$ … (1)

$6x + 3y = 8$ … (2)

This system of equations can be written is matrix form as

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Or AX = B … (3), where $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 12 - 12 = 0$$

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}(3) = 3, \quad A_{12} = (-1)^{1+2}(6) = -6$$

$$A_{21} = (-1)^{2+1}(2) = -2, \quad A_{22} = (-1)^{2+2}(4) = 4$$

$$\text{adj}A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}^{1} = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18-16 \\ -36+32 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \neq O = \text{null matrix}$$

Hence system has no solution.

i.e. given system of equations is inconsistent.

(ii) Given system of equations is

$3x + 6y - 4z = 3$ … (1)

$3x - z = 0$ … (2)

$12x - 6y - z = -3$ … (3)

This system of equations can be written in matrix forms as

$$\begin{bmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

Or AX = B … (4)

where $A = \begin{bmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{vmatrix}$$

Expanding along R_1

$$|A| = 3(0-6) - 6(-3+12) - 4(-18-0) = -18 - 54 + 72 = 0$$

$\therefore A^{-1}$ does not exist.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(0-6) = -6$

$A_{12} = (-1)^{1+2}(-3+12) = -9$

$A_{13} = (-1)^{1+3}(-18-0) = -18$

$A_{21} = (-1)^{2+1}(-6-24) = 30$

$A_{22} = (-1)^{2+2}(-3+48) = 45$

$A_{23} = (-1)^{2+3}(-18-72) = 90$

$A_{31} = (-1)^{3+1}(-6-0) = -6$

$A_{32} = (-1)^{3+2}(-3+12) = -9$

$A_{33} = (-1)^{3+3}(0-18) = -18$

$$\text{adj}A = \begin{bmatrix} -6 & -9 & -18 \\ 30 & 45 & 90 \\ -6 & -9 & -18 \end{bmatrix}' = \begin{bmatrix} -6 & 30 & -6 \\ -9 & 45 & -9 \\ -18 & 90 & -18 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} -6 & 30 & -6 \\ -9 & 45 & -9 \\ -18 & 90 & -18 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -18+0+18 \\ -27+0+27 \\ -54+0+54 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O = \text{null matrix}$$

Hence system has infinite many solutions and given by putting either x = k or y = k or z = k in any two equations given by (1), (2) and (3). Let us put z = k in (1) and (2), we get, where k is any real number

$3x + 6y = 3 + 4k$ …. (5)

$3x = k$ … (6)

$(6) \Rightarrow x = \dfrac{k}{3}$

Putting $x = k/3$ in (5), we get

$3(k/3) + 6y = 3 + 4k \Rightarrow 6y = 3 + 4k - k \Rightarrow 6y = 3 + 3k \Rightarrow y = (k+1)/2$

$\therefore x = \dfrac{k}{3}, y = \dfrac{k+1}{3}, z = k,$ where k is any real number.

Here is an exercise for you.

E 3) Solve the following system of equations by matrix method:

(i) $x + y = 3$, $4x - 3y = 5$ (ii) $2x - 3y = 3$, $6x - 9y = 5$ (iii) $2x + 3y = 5$, $4x + 6y = 10$

6.5 APPLICATION OF DETERMINANTS

There are many applications of determinants but here we shall discuss as to how the concept of determinant provides the solution of a given system of linear equations. The procedure which we will use is known as Cramer's rule.

Cramer's Rule: Cramer's rule can be used in any system of n linear equations in n variables. But here we shall discuss Cramer's rule only for system of 2 (or 3) equations in 2 (or 3) variables respectively. Suppose we are given following system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Cramer's rule suggests the following:

(i) Write the determinant for the coefficients. For the above given system of equations, it is $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta(\text{say})$.

(ii) Write determinant by interchanging first column of Δ with the right side constants of the given equations. Let this determinant denoted by Δ_1.

$$\therefore \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

Similarly, writing the determinants by interchanging the second and third columns of Δ with the right side constants and denoting them by Δ_2, Δ_3 respectively, we have

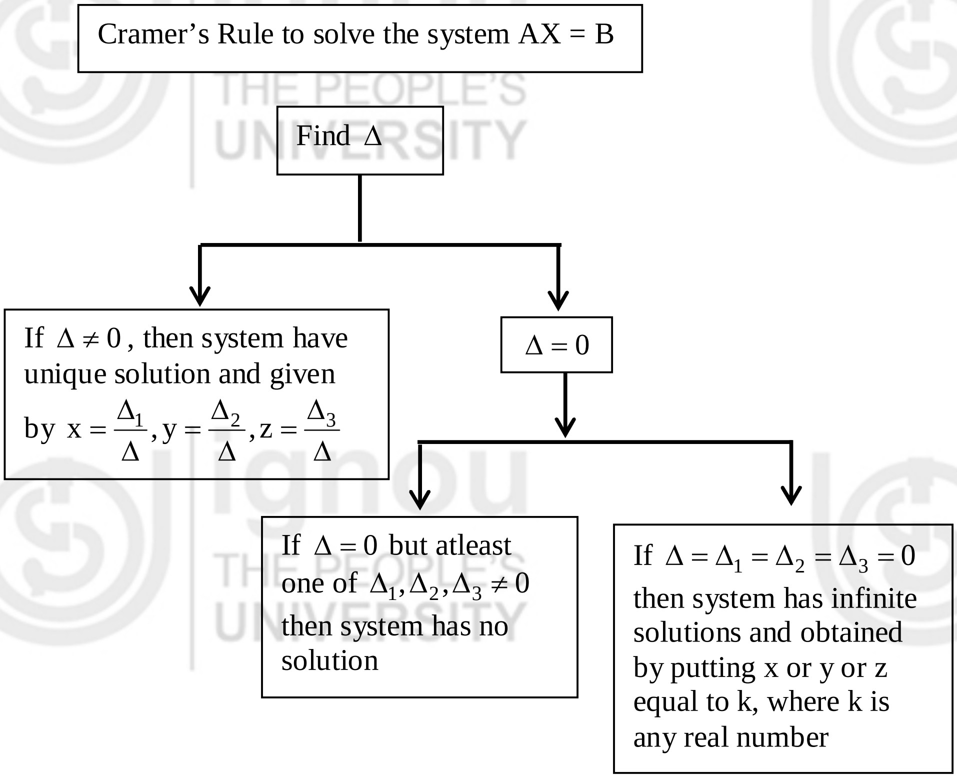
$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

(iii) If $\Delta \neq 0$, system has unique solution and is given by

$$x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta}, \; z = \frac{\Delta_3}{\Delta}.$$

If $\Delta = 0$ but at least one of the other determinants is non-zero then system has no solution. If $\Delta = 0$ and other determinants are also zero then system has infinitely many solutions given by putting x or y or z equal to k, where k is any real number.

Above discussion can be summarised in the following diagram.



Let us now consider some examples wherein Cramer's rule is applied.

Example 5: Solve the following system of equations:

2x + 3y + z = 4
x – y +2z = 9
3x + 2y –z =1
using Cramer's rule

Solution: Given system of equations is

2x + 3y + z = 4 … (1)
x – y + 2z = 9 … (2)
3x + 2y – z = 1 … (3)

$$\text{Here, } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = 2(1-4) - 3(-1-6) + 1(2+3) = -6 + 21 + 5 = 20 \neq 0$$

$\therefore$ system has unique solution.

$$\Delta_1 = \begin{vmatrix} 4 & 3 & 1 \\ 9 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

Expanding along R_1

$$\Delta_1 = 4(1-4) - 3(-9-2) + 1(18+1) = -12 + 33 + 19 = 40$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 & 1 \\ 1 & 9 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

Expanding along R_1

$$\Delta_2 = 2(-9-2) - 4(-1-6) + 1(1-27) = -22 + 28 - 26 = -20$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 9 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$$\Delta_3 = 2(-1-18) - 3(1-27) + 4(2+3) = -38 + 78 + 20 = 60$$

$\therefore$ by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{40}{20} = 2,\ y = \frac{\Delta_2}{\Delta} = \frac{-20}{20} = -1,\ z = \frac{\Delta_3}{\Delta} = \frac{60}{20} = 3$$

$\therefore x = 2,\ y = -1,\ z = 3$

Example 6: Solve the following system of equations:

$x + y + 2z = 4$

$x - y + 3z = 3$

$2x + 2y + 4z = 7$

using Cramer's rule.

Solution: Given system of equations is

$x + y + 2z = 4$ …(1)

$x - y + 3z = 3$ …(2)

$2x + 2y + 4z = 7$ … (3)

$$\text{Here, } \Delta = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 4 \end{vmatrix}$$

Taking 2 common from R_3

$$\Delta = 2\begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left[\begin{array}{l}\because \text{if a row is multiplied with some} \\ \text{number, the whole determinant} \\ \text{is multiplied with that number.}\end{array}\right]$$

$= 2(0) = 0$ $\qquad [\because R_1 \text{ and } R_3 \text{ are identical}]$

$\therefore$ system has either no solution or infinite many solutions.

$$\Delta_1 = \begin{vmatrix} 4 & 1 & 2 \\ 3 & -1 & 3 \\ 7 & 2 & 4 \end{vmatrix}$$

Expanding along R_1

$$\Delta_1 = 4(-4-6) - 1(12-21) + 2(6+7) = -40 + 9 + 26 = -5 \neq 0$$

$\therefore$ systems have no solution.

Example 7: Solve the following system of equations:

$x + 3y + 2z = 6$

$-x + 4y + 5z = 8$

$2x + 5y + 3z = 10$

Solution: Given system of equations is

$x + 3y + 2z = 6$ … (1)

$-x + 4y + 5z = 8$ … (2)

$2x + 5y + 3z = 10$ … (3)

Here, $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 4 & 5 \\ 2 & 5 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta = 1(12-25) - 3(-3-10) + 2(-5-8) = -13 + 39 - 26 = -39 + 39 = 0$

$\therefore$ system has either no solution or infinite many solutions.

Now, $\Delta_1 = \begin{vmatrix} 6 & 3 & 2 \\ 8 & 4 & 5 \\ 10 & 5 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta_1 = 6(12-25) - 3(24-50) + 2(40-40) = -78 + 78 + 0 = 0$

$\Delta_2 = \begin{vmatrix} 1 & 6 & 2 \\ -1 & 8 & 5 \\ 2 & 10 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta_2 = 1(24-50) - 6(-3-10) + 2(-10-16) = -26 + 78 - 52 = -78 + 78 = 0$

$\Delta_3 = \begin{vmatrix} 1 & 3 & 6 \\ -1 & 4 & 8 \\ 2 & 5 & 10 \end{vmatrix}$

Expanding along R_1

$\Delta_3 = 1(40-40) - 3(-10-16) + 6(-5-8) = 0 + 78 - 78 = 0$

As $\Delta = 0$ and also $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow$ system has infinite many solutions which are given by putting z = k (where k is any real number) in any of the two equations given by (1), (2) and (3).

Let us put z = k in (1) and (2), we get

$x + 3y = 6 - 2k$ … (4)

$-x + 4y = 8 - 5k$ … (5)

Again for this system of equations

$\Delta = \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} = 4 + 3 = 7$

$\Delta_1 = \begin{vmatrix} 6-2k & 3 \\ 8-5k & 4 \end{vmatrix} = 24 - 8k - 24 + 15k = 7k$

$\Delta_2 = \begin{vmatrix} 1 & 6-2k \\ -1 & 8-5k \end{vmatrix} = 8 - 5k + 6 - 2k = 14 - 7k$

$\therefore$ by Cramer's rule

$x = \frac{\Delta_1}{\Delta} = \frac{7k}{7} = k$, $y = \frac{\Delta_2}{\Delta} = \frac{14-7k}{7} = 2 - k$

$\therefore x = k, \; y = 2 - k, \; z = k,$ where k is any real number

Remark 2: Here, in the above example, we have taken z = k. You may also take y = k or x = k and then can solve by taking any two equations out of (1), (2), (3) in remaining two unknowns using Cramer's rule.

