Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Block

5

INTEGRABILITY OF FUNCTIONS

Course Design Committee*

* The course design is based on the recommendations of the Programme Expert Committee and the UGC-CBCS template

Block Preparation Team

H.L. Vasudeva (Editor)
Professor (Retd.)
Deptt. of Mathematics
Panjab University
Chandigarh

Prof. Sujatha Varma
School of Sciences
IGNOU

Course Coordinator: Prof. Sujatha Varma

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BLOCK INTRODUCTION

In the previous block you studied the differentiation of real-valued functions. In this unit we study the notion of integrability of a real-valued function.

You already know from the Calculus course BMTC-131 that there are essentially two ways of describing the operation of integration. One way is to view it as the inverse operation of differentiation. The other way is to treat it as a limit of a sum. Here we first discuss the second method. We will follow a similar procedure studied in the calculus course. For a given real-valued function defined a closed bounded interval $[a,b]$, we form a sum known as Riemann sums using portions of $[a,b]$. If the limit of these sums exist when the partition becomes finer and finer, then that limit is called Riemann integral.

You recall that in the calculus course we introduced the notions of upper sums and lower sums of a bounded function f on an invterval $[a,b]$. These are sometimes referred to as Darboux sums. Here we will study that if a function is integrable, then both the limits give the same value as integral. As compared to the method of integration discussed in the Calculus course, the method discussed in this block has the advantage that it extends to the complex-valued functions.

The material covered in this block is divided into three units. In unit 1 we introduce the notion of Riemann integral of a function defined on a closed and bounded interval $[a,b]$. We begin with the definition of portion of $[a,b]$, norm of a partition related concepts. Using these concepts we define Riemann sums for a function defined $[a,b]$. The limit of Rimann sums of a function as the norm of the partion tends to 0 is, if it exists, is called the Riemann integral. We also discuss the Riemann integrability of certain standard functions. After that we shall discuss a criteria to decide integrability of a function known as Cauchy criteria for integrability.

In the next unit, Unit 15, we shall consider the algebra of integrable functions. In the previous unit you have seen that there are some integrable functions as well as some non-integrable functions also. Here you will study that the set of all Riemann integrable functions, denoted by $R[a,b]$ is closed under addition and multiplication by real numbers, and that integral of a sum equals the sum of the integrals. You will also see that the difference, product and quotient of two integrable functions is also integrable. Then we shall establish the integrability of several important class of functions: step function, continuous functions and monotone functions. The notion of integral as a limit of sums allows us to compute the integral in some cases. Nevertheless, it is not convenient for large class of function. We do require the process of differentiation to compute the integrals for certain class of functions. What is the relationship between the notions of differentiability and integrability? In the case of continuous functions, this relationship is expressed in the form of an important theorem called the Fundamental Theorem of Calculus, which is the main content of Unit 16. In this unit we shall have two additional theorems known as Mean-value Theorems of integrability which is analogous to the Mean-values Theorems of differentiability.

Notations and Symbols (used in Block 5)
(Also see the notations used in Volume I)

$\int f(x)\,dx$	Riemann integral function of f
$R(a,b)$	Class of Riemann integrable function
$\mathcal{P}([a,b])$	set of partitions on $[a,b]$
t_i	tags in a partition of (a,b)

UNIT 14

THE RIEMANN INTEGRAL

Structure **Page No.**

14.1 INTRODUCTION

You have learnt from Block 4 of the Calculus Course BMTC-131 that the indefinite integral $\int g(x)\,dx$ of a real-valued function g defined on some interval $[a,b]$ is the function G for which $G'(x) = g(x)$. Then the definite intergral $\int_a^b g(x)\,dx$ is defined as the real number $G(b) - G(a)$. This idea is highly useful in computing the area of many geometric objects in the plane, by identifying them as regions enclosed by graphs of appropriate functions. Later the work of Mathematician and Physicist **J. Fourier** (1768-1830) on analytical theory of heat conduction led to the beautiful theory on definite integrals. The first of which was due to the famous German mathematician **G.F.Bernhard Riemann** (1826 - 1866) and the second was due to the French mathematician **Henri L. Lebesgue** (1875-1941). In this unit we shall introduce you to Riemann Theory of Integration.