Now, you can try the following exercise.

E 4) Solve the following system of equations using Cramer's rule:

(i) $3x + 5y = -11$, $2x - 3y = 18$ (ii) $x - 2y = 5$, $-2x + 4y = 8$ (iii) $2x - y = 6$, $-6x + 3y = -18$

6.6 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Adjoint of a square matrix.
2) Inverse of a square matrix.
3) Application of matrices for solving a given system of linear equations, i.e. matrix method.
4) Application of determinants for solving a given system of linear equations, i.e. Cramer's rule.

6.7 SOLUTIONS/ANSWERS

E 1) $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(5) = 5, \quad A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(-3) = 3, \quad A_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \text{adj}A = \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 10 - (-12) = 10 + 12 = 22$$

$$A(\text{adj}A) = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 10+12 & 6-6 \\ 20-20 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

$$= 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 22I_2 = |A|I_2 \qquad \ldots (1)$$

$$(\text{adj}A)A = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10+12 & -15+15 \\ -8+8 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

$$= 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 22I_2 = |A|I_2 \qquad \ldots (2)$$

From (1) and (2), we get

$A(\text{adj}A) = (\text{adj}A)A = |A|I_2$

E 2) $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 8 + 3 = 11 \neq 0$$

$\therefore A^{-1}$ exists.

$$|B| = \begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} = -2 - 20 = -22 \neq 0$$

$\therefore B^{-1}$ exists.

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2+15 & 8-6 \\ -1+20 & -4-8 \end{bmatrix} = \begin{bmatrix} 17 & 2 \\ 19 & -12 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 17 & 2 \\ 19 & -12 \end{vmatrix} = -204 - 38 = -242 \neq 0$$

$\therefore (AB)^{-1}$ exists.

Let A_{ij}, B_{ij}, C_{ij} denotes the cofactors of $(i, j)^{th}$ element of matrices A, B, AB respectively.

$\therefore A_{11} = (-1)^{1+1}(4) = 4$, $\quad A_{12} = (-1)^{1+2}(-1) = 1$

$A_{21} = (-1)^{2+1}(3) = -3$, $\quad A_{22} = (-1)^{2+2}(2) = 2$

$B_{11} = (-1)^{1+1}(-2) = -2$, $\quad B_{12} = (-1)^{1+2}(5) = -5$

$B_{21} = (-1)^{2+1}(4) = -4$, $\quad B_{22} = (-1)^{2+2}(1) = 1$

$C_{11} = (-1)^{1+1}(-12) = -12$, $C_{12} = (-1)^{1+2}(19) = -19$

$C_{21} = (-1)^{2+1}(2) = -2$, $\quad C_{22} = (-1)^{2+2}(17) = 17$

$$\therefore \text{adj}A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}, \text{adj}B = \begin{bmatrix} -2 & -5 \\ -4 & 1 \end{bmatrix}' = \begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}$$

$$\text{adj}(AB) = \begin{bmatrix} -12 & -19 \\ -2 & 17 \end{bmatrix}' = \begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix},\ A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}(\text{adj}B) = \frac{1}{-22}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix} = \frac{-1}{22}\begin{bmatrix} -2 & -5 \\ -4 & 1 \end{bmatrix}'$$

$$(AB)^{-1} = \frac{1}{|AB|}(\text{adj}AB) = \frac{-1}{242}\begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix} \qquad \ldots (1)$$

$$B^{-1}A^{-1} = \frac{-1}{22}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{-1}{242}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} = \frac{-1}{242}\begin{bmatrix} -8-4 & 6-8 \\ -20+1 & 15+2 \end{bmatrix}$$

$$= \frac{-1}{242}\begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2), we have

$(AB)^{-1} = B^{-1}A^{-1}$

E3) (i) Given system of equations is

$x + y = 3$

$4x - 3y = 5$

This system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Or AX = B …. (1)

$$\text{where } A = \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} = -3 - 4 = -7 \neq 0$$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of $(i.j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}(-3) = -3, \quad A_{12} = (-1)^{1+2}(4) = -4$$

$$A_{21} = (-1)^{2+1}(1) = -1, \quad A_{22} = (-1)^{2+2}(1) = 1$$

$$\text{adj}A = \begin{bmatrix} -3 & -4 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}A) = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$\text{Equation (1)} \Rightarrow X = A^{-1}B = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -9-5 \\ -12+5 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By definition of equality of two matrices, we have

x = 2, y = 1

(ii) Given system of equations is

2x – 3y = 3

6x – 9y = 5

This system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Or AX = B … (1), where $A = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 \\ 6 & -9 \end{vmatrix} = -18 + 18 = 0$$

$\therefore A^{-1}$ does not exists.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of A.

$$\therefore A_{11} = (-1)^{1+1}(-9) = -9, \quad A_{12} = (-1)^{1+2}(6) = -6$$

$$A_{21} = (-1)^{2+1}(-3) = 3, \quad A_{22} = (-1)^{2+2}(2) = 2$$

$$\text{adj}A = \begin{bmatrix} -9 & -6 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} -9 & 3 \\ -6 & 2 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} -9 & 3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -27+15 \\ -18+10 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \end{bmatrix} \neq O = \text{null matrix}$$

Hence system has no solution.

(iii) Given system of equations is

$2x + 3y = 5$ … (1)

$4x + 6y = 10$ … (2)

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Or $AX = B$, where $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2\times 6 - 4\times 3 = 12 - 12 = 0$$

$\therefore A^{-1}$ does not exist.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(6) = 6$, $A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(3) = -3$, $A_{22} = (-1)^{2+2}(2) = 2$

$$\text{adj}A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 30-30 \\ -20+20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O = \text{ null matrix}$$

$\Rightarrow$ system has infinitely many solutions and given by putting y = k in (1), we get

$2x + 3k = 5 \Rightarrow x = \dfrac{5-3k}{2}$

$\therefore x = \dfrac{5-3k}{2}, y = k,$ where k is any real number.

E 4) (i) Given system of equations is

$3x + 5y = -11$

$2x - 3y = 18$

Here, $\Delta = \begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = -9 - 10 = -19 \neq 0$

$\Rightarrow$ system has unique solution.

$$\Delta_1 = \begin{vmatrix} -11 & 5 \\ 18 & -3 \end{vmatrix} = 33 - 90 = -57$$

$$\Delta_2 = \begin{vmatrix} 3 & -11 \\ 2 & 18 \end{vmatrix} = 54 + 22 = 76$$

$\therefore$ by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-57}{-19} = 3, \; y = \frac{\Delta_2}{\Delta} = \frac{76}{-19} = -4$$

$\therefore x = 3, \quad y = -4$

(ii) Given system of equations is

$x - 2y = 5$
$-2x + 4y = 8$

Here, $\Delta = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ system has either no solution or infinite many solutions.

$$\Delta_1 = \begin{vmatrix} 5 & -2 \\ 8 & 4 \end{vmatrix} = 20 + 16 = 36 \neq 0$$

$\therefore \Delta = 0$ but $\Delta_1 = 36 \neq 0$

$\Rightarrow$ system has no solution.

(iii) Given system of equations is

$2x - y = 6$ … (1)
$-6x + 3y = -18$ … (2)

Here, $\Delta = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 6 - 6 = 0$

$\Delta_1 = \begin{vmatrix} 6 & -1 \\ -18 & 3 \end{vmatrix} = 18 - 18 = 0, \; \Delta_2 = \begin{vmatrix} 2 & 6 \\ -6 & -18 \end{vmatrix} = -36 + 36 = 0$

$\therefore \Delta = 0, \quad \Delta_1 = \Delta_2 = 0$

$\Rightarrow$ system has infinite many solutions and given by putting either x or y equal to some arbitrary constant.

Let y = k, where k is any real number.

Equation (1) $\Rightarrow 2x = 6 + k \Rightarrow x = \dfrac{6+k}{2}$

$\therefore x = \dfrac{6+k}{2}, \quad y = k$, where k is any real number

UNIT 11 INTRODUCTION TO STATISTICS

Structure

11.1 INTRODUCTION

As you know every subject has its origin, development stages, scope, uses and limitations.

In this unit, we will discuss origin and development, definition, scope and uses, and limitations of statistics. Different measurement scales and different types of data also have been discussed in this unit.

Objectives

After completing this unit, you should be able to:

- know origin and development stages of statistics;
- know definition, scope, uses and limitations of statistics;
- get an idea of different types of measurement scales; and
- get an idea of different types of data.

11.2 ORIGIN AND DEVELOPMENT OF STATISTICS

As we know that every subject, race, machine, etc have its own origin and development stages. Similarly, Statistics also have its own origin and development stages. In fact, in a single sentence it can be said that human society has been using Statistics knowingly or unknowingly right from the beginning of its existence. This is because (i) most of the decisions taken by a human being are based on the past experience (i.e. based on statistical data he/she has experienced actually), and (ii) future events are also predicted by using/examining the past behavior of that particular event.

The word "Statistics' seems to have been derived from Latin word 'Status' or Italian word 'Statista' or German word 'Statistik'. But according to the observations of great John Graunt (1620-1674), the word 'Statistics' is of Italian origin and it is derived from the word 'Stato' and statista means a person who deals with affairs of the state. That is, initially kings or monarchs or governments used it to collect the information related to the population, agricultural land, wealth, etc. of the state. Their aim behind it was just to get an

State Craft: The art of managing state affairs.

idea about the men power of the state, force needed for the purpose of a war and necessary taxes to be impose to meet the financial need of the state.

So, it indicates that initially it was used by kings or monarchs or governments for administrative requirements of the state. That is why its origin lies in the state craft.

On the basis of evidences form papyrus manuscripts and ancient monuments in pharaonic temples, it is assumed that first census in the world was carried out in Egypt in 3050 BC. Yet, China's census data around 2000 BC is considered as the oldest surviving census data in the world.

Statistics in India

Though the use of Statistics, knowingly or unknowingly, has been there in India from ancient times, yet if we talk about its origin on the basis of evidences, it takes us in 3rd century BC when "Arthashastra" came into existence written by one of the greatest geniuses of political administration, Kautilya. In it, he had described the details related to conduct of population, agriculture and economic census. An efficient system of collecting official and administrative statistics was in use during the reign of Chandra Gupta Maurya (324-300 BC) under the guidance of Kautilya. Many things like taxation policy of the state, governance and administration, public finance, duties of a king, etc. had also been discussed in this celebrated Arthashastra. Another evidence that statistics was in use during Emperor Akbar's empire (1556-1605) is in the form of "Ain-I-Akbari" written by Abul Fazl, one of the nine jems of Akbar. Raja Todar Mal, Akbar's finance minister and another one of the nine jems of Akbar, used to keep very good records of land and revenue and he developed a very systematic revenue collections system in the kingdom of Akbar by using his expertise and the recorded data. Revenue collection system developed by Raja Todar Mal was so systematic that it became a model for future Mughals and later on for British.

British Government, after transfer of the power from East India Company to it, started a publication entitled 'Statistical Abstract of British India' as a regular annual feature in 1868 in which all the useful statistical information related to local administrations to all the British Provinces was provided. In between some census reports were coming on based on a particular area, but not at the national level. The first attempt to get detailed information on the whole population of India was made between 1867 and 1872. First decennial census was undertaken on 17th February 1881 by W.W. Plowden, first census commissioner of India. After that a census has been carried out over a period of 10 years in India. 2011 census was the 15th census in India.

Credit of establishing Statistics as a discipline in India goes to Prasanta Chandra Mahalanobis (P.C. Mahalanobis). He was a professor of physics in the Presidency College in Kolkata. During his study at Cambridge he got a chance to go through the work of Karl Pearson and R. A. Fisher. Continuing his interest in Statistics, he established a Statistical laboratory in the Presidency College Kolkata. On 17 December 1931, this statistical laboratory was given the name Indian Statistical Institute (ISI) mainly to promote the study, dissemination of knowledge of Statistics, research in it and to develop statistical techniques which play major role in addressing various problems of planning of national development and social welfare in the country. P.C. Mahalanobis was the founder director of ISI. Some well known personalities other than P.C. Mahalanobis associated to ISI whose research work made the institute unique internationally are Professor C.R. Rao, Professor R.C. Bose, Professor S.N. Roy, etc.

First post graduate course in Statistics was started by Kolkata University in 1941, while first under graduate course in Statistics was started by the Presidency College Kolkata. With the passage of time some more universities/institutes came up with courses in Statistics. Some of these are University of Mumbai, University of Pune, University of Madras, University of Mysore, University of Kerala and University of Lucknow. This list of institutions went on increasing with time and at present more than 1100 institutes are there in the country, which are offering under graduate or post graduate courses in statistics.

Statistics in World

Without going into more details, we will concentrate on only some major discoveries in the area of Statistics at international level. A lot of theoretical development in different areas of statistics took place in seventeenth and eighteenth centuries in many countries of the world. John Graunt (1620-1674 born in London), being a haberdasher by profession, has the credit of producing the first life table with probabilities of survival to each age. Due to this great achievement, he is known as father of vital statistics. This was the period when some other persons also did their contribution in the same area such as Edmund Haller (1656-1742) prepared a life table on the basis of the data collected by Casper Newman in 1691, relating to death records of Breslau. Sir William Petty (1623-1687) also prepared mortality tables and calculated expectation of life at different ages. G.F. Knapp (1842-1926) and W. lexis (1837-1914) also did valuable work on the statistics of mortality. Study of probability was also found to be very important in the area of Statistics, quantitative measure of which was given by Galileo (1564-1642), an Italian mathematician[For detail discussion on development of Probability Sec 1.1 of Unit 1 of MST-003 may be referred to]. Guass (1777-1855) gave the principal of least square and normal law of errors. J. Bernoulli (1654-1705) was the first person who states the law of large numbers in his great work Ars conjectandi published eight years after his death. Statistical methods in the field of biometry were first introduced of Sir Francis Galton (1822-1911). Later on Professor Karl Pearson (1857-1936) followed up the work of Galton and did significant contribution to Anthropology and correlation coefficient theories. Karl Pearson was also the founder of Statistical Research Laboratory in the university college, London in 1911. Credit of discovery of Chi-Square test also goes to Karl Pearson. Credit of discovery of ‘t test’ or ‘student t’ test goes to W.S. Gosset who wrote under the pseudonym of student’s ‘t’.

List of contributors in the area of Statistics did not end here but we conclude by throwing some light on the work done by Fisher. Credit of discovering of very powerful test known as Analysis of Variance (ANOVA) goes to Sir Ronald A. Fisher (1890-1962). Fisher also did a lot of work in the area of point estimation. Due to his remarkable contribution in the field of Statistics, he is known as Father of Statistics.

11.3 DEFINITION OF STATISTICS

In previous section of this unit, we have seen that statistics is a very old science and it has developed/grown up through ages. So, it is not surprising that through its long journey its definitions given by different authors time to time may also vary. In fact, today statistics is quite different that of earlier times. Let us first see some definitions by different authors and then it will be clear that which definition is more broad and may be consider as a good definition of statistics.

- "Statistics is the science of counting". – **A.L. Bowly.**
- "Statistics is the science of average." – **A.L. Bowly.**
- Statistics is "The science of the measurement of the social organism, regarded as a whole, in all its manifestations." – **A.L. Bowly.**
- "Statistics are the numerical statements of facts in any department of enquiry placed in relation to each other." – **A.L. Bowly**.
- "By statistics we mean quantitative data affected to a marked extent by multiplicity of causes." **– Yule and Kendall.**
- "Science of estimates and probabilities." – **Boddington.**
- "The method of judging collective natural or social phenomena from the results obtained by the analysis of an enumeration or collection of estimaties." – **W.I. King.**
- "Statistics is the science which deals with collection classification and tabulation of numerical facts as the basis for explanation description and comparison of phenomenon". – **Lovitt.**
- "The science which deals with the collection, tabulation, analysis and interpretation of numerical data." – **Croxton and Cowden.**

From the list of above definitions given by different authors, and various other definitions the comprehensive definition of Statistics may be given as:

"Statistics is a branch of science which deals with collection, classification, tabulation, analysis and interpretation of data."

11.4 SCOPE AND USES OF STATISTICS

In present times, statistics is not known as only collection of data, but it is regarded as a science having sound techniques of even handling huge data and providing valuable conclusions. Today statistical methods are universally applied. That is, statistics find its application in almost every sphere of human activity such as economics, commerce, management, information technology, education, planning, banking, insurance sector, medical science, biology, industrial, agriculture, market research, etc. That is, scope of statistics is very wide and in single sentence we can say that statistics is the queen of all sciences.

In the above paragraph, we have listed many fields where statistics has its application. Let us see its uses in some of these fields.

Statistics and Industry

In industrial line, statistics plays an important role such as in quality control and production engineering various control charts are used to maintain a certain quality level and different inspection plans are used in production engineering. Even to find average life of some products such as electric bulb, sampling technique is used. Those learners who will opt Industrial Statistics specialisation will get these terms in detail in courses MSTE-001 and MSTE-002.

Biology and Statistics

Professor Karl Pearson has stated that the whole doctrine of heredity rests on statistical basis. This is generally said that height of the child is associated with the height of the father. To test this type of hypothesis, statistics is the only science which provides the scientific methods. Vital statistics is totally devoted to the different aspects of human life like average life of men and women, birth and death rates, etc. Those learners who will opt Bio-Statistics specialisation will get these terms in detail in courses MSTE-003 and MSTE-004.

Statistics and Medicine

Statistics also plays an important role in the field of medicine. The hypothesis of the types:

(i) Drug A is better than drug B.

(ii) Smoking and cancer are associated.

(iii) Smoking and TB are associated.

All are tested using t-test or χ^2-test as the case may be. Statistics also find its application in clinical trials.

Statistics and Planning

Every institution/organisation plans for its future targets. Now, a days for a good planning, it has became necessary to analysis the statistical data according to the field of interest such as availability of raw material, consumption, investment, resources available, income, expenditure, quality needed, etc. In order to analysis these types of data, one has to totally depend on the statistical techniques. Thus statistics is essential for planning.

Statistics and Commerce

In present times, there is a very tough competition in almost every business. Also fashions, likings/tastes, requirements, trends, levels of qualities, technologies, etc. are changing very fast. So for the success of the business, it has become necessary for a business man to know the coming trend of market in advance or as soon as possible. This can also be achieved only with the help of market survey, which requires statistical techniques.

Statistics and Agriculture

Presently there are a number of varieties of seeds for a particular crop. Also different types of fertilizers are available in the market. For a good yield, it has become necessary to know that which one is better. This job is again done by a very popular and widely used test known as Analysis of variance (ANOVA) discovered by Professor R.A. Fisher. You will learn about ANOVA in more detail in block 2 of course MST-005. Complete Block 2 of MST-005 is totally devoted to ANOVA.

Statistics and Insurance Sector

Whole insurance sector totally depends on the statistical data and different concepts of probability theory. Life tables lies in the heart of human insurances. Curtate future life time and complete future life time, of a life are calculated using concept of random variables and their expected values. (you will learn in detail about random variables and their expected values in block 2 of course MST-003). Due to the large use of statistics in insurance sector, a new branch of statistics known as Actuarial statistics has been started in some institutes throughout the world.

Statistics and Research

Research is very important aspect in every discipline. In many disciplines such as psychology, tourism, education, M.B.A., etc. one has to collect the data on the characteristics of interest under study. Now a very important question arises, related to the measurement of scale to be used and appropriate test to be used. This requires the knowledge of different types of measurement scales and accordingly suitable statistical tool need. (Different types of measurement scales-nominal, ordinal, interval and ratio have been discussed in detail in Sec 11.6 of this unit). Also appropriate statistical tool to be needed in a given situation have been listed in Table 11.1.

Statistics and Economics

In order to know about the development of a country, it has become necessary to obtain the data related to its economical growth. Again, statistical tools are needed to collect relevant data (such as related to agricultural, industrial, literacy, etc) and for its analysis.

Statistics and Common Man

Statistics also plays an important role in the welfare of common man. Common man of any country faces lot of problems in his routine life such as food shortage, hygienic drinking water, unemployment, poverty, medical, shortage of public transport, etc. Time to time statistical figures on these issues enables the government to think and sort out these problems.

Statistics helps the common man in their day to day life in another way also, e.g. in purchasing any good he/she used his/her past experience (actually based on the data he/she faced/experienced) and take the decision to buy or not buy a particular object. Similarly, a farmer decide about the crop to be yield based on his past experience (actually based on the data he has faced) and labourer choose one of the works which gives him more wages based on his past experienced (actually based on the data he has faced).

List of fields/areas where statistics is used does not end here. We have just touch some of the areas where statistics has its application. We close this section by saying that there is hardly any field where statistics cannot be used. Infact, statistics can be used in any field of interest.

11.5 LIMITATIONS OF STATISTICS

In previous section of this unit, you have seen wide range of application of statistics. Being the queen of all sciences, statistics also have its own limitations, some of them are described as follows:

(1) Indirect Approach Towards Qualitative Characteristic

Science of statistics basically deals with numerical data. Therefore statistical tools are applicable only for quantitative measures. But many a times characteristic under study is qualitative in nature such as honesty, beauty, intelligence, boldness, drinking, smoking, etc. So any statistical tool cannot be directly applied on these types of characteristics. However, study of these types of characteristics can be made possible by first converting the characteristic under study into numerical figures based on some uniform criteria. For example, intelligence can be converted into numerical figures with the help of the marks obtained by the individuals in a common test.

(2) Dealingness with a Group

Science of statistics deals with aggregates of objects not with individuals. The individual's figures, when taken separately do not come under the category of statistical data. So, applicability of any statistical tool becomes meaningless. For example, salary of one employee of an institute does give any message related to the salaries of the employees of that particular institution.

(3) Lack of Exactness

Statistical results are not exactly true, but they are true on an average.

For example,

(i) If a statistical report says that 70% population of India lived in rural area. It does not imply that if you visit at public place like bus stand, railway station, etc. and asked the people about their living place. Results may surprise you and may highly differ with the above figure. But you may note that as sample size increases, the result will also come nearer and nearer to exact figure 70%.

(ii) Consider another example, suppose past data show that 90% operations of a doctor are successful. It does not imply that out of the next 100 operations, exactly 90 will be successful. It may happen that figure that will obtain in future may be 90%, 80%, 87%, 95%, etc. But there are sciences like mathematics where exactness is maintained. For example, if a book seller get 5% profit on selling a particular book. Then it is sure that if sell of that particular book is of Rs 200 he/she will get Rs 10 as profit and in case of sale of Rs 300 profit will be Rs 15 and so on.