In Section 14.2 we familiarize you with some preliminaries required for defining Riemann intergrals. We define the notions of partitions, tag points and the corresponding tagged partitions of an interval $[a,b]$ and use these notions to define the Riemann sums of a function defined on $[a,b]$.

In Section 14.3 we define the Riemann integral of a function defined on $[a,b]$ as the limit of Riemann sums. Some examples of Riemann integrable functions are discussed.

In Section 14.4 a criterion for checking the Riemann integrability of a given function is considered. We state and prove a theorem known as Cauchy criterion for Riemann integrability. You will learn that this criteria generates a class of functions which are not Riemann integrable. In this unit we have also discussed the connection between the theory of integration established by Riemann sums and the theory developed using the upper sums and lower sums explained in Block 5 of the Calculus Course. The latter integral is historically known as Darboux integral due to the Mathematician (Darboux). We observe that the value of the integral remains the same, no matter how one evaluate the integral.

Objectives

After working through this unit you should be able to

- explain the concept of partition, norm of a partition and tagged partition;
- define and compute the Riemann sums for a function;
- check the Riemann integrability of functions using Riemann sums;
- state, prove and apply the Cauchy Criterion for Riemann Integrabiltiy;
- state the connection between Riemann integral and Darboux integral.

14.2 PARTITIONS AND TAGGED PARTITIONS

In this section, we discuss some preliminary concepts which are required to define an integral of a real valued function, defined on an interval $[a,b]$. Recall that in the Calculus course you have learnt that an integral represents an area of the region between a graph $y = f(x), x \in [a,b]$ and x-axis and the lines $x = a$ and $x = b$. (See Fig. 1). For instance consider the function $f(x) = 4x^3 - 12x^2 + 9x + 1, \quad x \in [0,1]$ for which the integral of $f(x)$ is given by the area of the shaded region, shown in Fig. 1.

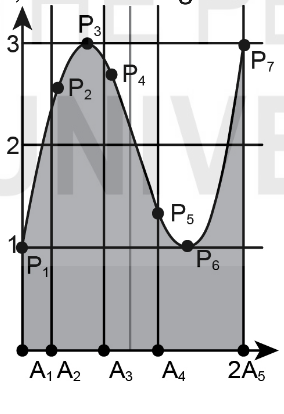

Fig. 1

Suppose we are in need of evaluating the definite integral $\int_0^3 g(x)dx$ where the function $g:[0,3] \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} 2, & for\ 0 \le x \le 1 \\ 3, & for\ 1 < x \le 3. \end{cases}$$

Note that g is not continuous at $x = 1$. From the Calculus course you have learnt that

$$\int_0^3 g(x)\,dx = \int_0^1 g(x)\,dx + \int_1^3 g(x)\,dx = \int_2^1 2\,dx + \int_1^3 3\,dx = 2\times(1-0) + 3\times(3-1) = 8$$

On the other hand, the graph of the function g, given in Fig. 2, suggests that this value of the integral is the sum of the areas of the rectangles of heights 2 and 3 on the subintervals $[0,1]$ and $(1,3]$ respectively (of the domain of g, namely, the interval $[0,3]$) which is given by $1\times 2+2\times 3=8$.

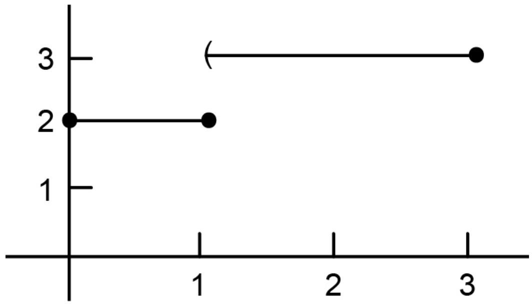

Fig. 2: Graph of g

This motivates the basic idea of the Riemann theory of the definite integral of a real valued real function. Note that, the splitting of the integral into two parts, as above, is a result of partitioning the base of the interval $[0,3]$ into two subintervals $[0,1]$ and $[1,3]$. For convenience, we denote this partition as the set $\mathrm{P}=[0,\ 1,\ 3]$. This leads to the following definitions.