(4) Requirement of Experts Hands for Effective Use

Requirement of experts' hands for effective and appropriate use is one of the main draw backs of the science of statistics. There are many statistical tools of similar type.

For example,

(i) To find average in a particular situation, which of the possible tools likes mean, median, mode, geometric mean, harmonic mean, etc. is appropriate needs the hands of experts.

(ii) Similarly to test a given statistical hypothesis which of the possible tools like Z-test, t-test, χ^2-test, F-test, ANOVA, median test, run test, sign test, etc. is appropriate again needs the hands of experts. This limitation of the statistics limits the range of its effective users.

11.6 MEASUREMENT SCALES

Two words "counting" and "measurement" are very frequently used by everybody. For example, if you want to know the number of pages in a note book, you can easily count them. Also, if you want to know the height of a man, you can easily measure it. But, in Statistics, act of counting and measurement is divided into 4 levels of measurement scales known as

(1) Nominal Scale

(2) Ordinal Scale

(3) Interval Scale

(4) Ratio Scale

Let us discuss these scales of measurement one by one:

(1) Nominal Scale

In Latin, 'Nomen' means name. The word nominal has come from this Latin word, i.e. 'Nomen'. Therefore, under nominal scale we divide the objects under study into two or more categories by giving them unique names. The classification of objects into atleast two or more categories is done in such a way that

(a) Each object takes place only in one category, i.e. each object falls in a unique category, i.e. it either belongs to a category or not. Mathematically, we may use the symbol ("=", "$\neq$") if an object falls in a category or not.

(b) Number of categories must be sufficient to include all objects, i.e. there should not be scope for missing even a single object which does not fall in any of the categories. That is, in statistical language categories must be mutually exclusive and exhaustive.

Generally nominal scale is used when we want to categories the data based on the characteristic such as gender, race, region, religion, etc.

To get more familiar with the idea of nominal scale, let us consider some examples:

(i) Classification into Different Categories Based on Gender

This can be done by dividing the population into two categories male 'M' and female 'F'

Category	**Name/Code**
Male	M
Female	F

Here we have named male as 'M' and female as 'F'. This is not the only way, we can also code male by '0' and female by '1' or we may use any other convenient symbols. So, we note that main thing is that we have to give a unique name to each category.

(ii) Classification into Different Categories Based on Caste

Different categories	Code alloted /Name given
General	Gen
Scheduled caste	SC
Scheduled tribes	ST
Backward class	BC
Others	'O'

Here also we can give a code to general, scheduled caste, scheduled tribes, backward class and other categories by '0', '1', '2', '3', '4' respectively.

(iii) Classification into Different Categories Based on Region

28 states and 7 union territories together classified India into 35 categories which can be coded by their usual names or may be coded by using some other symbols.

(iv) Classification into Different Categories Based on Religion

Population of India can be broadly categorised based on the following different religions:

Different categories	Codes allowed /Names given
Hindu	1
Muslim	2
Sikh	3
Isaiah	4
Others	5

(v) Classification into Different Categories Based on Number Allotted

In a sport event, the numbers allotted to the participants also come under nominal scale.

Note 1: We note that in nominal scale we have just coded the objects. Sign of less than or greater than does not make any sense in nominal scale. That is here we have coded Hindu, Muslim, by '1' and '2' respectively. But Hindu > Muslim or Muslim > Hindu does not make any sense.

Similarly, male > female or female > male does not make any sense. That is, we cannot talk about the order between two categories in case of nominal scale.

If in a measurement scale orders also make sense then, this scale comes under the heading ordinal scale discussed below.

(2) Ordinal Scale

We have seen that order does not make any sense in nominal scale. As the name ordinal itself suggests that other than the names or codes given to the different categories, it also provides the order among the categories. That is, we can place the objects in a series based on the orders or ranks given by using ordinal scale. But here we cannot find actual difference between the two categories.

Generally ordinal scale is used when we want to measure the attitude scores towards the level of liking, satisfaction, preference, etc. Different designation in an institute can also be measured by using ordinal scale.

To get more familiar with the concept of ordinal scale let us consider some examples:

(i) Opinion of persons about proposal of introducing co-education in a college comes under this scale. Suppose we assign '1'to strongly disagree, '2' to disagree '3' to indifferent (or neutral) '4' to agree and '5' to strongly agree. Here, the order also matters and mathematically we may use the symbols >, < in addition to those used for nominal scale, i.e. =, $\neq$ as here strongly agree opinion comes first in order as compared to agree and so on, i.e. 5 > 4 > 3 > 2 > 1 or 1 < 2< 3 < 4< 5. But actual difference between different categories in this Likert Scale is not possible. This is because, suppose "strongly agree" means he/she gives marks from 75% to 100% for the co-education to be introduced and suppose "agree" means the marks are given in the range say 50% to 75%. Now, the actual difference between "strongly agree" and "agree" is not feasible in this sense.

$\left[\begin{array}{l}\because \text{"strongly agree" may have the marks percentage as} \\ \text{80\% and "agree" marks 74\%. Similarly in other} \\ \text{case these values may be 90\% and 70\% respectively.}\end{array}\right]$

(ii) Suppose, a school boy is asked to list the name of three ice-cream flavours according to his preference. Suppose he lists them in the following order:

Vanilla
Straw berry
Tooty-frooty

This indicates that he likes vanilla more compared to straw berry and straw berry more as compared to tooty-frooty. But the actual difference between his liking between vanilla and straw berry cannot be measured.

(iii) In sixth pay commission, teachers of colleges and universities are designated as Assistant Professor, Associate Professor and Professor. The rank of Professor is higher than that of Associate Professor and designation of Associate Professor is higher than Assistant Professor. But you cannot find the actual difference between Professor and Associate Professor or Professor and Assistant Professor or Associate Professor and Assistant Professor. This is because, one teacher in a designation might have served certain number of years and have done a good quality of research work, etc. and other teacher in the same designation might have served for lesser number of years have done unsatisfactory research work, etc. So, the actual difference between one designation and other designation cannot be found. So one may be very near to his next higher designation and other may be very far from it depending on their quality of teaching/research.

(iv) Based on economic condition of a family, generally families of a society are divided into three categories:

Higher class family
Middle class family
Lower class family

Every body knows that economic condition of higher class family is better than middle class family and middle class family is in a better condition compare to lower class family. But the actual difference between the economic condition of a higher class family and middle class family or between middle class family and lower class family cannot be measured.

That is, we can only give order/rank to the three classes of the families but actual difference cannot be measured. In all the above examples, the actual difference is not possible because, all the ranks are on the ranges and not on fixed points.

Now, we are going to study the next higher level of measurement wherein the actual differences can be found. This scale is known as interval scale.

(3) Interval Scale

You have become familiar with the concept of interval and its length in Sec. 2.2 of Unit 2 of this course. If I = [4, 9] then length of this interval is $9-4=5,$ i.e. difference between 4 and 9 is 5, i.e. we can find the difference between any two points of the interval. For example, 7, $7.3 \in$ I and difference between 7 and 7.3 is 0.3. Thus we see that property of difference holds in case of intervals. Similarly, third level of measurement, i.e. interval scale possesses the property of difference which was not satisfied in case of nominal and ordinal scales.

Nominal scale gives only names to the different categories, ordinal scale moving one step further also provides the concept of order between the categories and interval scale moving one step ahead to ordinal scale also provides the characteristic of the difference between any two categories.

Interval scale is used when we want to measure years/historical time/calendar time, temperature (except in the Kelvin scale), sea level, marks in the tests where there is negative marking also, etc. Mathematically, this scale includes +, – in addition to >, < and $=, \neq$.

To get more familiar with the concept of interval scale, let us consider some examples:

(i) The measurement of time of an historical event comes under interval scale because there is no fixed origin of time (i.e. '0' year). As'0' year differ calendar to calendar or society/country to society/country e.g. Hindus, Muslim and Hebrew calendars have different origin of time, i.e. '0' year is not defined. In Indian history also, we may find BC (Before Christ).

(ii) Measurement of temperature in degree Celsius (^0C) assumes 0^0C when water starts freezing to ice and it becomes ice at $-40^\circ C$. So, in degree Celsius origin is arbitrary that's why measurement of temperature in degree Celsius comes in interval scale. Because in degree Celsius origin is arbitrary, so we cannot say that 30^0C is twice as hot as 15^0C. Because if it is so then can we say that $4^\circ C$ is -1 times $-4^\circ C$? No it is meaningless. Similarly, measurement of temperature in Fahrenheit comes in the interval scale.

(iii) Mean sea level (MSL) also have arbitrary origin because it is mean of two means, mean high tide and mean low tide(and mean high tide and mean low tide vary according to high and low pressure zones). Further it also varies place to place and time to time. So measurement of sea level also comes in the interval scale.

(4) Ratio Scale

Ratio scale is the highest level of measurement because nominal scale gives only names to the different categories, ordinal scale provides orders between categories other than names, interval scale provides the facility of difference between categories other than names and orders but ratio scale other than names, orders and characteristic of difference also provides natural zero (absolute zero). In ratio measurement scale values of characteristic cannot be negative.

Ratio scale is used when we want to measure temperature in Kelvin, weight, height, length, age, mass, time, plane angle, etc. Ratio scale includes $\times$, $\div$ in addition to $+, -, >, <, =, \neq$. But be careful never take '0' in the denominator while finding ratios. For example, $\frac{4}{0}$ is meaningless.

To get more familiar with the concept of ratio scale let us consider some examples, where ratio scale is used:

(i) Measurement of temperature in Kelvin scale comes under ratio scale because it has an absolute zero which is equivalent to -273.15^0C. This characteristic of origin allows us to make the statement like 50K ('50K' read as 50 degree Kelvin) is 5 time hot compare to 10K.

(ii) Measurement of money also comes under ratio scale because it satisfies all the requirement of interval scale and has a natural zero. For example, suppose there are 60 teachers in a particular school in Delhi. If we associate a unique number to each teacher related to the cash (in rupees) he/she has with him/her at the time of investigation. Then we have a fixed whole number corresponding to each teacher. Of course two or more teachers may have same cash (in rupees). These teachers will be allotted the same whole number and will fall in one category. Here we note that, the whole numbers allotted to the teachers can be ordered, have an actual difference and also have origin (i.e. absolute zero '0'). Here natural zero indicates the absence of money in the pocket of the teacher. If a teacher has Rs 500 and another teacher has Rs 100 then we can say that the teacher having Rs 500 has 5 times amount than a teacher having Rs 100. Thus it satisfies all the requirement of ratio scale.

(iii) Both height (in cm.) and age (in days) of students of M.Sc. Statistics of a particular university satisfy all the requirements of a ratio scale. Because height and age both cannot be negative (i.e have an absolute zero).

Permissible Statistical Tools

One of the advantages of measurement scale is that these help us to decide which statistical tool should be used in a given situation.

Table 11.1 shows the list of permissible statistical tools in case of nominal, ordinal, interval and ratio scales. Based on information provided by these scales, their levels from lowest to height are nominal, ordinal, interval and ratio (see Fig 11.1). That is why all the Statistical tools applicable on the lower scale will automatically be applicable on the next level scale. So, we will not repeat the permissible statistical tools used in lower level scale. It is understood that statistical tools which are permissible for nominal will be permissible in case of ordinal and so on.