Definition 1: A partition of a closed and bounded interval $I=[a,b]$ in $\mathbb{R}$ is defined as a finite, ordered set $\mathcal{P}=(x_0,x_1,...,x_{n-1},x_n)$ of points in I such that

$$a=x_0<x_1<...<x_{n-1}<x_n=b.$$

Obviously, the points of $\mathcal{P}$ divides I into **subintervals**

$$I_1=[x_0,x_1], I_2=[x_1,x_2],...,I_n=[x_{n-1},x_n]$$

of non-overlapping interiors (i.e. intersecting only at some end points.)
The typical closed subinterval $[x_{i-1},x_i]$ is called the ***i*-th subinterval** of the partition $\mathcal{P}$.

Definition 2: The **length of the *i*-th subinterval** $[x_{i-1},x_i]$ is the difference x_i-x_{i-1} and we denote this by

$$\Delta x_i=x_i-x_{i-1} \qquad (i=1,2,...,n)$$

In situations where no confusion arises, we may denote, for convenience, the partition $\mathcal{P}$ by $\{[x_{i-1},x_i]\}_{i=1}^{n}$ and the subinterval $[x_{i-1},x_i]$ by its length Δx_i itself.

The following figure explain this (See Fig. 3).

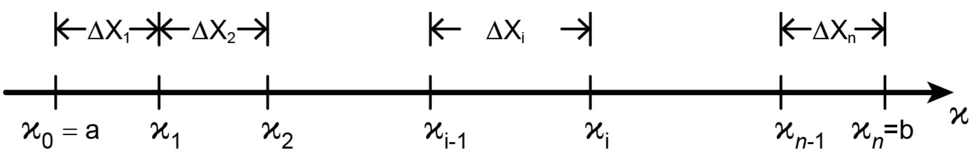

Fig 3: Partition, subintervals, lengths of different subintervals

When all the subintervals are of equal length, the partition is called a **standard partition.**

Definition 3: For the partition $\mathcal{P}$, of an interval $[a,b]$ given by $\mathcal{P}=\{a=x_0,x_1,...,x_n=b\}$, we define the **norm** (or **mesh**) of $\mathcal{P}$ is denoted and defined by

$$\|\mathcal{P}\|=\max\{x_1-x_0,x_2-x_1,...,x_n-x_{n-1}\}.$$

Obviously, the norm of a partition is the length of the largest subinterval into which the partition divides $[a,b]$.

We make some remarks here.

Remark 1: Clearly, the norm is a real valued function of the partition (i.e., $\|\ \|:\{$ set of all partitions on $[a,b]\}\to\mathbb{R}\}$.. However, many partitions can have the same norm. For instance, $\mathcal{P}_1=(1,2,3,7)$ and $\mathcal{P}_2=(1,5,7)$ are different partitions of $[1,7]$ having the same norm 4.

Now, we are going to build a theory to calculate the integral (i.e., the area which it represents) for some functions. Let us consider the function given in Fig. 4. The partitions are giving the base of the rectangles but we still need to determine the heights of the rectangles for calculating the area. Now, for the function given in Fig. 4 we need to determine the height. This has to be a value that the function takes (assumes) within the intervals of partition. Let $t_1,t_2,...,t_6$ be points within the intervals of partition (see Fig. 4).

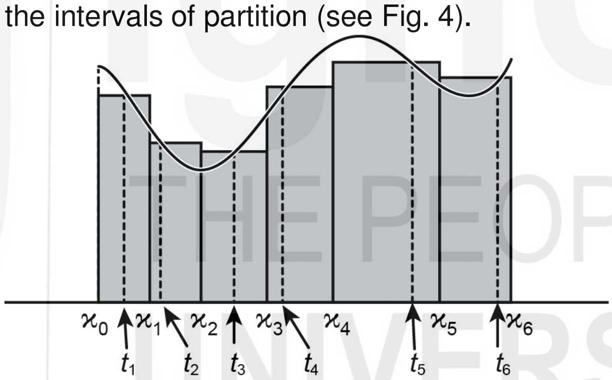

Fig. 4:

Now, if you carefully observe Fig. 4, you will see the shaded region does not represent the exact area which we need to calculate to find the integral in general. So, at this stage, the sum of areas of the rectangles with subintervals $[x_{i-1},x_i]$ as their bases and the values $f(t_i)$ as height is called a **Riemann sum** (instead of calling it integral).

Points t_i selected from each subinterval $I_i=[x_{i-1},x_i]$, is called a **tag** of the subinterval I_i. For a given partition $\mathcal{P}$, a set $\hat{\mathcal{P}}$ consisting of ordered pairs (I_i,t_i) of subintervals and their corresponding tags is called a **tagged partition** of I. Thus, a tagged partition of I is the set $\hat{\mathcal{P}}=\{(I_i,t_i):t_i\in I_i=[x_{i-1},x_i],$ $i=1,2,...,n\}$. (Carefully note the cap over $\mathcal{P}$ which indicates that a tag has been chosen for each subinterval.) The tags can be chosen arbitrarily. One can choose the tags to be the left endpoints, or the midpoints of the subintervals, etc (See Fig. 4 (a)). Note that an endpoint x_i of a subinterval $[x_{i-1},x_i]$ can be used as a tag for both the consecutive subintervals $[x_{i-1},x_i]$ and $[x_i,x_{i+1}]$. Since a point can be chosen from a subinterval in infinitely many ways, each tag can be chosen in infinitely many ways. Consequently, each partition can be tagged in infinitely many ways.

We make a remark here.

Remark 2: Since the length of a subinterval $[x_{i-1}, x_i]$ does not depend on the choice of any of its tag t_i, the norm of a tagged partition $\hat{\mathcal{P}}$ is defined as the norm of the corresponding ordinary partition $\mathcal{P}$.

We formally make a definition now.

Definition 4: The **Riemann sum** of a function $f:[a,b] \to \mathbb{R}$ corresponding to a tagged partition $\hat{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ of $[a,b]$ is the number $S(f;\hat{p})$ defined by

$$S(f;\hat{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \qquad \ldots (1)$$

The idea behind defining the Riemann sums will be clear to you in the next section. At present let us have a close look at the computation of Riemann sums given by Eqn. (1).

We now consider some examples to give you some practice for dealing with partitions and finding norms,.

Example 1: Find the norm of the partition given below of $[0,1]$.

$$\mathcal{P} = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}.$$

Solution: Here $\Delta x_1 = \frac{1}{2} - 0 = \frac{1}{2}, \Delta x_2 = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}, \Delta x_3 = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$ and $\Delta x_4 = 1 - \frac{3}{4} = \frac{1}{4}$. The norm of the partition is given by

$$\|\mathcal{P}\| = \max\left\{\frac{1}{2}, \frac{1}{10}, \frac{3}{20}, \frac{1}{4}\right\} = \frac{1}{2}.$$

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Example 2: Let $f(x) = x, x \in [0,1]$. Let $\mathcal{P}_n$ be the tagged partition formed by the subinterval

$I_1\left[0, \frac{1}{n}\right], I_2\left[\frac{1}{n}, \frac{2}{n}\right], I_3 = \left[\frac{2}{n}, \frac{3}{n}\right], \ldots\ldots, I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right], \ldots, I_n = \left[\frac{n-1}{n}, 1\right]$ where the tags are given by $t_i = \frac{i}{n}, i = 1, \ldots, n.$ Calculate the Riemann sum $S(f, \hat{\mathcal{P}}_n)$.

Solution: Here $f(x) = x$ and $t_i = \frac{i}{n}, i = 1, \ldots, n, \Delta x_i = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}$ for all $i = 1, 2, \ldots, n.$ Therefore, by the definition of Riemann sums, we have

$$S(f;\hat{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$= \frac{1}{n^2} \frac{n(n+1)}{2}$$
$$= \frac{n+1}{2n}.$$

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You can solve some exercises on your own.