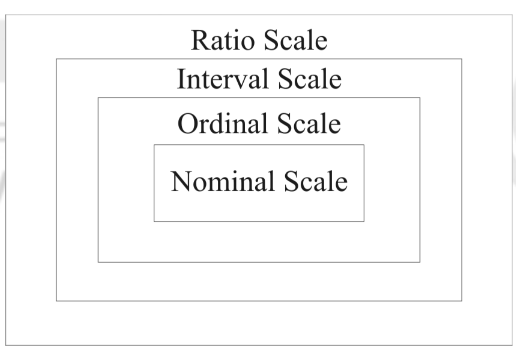


Fig. 11.1

Table 11.1

MEASUREMENT SCALE	PERMISSIBLE STATISTICAL TOOLS	LOGIC/REASON
NOMINAL SCALE	Mode, chi-square test and run test	Here counting is only permissible operation.
ORDINAL SCALE	Median all positional averages like quartile, Decile, percentile, Spearman's Rank correlation.	Here other than counting, order relation (less than or greater than) also exists.
INTERVAL SCALE	Mean , S.D., t-test, F-test, ANOVA, sample multiple and moment correlations, regression.	Here counting, order and difference operations hold.
RATIO SCALE	Geometric mean (G.M.), Harmonic mean (H.M.), Coefficient of variation.	Here counting, order, difference and natural zero exist.

Before closing this section let us consider some situations and appropriate measurement scale that can be used with the help of some examples followed by some exercises.

Example 1: If you want to collect the data based on the characteristic of literacy then which scale will be used? Explain with reasons.

Solution: Appropriate scale is nominal scale because population can be categorised in two categories literate (L) and illiterate (I). The symbols for literate and illiterate can be used according to our choice like 0, 1 or A, B or X, Y, etc.

Example 2: At a picnic spot in India, 1000 tourists visit over a period of 7 days. Each tourist is asked the name of the country of his/her birth. Then the data thus obtained come under which measurement scale.

Solution: Nominal scale, because the characteristic 'name of the country' divides the tourists into different categories each labels with the name of his/her country.

Example 3: Answer the following questions:

(i) Which scale is at lowest level?

(ii) Which scale is at highest level?

(iii) Which scale has absolute zero?

(iv) Which scale is used to find the mean sea level (MSL)?

Solution:

(i) Nominal scale is at lowest level, because it has only one permissible operation counting.

(ii) Ratio scale is at highest level, because it has all the four operations counting, order, distance and absolute zero.

(iii) Ratio scale is only scale out of the four measurement scales nominal, ordinal, interval and ratio scales which has absolute zero.

(iv) Because sea level has no absolute zero, so interval scale is used to find the mean sea level (MSL).

Example 4: Answer the following questions:

(i) In which scale median is not permissible?

(ii) In which scale(s) mean is not permissible?

(iii) In which scale(s) geometric mean and harmonic mean are not permissible?

(iv) In which scale geometric mean and harmonic mean are permissible?

Solution:

(i) In order to find median, we have to arrange the data in ascending or descending order of magnitude. But in nominal scale order operation is not present. So, in case of nominal scale data, median is not permissible.

(ii) In order to find mean, each observation of the data must be associated with a numerical quantity (which exactly measure the quantity of the characteristic). But, this requirement is not fulfilled by nominal and ordinal scales data. So, mean is not permissible in case of nominal and ordinal data.

(iii) In order to find geometric mean (G.M.) and harmonic mean (H.M.) absolute zero must be defined so that one can talk of quotient/ratio of two numbers. But as absolute zero is defined only in ratio scale, so G.M. and H.M. are not permissible in nominal, ordinal and interval scales data.

(iv) As discussed in (iii), G.M. and H.M. are defined only in ratio scale.

Now, here are some exercises for you.

E 1) Answer the following questions:

(i) Which scale is considered as the best scale of measurement so far as criteria of information provide is concerned?

(ii) Which scale is used in case of measurement of height, weight and age?

(iii) Allotment of license plates to different car comes under which scale of measurement?

(iv) Characteristic of equal distance between any two observations is maintained by which scale(s) of measurement?

E 2) Measurement of blood group comes under which scale of measurement?

11.7 TYPES OF DATA

Data play the role of raw material for any statistical investigation and defined in a single sentence as

"The values of different objects collected in a survey or recorded values of an experiment over a time period taken together constitute what we call data in Statistics"

Each value in the data is known as observation. Statistical data based on the characteristic, nature of the characteristic, level of measurement, time and ways of obtaining it may be classified as follows:

- Quantitative data
- Qualitative data

} based on the characteristic

- Discrete data
- Continuous data

} based on nature of the characteristic

- Nominal data
- Ordinal data
- Interval data
- Ratio data

} based on level of measurement

- Time Series data
- Cross Sectional data

} based on time component

- Primary data
- Secondary data

} based on the ways of obtaining the data

Let us discuss different types of data one by one:

Quantitative Data

As the name quantitative itself suggests that it is related to the quantity. In fact, data are said to be quantitative data if a numerical quantity (which exactly measure the characteristic under study) is associated with each observation.

Generally, interval or ratio scales are used as a measurement of scale in case of quantitative data. Data based on the following characteristics generally gives quantitative type of data. Such as weight, height, ages, length, area, volume, money, temperature, humidity, size, etc.

For example,

(i) Weights in kilogram (say) of students of a class.

(ii) Height in centimeter (say) of the candidates appearing in a direct recruitment of Indian army organised by a particular cantonment.

(iii) Age of the females at the time of marriage celebrated over a period of week in Delhi.

(iv) Length (in cm) of different tables in a showroom of furniture.

Here, is an exercise for you

E 3 Provide an example based on each of the following characteristic:
(i) Area (ii) Volume (iii) Money (iv) Temperature (v) Humidity (vi) Size

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Qualitative Data

As the name qualitative itself suggests that it is related to the quality of an object/thing. It is obvious that quality cannot be measured numerically in exact terms. Thus, if the characteristic/attribute under study is such that it is measured only on the bases of presence or absence then the data thus obtained is known as qualitative data.

Generally nominal and ordinal scales are used as a measurement of scale in case of qualitative data. Data based on the following characteristics generally gives qualitative data. Such as gender, marital status, qualification, colour, religion, satisfaction, types of trees, beauty, honesty, etc.

For example,

(i) If the characteristic under study is gender then objects can be divided into two categories, male and female.

(ii) If the characteristic under study is marital status then objects can be divided into four categories married, unmarried, divorcee, widower.

(iii) If the characteristic under study is qualification (say) 'matriculation' then objects can be divided into two categories as 'Matriculation passed' and 'not passed'.

(iv) If the characteristic under study is 'colour' then the objects can be divided into a number of categories Violet, Indigo, Blue, Green, Yellow, Orange and Red.

Here, is an exercise for you.

E 4 Give an example based on the following characteristic:

(i) Religion (ii) Satisfaction

Discrete Data

If the nature of the characteristic under study is such that values of observations may be at most countable between two certain limits then corresponding data are known as discrete data (concept of countability have already been discussed in Sec 2.6 of Unit 2 of this course).

For example,

(i) Number of books on the self of an elmira in a library form discrete data. Because number of books may be 0 or 1 or 2 or 3,…. But number of books cannot take any real values such as 0.8, 1.32, 1.53245, etc.

(ii) If there are 30 students in a class, then number of students presents in a lecture forms discrete data. Because number of present students may be 1 or 2 or 3 or 4 or…or 30. But number of present students cannot take any real values between 0 and 30 such as 1.8675, 22.56, 29.95, etc.

(iii) Number of children in a family in a locality forms discrete data. Because number of children in a family may be 0 or 1 or 2 or 3 or 4 or…. But number of children cannot take any real values such as 2.3, 3.75, etc.

(iv) Number of mistakes on a particular page of a book. Obviously number of mistakes may be 0 or 1 or 2 or 3…. But cannot be 6.74, 3.9832, etc.

Continuous Data

Data are said to be continuous if the measurement of the observations of a characteristic under study may be any real value between two certain limits.

For example,

(i) Data obtained by measuring the heights of the students of a class of say 30 students form continuous data, because if minimum and maximum heights are 152cm and 175 cm then heights of the students may take any possible values between 152 cm and 175 cm. For example, it may be 152.2375 cm, 160.31326… cm, etc.

(ii) Data obtained by measuring weights of the students of a class also form continuous data because weights of students may be 48.25796…kg, 50.275kg, 42.314314314…kg, etc.

Here is an exercise for you.

E 5) Identify whether the data are discrete or continuous in the following cases:

(i) Number of people present in a party.

(ii) Length of leafs of a plant.

(iii) Lifetime in hours of an electrical bulb.

(iv) Number of cars standing in a showroom over a period of 7 days.

(v) Number of patients visited to a hospital on a particular day.

Nominal Data

Data collected using nominal scale is called nominal data.

Similarly, data collected using ordinal scale, interval scale and ratio scale are called **ordinal data, interval data** and **ratio data** respectively. These scales of measurement have already been discussed in detail in Sec. 11.6.

Time Series Data

Collection of data is done to solve a purpose in hand. The purpose may have its connection with time, geographical location or both. If the purpose of data collection has its connection with time then it is known as time series data. That is, in time series data, time is one of the main variables and the data collected usually at regular interval of time related to the characteristic(s) under study show how characteristic(s) changes over the time.

For example, quarterly profit of a company for last eight quarters, yearly production of a crop in India for last six years, yearly expenditure of a family on different items for last five years, weekly rate of inflation for last ten weeks, etc. all form time series data.

Yearly expenditures (in Rs) for a family on different items from 2006 to 2010 are given in the following table.

Year	Food	Education	Rent	Miscellaneous	Total
2006	40000	10000	36000	20000	106000
2007	45000	12000	40000	28000	125000
2008	54000	15000	45000	32000	146000
2009	60000	20000	50000	40000	170000
2010	70000	30000	55000	45000	2000000

Data given in above table is nothing but time series data.

Note 2: If the purpose of the data collection has its connection with geographical location then it is known as **Spatial Data.**

For example,

(i) Price of petrol in Delhi, Haryana, Punjab, Chandigarh at a particular time.

(ii) Number of runs scored by a batsman in different matches in a one day series in different stadiums.

Note 3: If the purpose of the data collection has its connection with both time and geographical location then it is known as **Spacio Temporal Data.**

For example, data related to population of different states in India in 2001 and 2011 will be Spacio Temporal Data.

Note 4: In time series data, spatial data and spacio temporal data we see that concept of frequency have no significance and hence known as **non-frequency**

data. For instance, in the example discussed in case of time series data, expenditure of Rs 40000 on food in 2006 is itself important, here its frequency say 3 (repeated three times) does not make any sense.

Note 5: Now consider the case of marks of 40 students in a class out of 10 (say). Here we note that there may be more than one student who score same marks in the test. Suppose out of 40 students 5 score 10 out of 10, it means marks 10 have frequency 5. This type of data where frequency is meaningful is known as **frequency data.**

Cross Sectional Data

Sometimes we are interested to know that how a characteristic (such as income or expenditure, population, votes in an election, etc.) under study at one point in time is distributed over different subjects (such as families, countries, political parties, etc.). This type of data which is collected at one point in time is known as cross sectional data.

For example, annual income of different families of a locality, survey of consumer's expenditure conducted by a research scholar, opinion polls conducted by an agency, salaries of all employees of an institute, etc.

Note 6:

(i) If you are interested to know the changes in a characteristic say expenditure of a family over a period of time then you have to use time series data.

(ii) If you are interested to know the changes in a characteristic say expenditure of different families at single point in time you have to use cross sectional data.