E1) Two partitions of the interval $[-1,1]$ are given by

i) $\mathcal{P}_1 = \left\{\left[-1,-\frac{1}{2}\right],\left[-\frac{1}{2},0\right],\left[0,\frac{1}{2}\right],\left[\frac{1}{2},1\right]\right\}$

ii) $\mathcal{P}_2 = \left\{\left[-1,-\frac{1}{4}\right],\left[-\frac{1}{4},\frac{1}{3}\right],\left[\frac{1}{3},1\right]\right\}$

Find the norms of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$.

E2) Let $f(x) = 3x^2$, calculate the Riemann sums where

i) $\hat{\mathcal{P}}_1 = \{0,1,2,4\}$ with tags at the left end points of the sub interval.

ii) $\hat{\mathcal{P}}_2 = \{0,2,3,4\}$ with the tags at the right end points of the sub interval.

You may note that a Riemann sum can only be an approximation to the area under the graph. The more narrower we make the rectangle, more close the Riemann sums should be to the actual area. So, we want a measure of how narrow the rectangles in a partition can be. In the next section, we shall explain this.

14.3 RIEMANN INTEGRATION

In this section we shall introduce the concept of Riemann integral of a real valued function f defined on an interval $[a,b]$, and discuss functions which are Riemann integrable. In what follows all functions considered will be bounded functions.

Definition 5: A function $f:[a,b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a,b]$ if there exists a number $L \in \mathbb{R}$ such that corresponding to each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ satisfying $|S(f;\mathcal{P}) - L| < \varepsilon$ for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$.

If such a number L exists for f, it is defined as the **Riemann integral** of f over $[a,b]$. In this case, we write $L = \int_a^b f$ or $L = \int_a^b f(x)\,dx$. The function f is called the **integrand** and a and b are called the **bounds of the integral** with 'a' being the lower bound and 'b' being the upper bound.

It should be understood that any letter other than x can be used in the expression $\int_a^b f(x)\,dx$, so long as it does not cause any ambiguity.

Theorem 1: If $f \in R[a,b]$, then the value of the integral is uniquely determined.

Proof: Assume that both L' and L'' satisfy the required condition in Definition 1 of Riemann integrability of f over $[a,b]$. Let $\varepsilon > 0$. Since L' satisfies this condition, there exists $\delta'_{\varepsilon/2} > 0$ such that

$$\left|S(f;\hat{\mathcal{P}}_1 - L)\right| < \frac{\varepsilon}{2} \qquad \dots (2)$$

for all tagged partitions $\hat{\mathcal{P}}_1$ with $\left\|\hat{\mathcal{P}}_1\right\| < \delta'_{\varepsilon/2}$. Since L'' also satisfies the required condition, there exists $\delta''_{\varepsilon/2}$ such that

$$\left|S(f;\hat{\mathcal{P}}_2) - L\right| < \frac{\varepsilon}{2} \qquad \dots (3)$$

for all tagged partitions $\hat{\mathcal{P}}_2$ with $\left\|\hat{\mathcal{P}}_2\right\| < \delta'_{\varepsilon/2}$.

Take $\delta_\varepsilon = \min\{\delta'_{\varepsilon/2}, \delta''_{\varepsilon/2}\}$. Clearly $\delta_\varepsilon > 0$ since both $\delta'_{\varepsilon/2}$ and $\delta''_{\varepsilon/2}$ are positive.