Primary Data

Data which are collected by an investigator or agency or institution for a specific purpose and these people are first to use these data, are called primary data. That is, these data are originally collected by these people and they are first to use these data. Primary data have been discussed in Sec. 12.2 of next unit (i.e. UNIT 12) of this course in detail.

For example, suppose a research scholar wants to know the mean age of students of M.Sc. Chemistry of a particular university. If he collects data related to the age of each student of M.Sc. Chemistry of that particular university by contacting each student personally then data so obtained by the research scholar is an example of primary data for the same research scholar.

Secondary Data

Data obtained/gathered by an investigator or agency or institution from a source which already exists, are called secondary data. That is, these data were originally collected by an investigator or agency or institution and has been used by them at least once. And now, these data are going to be used at least second time. Secondary data have been discussed in Sec. 12.3 of next unit (i.e. UNIT 12) of this course in detail.

For example, consider the same example as discussed in case of primary data. If the research scholar collects the ages of the students from the record of that particular university, then the data thus obtained is an example of secondary data. Note that, in both the cases data remain the same, only way of collecting the data differs.

11.8 SUMMARY

In this unit, we covered following topics:

1) Origin and development of statistics.
2) Definitions of statistics by different authors.
3) Scope and uses of statistics.
4) Limitations of statistics.
5) Different measurement scales and types of data.
6) Frequency and non frequency data.

11.9 SOLUTIONS/ANSWERS

E 1 (i) Ratio scale is considered as the best measurement scale so far as the criteria of information provide are concerned because all the four operations counting, order, distance and absolute zero are defined on the observations.

(ii) Measurement of height, weight, age requires absolute zero and only ratio scale has absolute zero. So, appropriate scale of measurement for height, weight, age is ratio scale.

(iii) Allotment of license plates to the different cars comes under nominal scale measurement, because license plates categories the cars or license plates only provide unique names to the cars. Further, the car remains the same if some other registration number is provided to it.

(iv) Characteristic of equal distance between any two observations is maintained by two scales of measurements interval and ratio scales. For example, distance between temperatures of 18K and 13K is same as distance between 100K and 105K.

E 2) Blood group just divides the objects/things into four categories named as A, B, AB, O. So it comes under nominal scale.

E 3) Answers are not unique. There are a number of examples for each part, here one answer is provided for each part.

(i) Area of each state (in km^2) of India.

(ii) Volume of different buckets available at a particular shop.

(iii) Income of each family over a period of one year in a particular locality.

(iv) Highest or lowest temperature of a place over a period of 50 days.

(v) Level of humidity of a particular place at each hour of a particular day.

(vi) Size of different shoes present at a particular showroom on a specified day.

E 4) (i) If the characteristic under study is ‘religion’ then the objects can be divided into five categories Hindu, Muslim, Sikh, Isaiah, and others.

(ii) If the characteristic under study is ‘satisfaction’ then the objects can be divided into five categories (Likert scale) as shown on the next page:

Highly satisfied	Satisfied	Neutral	Dissatisfied	Highly dissatisfied
5	4	3	2	1
OR				
2	1	0	–1	–2

E 5 (i) Number of people present in a party may be 2 or 3 or 4 or 5 or 6 and so on, but cannot be 2.3, 4.375, 9.62875, etc.
$\therefore$ it is an example of discrete of data.

(ii) Lengths of leafs of a plant form continuous data because lengths of leafs may be any real number, e.g. 3.75 cm, 2.959595… cm, etc.

(iii) It is an example of continuous data because life time of an electrical bulb may be any possible fraction of time. For example, 8 hours 8.76 hours, 100.25796 hours, 0.25 hours, etc.

(iv) It is an example of discrete data because number of cars may be 1 or 2 or 3 or 4 or 5 or 6 or 7 and so on, but cannot be 2.87, 5.687, etc.

(v) It is an example of discrete data because number of patients visited to a hospital on a particular day may be 0 or 1 or 2 or 3 or 4 and so on, but cannot be 2.8, 10.357, 7.856, etc.

UNIT 12 COLLECTION AND SCRUTINY OF DATA

Structure

12.1 INTRODUCTION

Recall the definition of statistics, given in Sec. 11.3 of previous unit "Statistics is a branch of science which deals with collection, classification, tabulation, analysis and interpretation of data".

In the above definition out of the five successive steps (i.e. collection, classification, tabulation, analysis and interpretation) used in any statistical investigation, the first step is collection of data, and last step is interpretation of data which ultimately depends on the collection of data. So, if collection of data is not done carefully and sincerely then goal of the statistical investigation is not achieved or objective(s) of the statistical investigation is/are not fulfilled or final outcomes of the investigation will not be satisfactory.

Therefore, it becomes very important to focus on the collection of data in detail. In this unit two main types of collection of data namely primary and secondary will be discussed in detail. Also we shall discuss their different methods of collection with their merits and demerits. Scrutiny of data is also discussed in this unit. Finally we conclude this unit by throwing some light on preparing a questionnaire.

Objectives

After completing this unit, you should be able to:

- define primary data and get familiar with the different methods of collection of primary data;
- define secondary data and get familiar with the different sources of collection of secondary data;
- get an idea about the scrutiny of data; and
- know some important points related to the preparation of a questionnaire.

12.2 PRIMARY DATA

In previous section of this unit, we have seen why collection of data is important in any statistical investigation. In fact there are mainly two types of data namely Primary Data and Secondary Data. Both have their own importance. In a statistical investigation which of the two is to be used, totally depends upon many factors such as nature of problem, purpose of investigation, time period in which conclusion required and finally another important factor is availability of money/resources.

In this section we will discuss concept of primary data, different methods of collection of primary data with their merits and demerits.

Let us first formally define primary data and secondary data. Here we will just define secondary data with an example. Next section is devoted to discuss secondary data in detail.

Primary Data

Data which are collected by an investigator or agency or institution for a specific purpose and these people are first to use these data, are called primary data. That is, these data are originally collected by these people and they are first to use these data.

For example, suppose a research scholar wants to know the mean age of students of M.Sc. Chemistry of a particular university. If he collects the data related to the age of each student of M.Sc. Chemistry of that particular university by contacting each student personally. The data so obtained by the research scholar is an example of primary data for the same research scholar.

Secondary Data

The data obtained/gathered by an investigator or agency or institution from a source which already exists, are called secondary data. That is, these data were originally collected by an investigator or agency or institution and have been used by them at least once. And now, these data are going to be used at least second time.

For example, consider the same example as discussed in case of primary data. If the research scholar collects the ages of the students from the record of that particular university, then the data thus obtained is an example of secondary data. Note that, in both the cases data remain the same, only way of collecting these data differs. Our aim of just defining secondary data in this section is over, because we just want to click the idea how primary and secondary data differ from each other and are discussed into two different sections.

Now we move towards the aim of this section, which is to focus on primary data. We have already define primary data. There are a number of methods of collection of primary data depending upon many factors such as geographical area of the field, money available, time period, accuracy needed, literacy of the respondents/informants, etc.

Here we will discuss only following commonly used methods.

(1) Direct Personal Investigation Method

(2) Telephone Method

(3) Indirect Oral Interviews Method

(4) Local Correspondents Method

(5) Mailed Questionnaires Method

(6) Schedules Method

Let us discuss these methods one by one with some examples, merits and demerits.

(1) Direct Personal Investigation Method

In this method, the investigator personally contacts the informants and collects the related data through face to face interviews of the informants. Due to face to face meeting of investigator and informants data collected under this method has maximum degree of accuracy. But the degree of accuracy depends upon the sincerity, honesty, unbiasness and expertness of the investigator, because it is the investigator, who ultimately gives the final shape to the information provided by the informants. This method of collection of primary data is recommended/suitable when field of enquiry is small, secrecy related to data is need, high accuracy is required and time as well as money is sufficient.

Following are some **merits** of this method:

(i) It is simple to apply.

(ii) It is convenient for both investigator as well as informants.

(iii) Data have high degree of accuracy.

(iv) Data are homogenous in nature as there is only one investigator.

(v) Due to the presence of both investigator and informants, there is flexibility to clear any doubt or some other modification are possible according to physibility.

(vi) Confidential information can also be obtained.

Having so many plus points, this method is not free from the demerits. Following are some **demerits** of this method:

(i) It is time consuming and costly.

(ii) It is not suitable when area of investigation is large.

(iii) It suffers from the biasness of the investigator.

(iv) Data may be misleading if the investigator do not collect the data sincerely and honestly.

(v) If the investigator does not have expertise, data again may be misleading.

(2) Telephone Method

In the direct personal investigation method investigator has to personally contact with the informants, but now a day's telephone is very good communication tool. If the information of the interest is collected through telephone then data so obtained come under telephone method.

Some merits and demerits of this method are listed below:

Merits

(i) All merits of method 1 are also the merits of this method. Some additional merits are given below.

(ii) It is easy to apply compare to method 1.

(iii) It is time and cost saving method compare to method 1.

(iv) It is suitable when area of investigation is large compare to method 1.

Demerits

(i) Information related to those informants who do not have telephone and those who actually have telephone but their number are not in the list will not be included.

(ii) This method also suffers from the demerits (iii) to (v) listed in method 1.

(3) Indirect Oral Interviews Method

In this method, investigator does not meet to the actual informants directly, but the related information is collected/obtained from other persons who are supposed to have the required type of information. The informants who provide the information about actual informants are known as 'witnesses'. The success of this method mainly depends upon the skill and experience of the investigator. Because it depends upon the way, sequence and trick of questions prepared and asked by the investigator from the informants. It also depends on the behavior and how much he/she is capable to create confidence in the informants. This method of collection of primary data is generally adopted to obtain the information related to the cases such as

(i) murder

(ii) theft

(iii) in the cases where a person hesitates to provide correct information.

For example, if a researcher wants to collect the data on the smoking habit of students of a particular class of a college then it may happen that actual informants does not provide correct information. So, data can be collected with the help of class met or college met or from the neighbours.

Following are some **merits** of this method:

(i) As far as time and money is concerned, it is economical compare to direct personal investigation.

(ii) This method is easy to use, even if the area of investigation is large.

(iii) Informants being a third person, so it is free from the biasness of both actual informants and investigator.

Following are some **demerits** of this method:

(i) Here data totally depends upon the information provided by the third person. So, data suffer from the biasness of the third person.

(ii) If the investigator is not experienced and well behaved, then data will not be reliable.

(iii) It also depends on the honesty of the investigator.

(4) Local Correspondents Method

In this method, first of all some local correspondents or agents are appointed by the investigator or agency or institute to collect data. These local agents directly meet to the informants and collect data related to the required purpose in hand. Data collected by these local agents have high degree of accuracy, because they are familiar with the local language, traditions, general behaviour of the people, etc. of that particular area.

This method of collection of primary data is suitable if

- Area of investigation is large, e.g. news channels have their reporters throughout India.
- Time period in which information is needed is very short. For example, news related to the happening of any special incident can be easily seen on the news channels in very short time period after its happening.
- Information is needed on regular basis. For example, news are provided daily using this method.

Below some **merits** of this method are listed.

(i) It is very economical in terms of money, time and man power.

(ii) Time period in which information are provided by this method is very short.

(iii) Large area and heterogeneity of the informants can easily handled by this method.

(iv) This method can provide regular basis information.

Having so many merits, this method also has some **demerits**, listed below.

(i) Data suffer from the biasness of the local agents.

(ii) If local agents do not perform their duties with honesty and sincerity, then data will not be reliable.