Let $\hat{\mathcal{P}}$ be any tagged partition of $[a,b]$ with $\left\|\hat{\mathcal{P}}\right\| < \delta_\varepsilon$. Then $\left\|\hat{\mathcal{P}}\right\| < \delta'_{\varepsilon/2}$ as well as $\left\|\hat{\mathcal{P}}\right\| < \delta''_{\varepsilon/2}$, since $\delta_\varepsilon \le \delta'_{\varepsilon/2}$ and $\delta_\varepsilon \le \delta''_{\varepsilon/2}$. Now from (2) and (3), we get $\left|S(f;\hat{\mathcal{P}}) - L'\right| < \varepsilon/2$ and $\left|S(f;\hat{\mathcal{P}}) - L''\right| < \varepsilon/2$. An application of triangle inequality as can be seen in the second step below gives

$$\begin{aligned}\left|L' - L''\right| &= \left|L' - S(f;\hat{\mathcal{P}}) + S(f;\hat{\mathcal{P}}) - L''\right| \\ &\le \left|L' - S(f;\hat{\mathcal{P}})\right| + \left|S(f;\hat{\mathcal{P}}) - L''\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $L' = L''$. ■

We look at some simple examples to understand the definition of Riemann integral.

Example 3: Show that every constant function on $[a,b]$ is Riemann integrable and, find its integral.

Solution: Consider a constant function f defined by $f(x) = \alpha$ for all $x \in [a,b]$, where $\alpha \in \mathbb{R}$ is fixed. Then for any tagged partition $\hat{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ of $[a,b]$, we have $f(t_i) = \alpha$ for all $i = 1,\dots,n$ and hence

$$\begin{aligned}S(f;\hat{\mathcal{P}}) &= \sum_{i=1}^{n} \alpha(x_i - x_{i-1}) \\ &= \alpha\left[(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + (x_{n-2} - x_{n-3}) + \dots + (x_1 - x_0)\right] \\ &= \alpha(x_n - x_0)\end{aligned}$$

$$= \alpha(b-a).$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon = 1$ (or any positive real number) so that if $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left|S(f;\hat{\mathcal{P}}) - \alpha(b-a)\right| = 0 < \varepsilon$$

Since this is true for every $\varepsilon > 0$, we conclude, by taking $L = \alpha(b-a)$, that $f \in R[a,b]$. Further,

$$\int_a^b f(x)d(x) = \alpha(b-a).$$

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Example 4: Consider our opening example $g:[0,3] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2, & for\ 0 \le x \le 1 \\ 3, & for\ 1 < x \le 3 \end{cases}.$$

Show that g is Riemann integrable, and find its integral.

Solution: Our earlier experience hints that the Reimann integral of g might be 8.

Consider any tagged partition $\hat{\mathcal{P}}$ of $[0,3]$ with $\|\hat{\mathcal{P}}\| < 1$. Let $\hat{\mathcal{P}}_1$ be the subset of $\hat{\mathcal{P}}$ having its tags in $[0,1]$ where $g(x) = 2$, and let $\hat{\mathcal{P}}_2$ be the subset of $\hat{\mathcal{P}}$ with its tags in $]1,3]$ where $g(x) = 3$. Then

$$S(g;\hat{\mathcal{P}}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= \sum_{t_i \in [0,1], i=1}^{n} g(t_i)(x_i - x_{i-1}) + \sum_{t_i \in]1,3], i=1}^{n} g(t_i)(x_i - x_{i-1})$$

Hence $S(g;\hat{\mathcal{P}}) = S(g;\hat{\mathcal{P}}_1) + S(g;\hat{\mathcal{P}}_2)$ … (4)

Let $\|\hat{\mathcal{P}}\| < \beta$. Then $\beta < 1$ since, by assumption, $\|\hat{\mathcal{P}}\| < 1$. We claim that the interval $[0, 1-\beta]$ is contained in the union of all subintervals in $\hat{\mathcal{P}}$ with tags $t_i \in [0,1]$. To prove this, let $u \in [0, 1-\beta]$. Obviously, $u \le 1-\beta$. Since $\hat{\mathcal{P}}_1$ is a subset of $\hat{\mathcal{P}}$ having its tags in $[0,1], u$ lies in some subinterval $[x_{i-1}, x_i]$ with tag $t_i \in [0,1]$. But then $x_{i-1} \le u$ obviously. This together with $u \le 1-\beta$ imply that $x_{i-1} \le 1-\beta$ so that $x_{i-1} + \beta \le 1$. But $x_i - x_{i-1} \le \|\hat{\mathcal{P}}\| = \beta$. Hence $x_i < x_{i-1} + \beta \le 1$, and in such a case, $[x_{i-1}, x_i] \subset [0,1]$ so that the tag $t_i \in [0,1]$. Thus each $u \in [0, 1-\beta]$ lies in some subinterval of $\hat{\mathcal{P}}$ with tag $t_i \in [0,1]$. Consequently, the interval $[0, 1-\beta]$ is contained in the union U of all subintervals in $\hat{\mathcal{P}}$ having their tags $t_i \in [0,1]$. Hence the claim. Further, $[x_{i-1}, x_i] \subset [0,1] \Rightarrow U \subset [0,1] \subset [0, 1+\beta]$. Since $g(t_i) = 2$ for all the tags $t_i \in [0,1]$, we have