(5) Mailed Questionnaires Method

In this method, first of all a list of questions related to the information required by the investigator is prepared. At the time of preparing the list of questions following points should keep in mind.

- Number of questions should not be too many.
- Each question should be related directly or indirectly to the objective(s) of the investigation.
- Each question should be clear.
- Generally objective type questions should be used, but if necessary multiple choice and open-ended questions can also be used.
- Language should be simple and effective.

Open ended questions are defined on page number 87 of this unit.

After preparing the 'final list' of questions known as questionnaire, it is sent through mail to the informants. With this questionnaire a covering letter in which it is requested to the informants that please sent it back after completing and a brief introduction about the objective of the investigation is also attached.

This method is suitable when the informants are literate and area of investigation is large.

Following are some **merits** of this method:

(i) This method is very useful if area of investigation is large.

(ii) This method is very economical as far as time, money and labour is concerned.

(iii) This method provides very good results when informants are literate.

(iv) Informants have enough time to think and give correct information. Thus data obtained by this method have high degree of accuracy.

(v) Biasness of the investigator is not involved.

Having being very economical and suitable for large area, following are some **demerits** of this method.

(i) This method fails if informants are illiterate.

(ii) Generally percentage of responses are very less because people take less interest in filling up the questionnaires.

(iii) If informants do not fill up the questionnaire sincerely or honestly then biasness of the informants may mislead the investigator.

(6) Schedules Method

In this method first of all a list of questions based on the information to be required is prepared like mailed questionnaires method, known as schedule. After doing this whole area of investigation is divided into sub areas. Then a number of people are appointed to collect the information directly from the informants. The appointed people are known as enumerators. The exact figure of the enumerator to be appointed depends upon the area of investigation and the amount of information to be required. These enumerators meet with the informants face to face and after giving a brief introduction about the objectives of the investigation they ask the answer of each question listed in the schedule. The answers provided by the informants are filled up in the schedule by enumerator themselves. This is one of the main differences of the two methods namely mailed questionnaires method and schedules method. In mailed questionnaires method answers are filled up by the informants themselves while in schedules method this job is done by enumerators. This characteristic of the schedules method make it superior to mailed questionnaire method in the case when informants are illiterate or semi-illiterate. Here enumerators have to meet directly with the informants, therefore it becomes important that enumerators should be well behaved, honest sincere and unbiased in nature.

This method is suitable when area of investigation is large and informants are illiterate or semi-illiterate.

Following are some **merits** of this method:

(i) This method is suitable when informants are illiterate or semi-illiterate.

(ii) It is application whatever large the area is.

(iii) After every ten years census data is collected by using this method in India.

(iv) Data are least affected by the bias of the enumerators and investigators.

(v) Since the information is directly obtained from the informants, so data collected by this method are more reliable and have a high degree of accuracy.

(vi) Because enumerator is present in front of the informants so if informants have any doubt, he/she can easily clear it from the enumerator.

There are so many merits of this method, even though it is not free from the demerits. Following are some **demerits** of this method:

(i) It is very time-consuming and large amount of money is needed.

(ii) Because a large number of enumerators have to be appointed, so it becomes too difficult to get all well trained and experienced persons.

(iii) Training is also needed to the enumerators.

(iv) Even after providing the training, some enumerators may not do their responsibilities sincerely, honestly and efficiently.

(v) Accuracy of the data will suffer if enumerator is bias or not devoted.

12.3 SECONDARY DATA

Discussion in the previous section shows that collection of primary data requires lot of time, money, manpower, etc. But sometimes some or all these resources are not sufficient to go for the collection of primary data. Also, in some situations it may not be feasible to collect primary data easily. To overcome these types of difficulties, there is another way of collecting data known as secondary data. The data obtained/gathered by an investigator or agency or institution from a source which already exists, are called secondary data. That is, these data were originally collected by an investigator or agency or institution and has been used by them at least once and now, these are going to be used at least second time. Already existed data in different sources may be in published or unpublished form. So sources of secondary data can broadly be classified under the following two heads.

(1) Published Sources, and

(2) Unpublished Sources.

Let us discuss these two main types of sources one by one:

(1) Published Sources

When an institution or organisation publishes its own collected data (primary data) in public domain either in printed form or in electronic form then these data are said to be secondary data in published form and the source where these data are available is known as published source of the secondary data of the corresponding institution or organisation. Some of the published sources of secondary data are given below:

(a) International Publications

There are many international organisations including the governments' organisations of different countries which collect data regarding the characteristics under their objectives and publish these data. Some of these organisations or publications are:

(i) Annual Abstract of Statistics (United Kingdom)

(ii) Annual Reports of International Labour Organisation (ILO)

(iii) World Health Organisation (WHO)

(iv) World Bank

(v) UNESCO Institute for Statistics, etc.

(b) Government Publications in India

There are number of government organisations or bodies at national level or state level which collect and publish data on different characteristics of interest. Data published by these bodies play a significant role in the sources of secondary data. Some of these organisations or bodies or publications are:

(i) Office of Registrar General of India

(ii) Central Statistical Organisation (CSO)

(iii) National Sample Survey Organisation (NSSO)

(iv) Reserve Bank of India Bulletin

(v) Directorate of Economics and Statistics (DES), etc.

(c) Published Reports of Commissions and Committees

From time to time, Central government and State governments constitute or appoint some commissions and committees to get a road map on some issues of interest. Some of these (with time of appointment in brackets) are listed below:

(i) Sarkaria Commission (1983)

(ii) Sixth Pay Commission (2006)

(iii) Shunglu Committee (2010)

(iv) Kalodkar Committee (2010), etc.

(d) Research Publications

Published research work of research scholars in different journals throughout the world is another major source of published data. Most of these research works are carried out in universities or other research institutes. Some of the journals related to the subject Statistics are:

(i) International Journal of Probability and Statistics

(ii) American Journal of Mathematics and Statistics

(iii) Sankhya

(iv) Journal of Statistics & Management Systems

(v) Journal of Statistical Research, etc.

(e) Reports of Trade and Industry Associations

In India, there are number of trade and industry associations whose published reports are also secondary sources of published data. Some of these associations are given below:

(i) All India Association of Industries

(ii) The Indian Cotton Mill Association

(iii) All India Resort Development Association

(iv) All India Biotech Association, etc.

(f) Published Printed Sources

Books, directories, newspapers, magazines, etc. are published printed sources of secondary data. Some of these are:

(i) Statistical Year Book, India

(ii) Business Line

(iii) The Economic Times

(iv) Business Standard

(v) Business Today, etc.

(g) Published Electronic Sources

Today, internet is a huge source of published data because most of the published material is available on internet. Information of interest which is available on internet can easily be obtained in very short time. Some of the electronic sources of secondary data are listed below:

(i) Online databases

(ii) e-books

(iii) e-journals

(iv) Websites of different institutes or organisations or agencies, etc.

(2) Unpublished Sources

Collected information in term of data or data observed through own experience by an individual or by an organisation which is in unpublished form is known as unpublished source of secondary data. Some of the sources of unpublished secondary data are given on the next page:

(i) Records and statistics maintained by different institutions or organisations whether they are government or non-government

(ii) Unpublished projects works, field works or some other research related works submitted by students in their corresponding institutes

(iii) Records of Central Bureau of Investigation

(iv) Personal diaries, etc.

After discussing sources of secondary data, natural questions which may arise in your mind are:

(1) What are the precautions one should use before using secondary data?

(2) What are the advantages of secondary data?

(3) What are the limitations of secondary data?

Let us address these questions one by one.

(1) Precautions to be taken before using Secondary Data

Every investigation in hand has some specific objectives and data are collected keeping these objectives in view. So, secondary data which we are planning to use in our investigation may be collected for some different objectives. Therefore some precautions which are necessary before using the secondary data in our investigation are given below.

(i) Reliability of Data

Reliability of data is judged by:

- Reliability and experience of the investigator or institution for collecting data.
- Reliability of the source(s) from where data were collected.
- Whether the proper methods of collecting data were used. Whether the sample size was proper if sample technique was used in data collection?
- Whether data collected in normal times? That is, whether data were free from periods such as floods, famines, earthquakes, etc?
- Whether data were free from the biasness of the collecting investigator or institution?

If above criteria are met, we assume, generally, that reliability of data is all right.

(ii) Suitability of Data

Suitability of data is judged by:

- Comparing the nature, scope and objectives of the investigation at hand to the original one.
- Comparing definitions of different terms and units used in original investigation to the one at hand. For example, if word "large" is used as a measurement unit then what figure it represents such as 100-200 or 500-1000 or 10000 and above, etc.
- Checking uniformity of different terms or units, i.e. we have to check whether the definition of different terms and units is maintained throughout or not.

(iii) Adequacy of Data

Adequacy of data is judged by:

- Comparing geographical area covered and to be covered in original and one at hand investigations respectively. If variation between the areas of two investigations is large then data will not be adequate. For example, data collected for the purpose of estimating per person income of a state say Delhi cannot be used to estimate the per person income in India.
- Similar argument applies on time factor also. For example, if price of a commodity are available for a particular month of a year then on the basis of prices of one particular month one cannot accurately estimate the price of that commodity for whole year.

(2) Advantages of Secondary Data over Primary Data

Use of secondary data in an investigation has following advantages:

- It saves lot of time and money.
- It is easy to use.
- In some investigations primary data cannot be collected.
- The only source in case of historical documents.
- Longitudinal study can be possible.
- Secondary data complements primary data in many ways such as better understanding of the problem(s) in terms of what are gaps and deficiencies in the earlier investigation(s) which need to improve.

(3) Limitations of Secondary Data

Some of the limitations of the secondary data are given below:

- It is very difficult to get secondary data which is appropriate for all objectives of our investigation at hand.
- It is very difficult to get secondary data which meet all the requirements like reliability, suitability, adequacy and accuracy.
- Secondary data are generally not available for all types of investigations.
- Data may be beyond our reach.
- Available data may be out dated.

Example 1: Giving reason(s) in each of the following cases, specify whether data are primary or secondary?

(i) A Television channel telecasts the published survey report of an agency XYZ (say) based on the data collected by the agency before the general election in India to know the opinion of the people about casting their votes.

(ii) Data source in part (i) used by agency XYZ.

(iii) An Industrial Statistics student estimate the average life of electric bulbs of a company in which he/she works by observing the lives of a random sample of 100 bulbs.

(iv) A Bio-Statistics student collected data from the records of 10 hospitals of a state in order to conduct his/her study, whether smoking and T.B. are associated?

Solution:

(i) Secondary because TV channel used the data which already existed in a published form.

(ii) Primary source because agency itself collected data from the field.

(iii) Data used by the student were primary as data observed by him/her were original in character.

(iv) Student collected data from already existed sources (i.e. records of the hospitals) so data used by the student were secondary.

Now you can try the following exercises.

E 1) Give reasons in each case whether the data are primary or secondary?

i) 2011 census data published by Office of Registrar General of India and to be used by itself.

ii) 2011 census data to be used by a demographer in his study.

E 2) What are the differences between primary and secondary data?

12.4 SCRUTINY OF PRIMARY DATA

At the time of collecting primary data, information provided by the informants is recorded either in the form of questionnaire or schedule or by other means. Before tabulating primary data, it is necessary to scrutinise the questionnaires or schedules or other means used while collecting primary data to maintain

(i) Completeness of data

(ii) Consistency of data

(iii) Accuracy of data, and

(iv) Homogeneity of data

(i) Completeness of Data

We may find some questionnaires or schedules which are incomplete. Incomplete questionnaires or schedules, if possible, should be completed by revisiting to the informants otherwise we should reject them. Incomplete questionnaires or schedules should not be entertained.

(ii) Consistency of Data

Sometimes, the information provided by the informants either does not make any sense or may contradicts some other information. For example,

- If an informant puts his age 40 years while age of his son as 32 years. This information does not make any sense.
- If total age and date of birth provided by the informant do not match then these two types of information contradict each other. Again either these information should be corrected by revisiting to the informants or we should reject them.

(iii) Accuracy of Data

To maintain 100% accuracy is not an easy task. We can only do correction(s) related to certain figures by checking their sum, subtraction, etc. But we have no control if some informants provide wrong information. For example, some informants may provide wrong figure about their annual income.

(iv) Homogeneity of Data

Keeping the homogeneity of data is also an important part of editing/scrutiny of data. Here we have to check that the units of measurements used by the informants are same or different. For example,

- Some informants may put their heights in centimeters while others may put in inches.
- Some informants may put their monthly income while others may put yearly income, etc.

To maintain homogeneity of data, we have to converts all the informations provided by all the informants in the same unit(s).

12.5 PREPARATION OF QUESTIONNAIRE

We have already become familiar with the words questionnaire and schedule during our discussion of different methods of collecting primary data. It has also been mentioned there that the difference between questionnaire and schedule is that questionnaires are filled by informants themselves while schedules are filled by the enumerators/investigators. Both are formalised set of the questions for obtaining information from the informants. Questionnaire plays the role of an instrument for the investigator to get the information from the informants. So, it becomes necessary that questionnaire or schedule should be well drafted.

This whole section is devoted to discuss various aspects of the questionnaire. There are no hard and fast rules for preparing a questionnaire or schedule. Actually designing of a good questionnaire is an art not a science. Development of this art requires skill, experience, dedication and hard work. Following are some guidelines to improve the quality of the questionnaire.

(1) Identification of Objectives and Target Population

First of all, we identify objectives of the investigation at hand. Once our objectives are clear we can accordingly set the type of information needed and target population. Target population is very important because questions appropriate for one group may not be appropriate for other, e.g. questions appropriate for university students may not be appropriate for farmers.

(2) Selection of Method

After setting the information needed and the target group, we have to decide one of the methods of primary data collection from amongst those studied in Sec 12.2. The method used for the data collection also affects the questionnaire. For example, in direct personal interviews method, investigator personally contacts the informants and hence lengthy, complex and varied questions can be asked whereas in the telephone method we have to ask simple and short questions because on telephone lengthy questions may confuse the informants.

(3) Content of Individual Questions

Keep only those questions in the questionnaire which directly or indirectly contribute to the information needed or serve some specific purpose. Unnecessary questions should not be included in the questionnaire.

(4) Wording of the Questions

Whatever method of data collection is used, one should maintain the following features in the questionnaire:

- Language should be simple, i.e., use of technical terms should be avoided unless informants are technically trained.
- Questions should be short, simple and easy to understand.

- Questions should have clear meanings, i.e. words having multiple meaning should be avoided. For example, words like poor, rich, etc should be avoided because meaning of these words vary from person to person.
- Questionnaire should be self explanatory. That is, if a particular question needs some clarification, it should be provided with the help of footnote.
- Questions which are sensitive and personal in nature should be avoided.

(5) Sequence of the Questions

Arrangement of the questions in the questionnaire should be logical and all questions which are related to a particular topic should be asked before beginning of a new topic.

(6) Types of Questions

Main types of the questions which are generally used in a questionnaire are given below:

- **Dichotomous Questions:** Questions having only two alternatives like yes or no, agree or disagree, etc. are known as dichotomous questions. For example, Have you visited USA?
- **Multiple Choice Questions:** Questions having more than two alternatives are known as multiple choice questions. For example, which soap do you use for bath?
  - (i) Lux
  - (ii) Dettol
  - (iii) Dove
  - (iv) Kanti
  - (v) Other
- **Specific Information Questions:** Questions which are used to know some specific information from the informants are known as specific information questions. For example, which is your favourite soft drink?
- **Open Ended Questions:** Those questions which provide freedom to the informants to give their opinion are known as open ended questions. For example, what are your hobbies?
- **Scale Questions:** Consider following question:

  Are you satisfied with the service provided by your mobile company? Options for this type of questions are based on Likert scale as given below (Refer Unit 11):

Highly satisfied	Satisfied	Neutral	Dissatisfied	Highly dissatisfied
5	4	3	2	1
OR				
2	1	0	–1	–2

  Such types of questions based on Likert Scale or some other scale are known as scale questions.

  Any of the above types of questions can be used in a questionnaire depending on the suitability of the information required.

(7) Size of the Questionnaire

Generally people do not like to answer lengthy questionnaire, so we should try to make the number of questions in the questionnaire the smallest possible. Usually, the number of questions in a questionnaire lies between 15-25. But less than 15 and more than 25 questions may be used depending on the type of information needed. For example, the number of questions in the questionnaire of 2011 census data collected by Office of Registrar General of India was 21.

(8) Cross Questions

Some cross questions can also be put in a questionnaire to check the accuracy of the information provided by the informants. For example, we can put two questions:

- What is your date of birth?
- What is your age as on date?

These two questions put a cross check on each other.

(9) Format, Layout and Quality of paper of a Questionnaire

To make the questionnaire eye catching and to give it professional appearance, one has to focus on:

- Proper line spacing
- Size and colour of headings and sub-headings
- Quality of paper used.

(10) Pretesting of Questionnaire

This is the last step to give a final shape to a questionnaire. Pretesting means "testing the questionnaire on a small group of informants to get some valuable suggestion(s) from the informants to improve the questionnaire". After incorporating valuable suggestions of the informants, questionnaire becomes ready to use.

This final questionnaire is sent to informants with a covering letter, stating briefly the objectives of the enquiry and with the request that please fill up the questionnaire and return it to the said address:

Let us prepare a questionnaire for the following example:

Example 2: A market research scholar wants to study the behaviour of Indian consumers regarding food and grocery items. Prepare a questionnaire for this purpose.

Solution: Questionnaire for this purpose is given on the next page:

Dear Informants,

I am conducting a market research study to explore the store choice behaviour of Indian consumers regarding food and grocery items. Information provided by you in the following questionnaire will be used only for academic purpose. So, you are requested to provide the feedback to the best of your knowledge and experience. Questionnaire is divided into three parts.

PART A

Put tick (√) mark in the appropriate box.

1. **Gender:** ☐ Male ☐ Female

2. **Marital Status:** ☐ Single ☐ Married

3. **In which age (in years) group you belong?**

   ☐ 0 – 20 ☐ 20 – 30 ☐ 30 – 40 ☐ 40 – 50 ☐ above 50

4. **Education Level**

   ☐ Below 10th ☐ 10th ☐ 10 + 2

   ☐ Graduation ☐ PG and above

5. **Monthly Income in Rupees**

   ☐ Below 10000 ☐ 10000 – 20000 ☐ 20000 – 40000

   ☐ 40000 – 60000 ☐ Above 60000

6. **Employment Status**

   ☐ Employed ☐ Unemployed ☐ Business person

   ☐ House Wife ☐ Other (Please specify)

7. **Store where I often buy my most of food and grocery items**

   ☐ Kirana Store ☐ Kirana Store with self service

   ☐ Chain Store – Individual Outlet (Please specify)

   ☐ Chain Store – Located in Mall (Please specify)

PART B

8. **Please put a tick (√) mark according to your importance level for each of the following factors while selecting a food and grocery retail store.**

Factor	Not at all Important	Not Important	Sometimes Important	Important	Extremely Important
Quality of product					
Variety of products					
Proper product display					
Price					
Spaciousness of the store					
Distance from residence					
Knowledge of sales person					
Behaviours of the workers of the store					
Location of store					
Music					
Air conditioning facility					
Cleanliness of the store					
Store lighting					

Parking facility					
Payment options (debit card, credit card, cash)					
Checkout time/billing time					
Home delivery facility					
Easy return and exchange					
Complaint handling					
Long opening hours					
Low crowd size					
Clear indication of prices					
Advance communication of discounts and offers					
Facility for membership holders					
Various offers and schemes					

9. **Give your response relating to the following statements by putting a tick (√) mark in the appropriate box. Some of the statements may or not be applicable to you.**

Statement	**Strongly disagree**	**Disagree**	**Neither Agree nor Disagree**	**Agree**	**Strongly Agree**
I wait for special offers to buy food and grocery products					
I like to compare prices of different stores before buying the items					
I like to try the new grocery outlets					
I frequently look for new products					
I make unplanned visits to stores					
I usually shop from a nearest grocery store					
I tend to buy from a particular grocery store					
I like to shop with my family					
I like to shop alone					

Now you can try the following exercise.

E 3) Suppose you are a member of the team involved in preparing units of this block. Being a team member you want to get the feedback of the learners related to this material so that at the time of revising the material valuable suggestions of the learners can be incorporated. Prepare a questionnaire to mail your learners.

12.6 SUMMARY

In this unit, we have discussed the following topics:

1) Primary data and different methods of collection of primary data.
2) Secondary data and its major sources.
3) Scrutiny of primary data.
4) Preparation of questionnaires.

12.7 SOLUTIONS/ANSWERS

E 1) (i) 2011 census data will be considered as primary data for Office of Registrar General of India because census data were collected by this office after every 10 years.

(ii) 2011 census data will be secondary data for the demographer because for him it will be an already existed source.

E 2) Difference between primary and secondary data is explained in the following table:

Factor of Difference	**Primary Data**	**Secondary Data**
Time	Long time is required for collection	Less time is required for collection
Money	Needs more money	Needs less money
Reliability	More reliable	Less reliable
Suitability	More suitability	Less suitable
Adequacy	More adequacy	Less adequate
Hand	First hand data	Second hand data
Precaution	Needs no extra precautions	Needs many precautions like reliability, suitability, adequacy and accuracy
Manpower	Needs more manpower	Needs less manpower

E 3) Answer of this exercise is not unique. One of the possible answers is given below:

Dear Learner,

During the study of the units of this block you may have found certain portions of the material where you faced some difficulty to grasp. We believe that there is always a scope for improvement. So we wish to know your difficulties and valuable suggestions to improve the material. Therefore, we request you to fill out and send us the following questionnaire. If you find that space provided is insufficient, kindly use a separate sheet.

QUESTIONNAIRE

1. **Enrolment No** ☐☐☐☐☐☐☐☐☐

2. **Mathematical Background**

   ☐ Up to 10th ☐ Up to 10 + 2

   ☐ Up to Graduation ☐ Higher than Graduation

3. **How many hours did you need for studying the units?**

Unit Number	9	10	11	12
Number of hours				

4. **Provide your feedback unit wise by putting a tick (√) mark in the appropriate box based on your experience get at the time of studing the units of this block.**

Item	Unit No	Excellent	Very Good	Good	Poor	Very Poor	Give specific example(s), in case of poor and very poor
Unit structure	9 10 11 12	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	
Way of presenting the content	9 10 11 12	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	
Conceptual clarity	9 10 11 12	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	
Language and style used	9 10 11 12	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	
Examples, exercises used	9 10 11 12	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	☐ ☐ ☐ ☐	

5 **Number of diagrams and flow charts used**

Unit 9 ☐ Sufficient ☐ Insufficient

Unit 10 ☐ Sufficient ☐ Insufficient

Unit 11 ☐ Sufficient ☐ Insufficient

Unit 12 ☐ Sufficient ☐ Insufficient

6 **Any other comments**

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