$$2(1-\beta) \le S(g;\hat{\mathcal{P}}_1) \le 2(1+\beta) \qquad \ldots (5)$$

A similar argument shows that the union of all subintervals with tags $t_i \in (1,3]$ contains the interval $[1+\beta, 3]$ of length $2-\beta,$ and is contained in $[1-\beta, 3]$ of length $2+\beta$. Therefore,

$$3(2-\beta) \le S(g;\hat{\mathcal{P}}_2) \le 3(2+\beta) \qquad \ldots (6)$$

Adding these inequalities and using equation (6), we obtain

$$8-5\beta \le S(g;\hat{\mathcal{P}}) = S(g;\hat{\mathcal{P}}_1) + S(g;\hat{\mathcal{P}}_2) \le 8+5\beta$$

Thus $-5\beta \le S(g;\hat{\mathcal{P}}) - 8 \le +5\beta$ or equivalently,

$$\left|S(g;\hat{\mathcal{P}}) - 8\right| \le 5\beta = 5\left\|\hat{\mathcal{P}}\right\| \qquad \ldots (7)$$

Now, let $\varepsilon > 0$ be given. Choose δ_ε to be any positive number less than min $(1, \varepsilon/5)$. (For example, take $\delta_\varepsilon = (1/2)\min(1, \varepsilon/5)$. Then $\left\|\hat{\mathcal{P}}\right\| < \varepsilon/5.$ For all partitions $\hat{\mathcal{P}}$ of $[0,3]$ with $\left\|\hat{\mathcal{P}}\right\| < \delta_\varepsilon$ and hence by (7), $\left|S(g;\hat{\mathcal{P}}) - 8\right| < \varepsilon.$ Since $\varepsilon > 0$ is arbitrary, we see that $g \in R[0,3]$ and $\int_0^3 g = 8$, as expected.

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From the example above, you must have realized that it is not easy to use the definition to show that a function is Riemann integrable. In the example, above the function was a constant in the subpartitions i.e. $g(x) = 2$ in $[0,1]$ and $g(x) = 3$ in $[1,3]$. Therefore, we have concentrated only on the partitions whose norms go to zero, and did not worry too much about the tags in the partitions. Sometimes, we employ some tricks that enable us to guess the value of the integral by considering a particular choice of the tag points.

We shall illustrate this in the following example.

Example 3: Consider the continuous function $h:[0,1] \to \mathbb{R}$ defined by $h(x) = x$ for $x \in [0,1]$. Show that $h \in R[0,1]$, and $\int_0^1 h(x)dx = \frac{1}{2}.$

This assertion is an immediate consequence of a result which we will prove later that every continuous function $f:[a,b] \to \mathbb{R}$ is Riemann integrable.

Solution: Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be any given partition of $[0,1]$. Choose the tag of the interval $I_i = [x_{i-1}, x_i]$ to be the midpoint $q_i = \frac{1}{2}(x_{i-1} + x_i)$. The Riemann sum $S(h;\hat{\mathcal{Q}})$ corresponding to the tagged partition $\hat{\mathcal{Q}} = \{(I_i, q_i)\}_{i=1}^n$ is calculated as follows:

$$S(h;\hat{\mathcal{Q}}) = \sum_{i=1}^n h(q_i)(x_i - x_{i-1}) = \sum_{i=1}^n \mathcal{Q}_i(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{2}(x_i^2 - x_{i-1}^2) = \frac{1}{2}(1^2 - 0^2)$$
$$= \frac{1}{2}$$

Let $\hat{\mathcal{P}} = \{(I_i, t_i)\}_{i-1}^{n}$ be an arbitrary tagged partition of $[0,1]$ with $\|\hat{\mathcal{P}}\| = \beta$. Then $x_i - x_{i-1} \le \beta$ for all $i = 1,...,n$. Using the same class of intervals in the basic partition $\mathcal{P}$, form a new tagged partition $\hat{\mathcal{Q}}$ by choosing the tags q_i to be the midpoint of the intervals I_i. Since both t_i and q_i belong to the same interval I_i, we have $|t_i - q_i| < \beta$ for each $i = 1,...,n$. Then

$$\left|S(h;\hat{\mathcal{P}}) - S(h;\hat{\mathcal{Q}})\right| = \left|\sum_{i=1}^{n} t_i(x_i - x_{i-1}) - \sum_{i-1}^{n}(x_i - x_{i-1})\right|$$
$$= \left|\sum_{i=1}^{n}(t_i - \mathcal{Q}_i)(x_i - x_{i-1})\right| \le \sum_{i=1}^{n}|t_i - \mathcal{Q}_i|(x_i - x_{i-1})$$
$$\le \beta \sum_{i=1}^{n}(x_i - x_{i-1}) = \beta(x_n - x_0)$$
$$= \beta(1-0) = \beta = \|\hat{\mathcal{P}}\|.$$

But, as noted earlier, $S(h;\hat{\mathcal{Q}}) = \frac{1}{2}$.

Hence $\left|S(h;\hat{\mathcal{P}}) - \frac{1}{2}\right| \le \|\hat{\mathcal{P}}\|$ … (8)

Now, for any given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then for every tagged partition $\hat{\mathcal{P}}$ of $[0,1]$ with $\|\hat{\mathcal{P}}\| < \delta$, from (8), we obtain $\left|S(h;\hat{\mathcal{P}}) - \frac{1}{2}\right| < \varepsilon$. Hence $h \in R[0,1]$ and $\int_0^1 h(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$.

***

Let us see some more examples.

Example 4: Let $F:[0,1] \to \mathbb{R}$ be defined by $F(x) = 1$ for $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and $F(x) = 0$ elsewhere. Show that $F \in R[0,1]$, and $\int_0^1 f(x)dx = 0.$

Solution: We first note that each of the points $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ in $[0,1]$ at which the function F is not 0 can belong to at most two subintervals in a given tagged partition $\hat{\mathcal{P}}$. Since there are 4 such points, there are at most 8 subintervals in $\hat{\mathcal{P}}$ only can make non zero contributions to $S(F;\hat{\mathcal{P}})$. Therefore, for a given $\varepsilon > 0$, we choose $\delta_\varepsilon = \varepsilon/8$.

Let $\hat{\mathcal{P}}$ be a tagged partition of $[0,1]$ with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$. If none of the points $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ is a tag in $\hat{\mathcal{P}}$, then $F(t_i) = 0$ at all the tags and hence $S(F;\hat{\mathcal{P}}) = 0$.

Otherwise, let $\hat{\mathcal{P}}_0$ be the subset of $\hat{\mathcal{P}}$ with tags different from $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and let $\hat{\mathcal{P}}_1$ be the subset of $\hat{\mathcal{P}}$ with tags at these points. Since $S(F;\hat{\mathcal{P}}_0)=0$, we see that

$$S(F;\hat{\mathcal{P}})=S(F:\hat{\mathcal{P}}_0)+S(F:\hat{\mathcal{P}}_1)=S(F:\hat{\mathcal{P}}_1).$$

Since there are at most 8 terms in the sum $S(F;\hat{\mathcal{P}}_1)$ and each term is less than δ_ε, we see that $0\le S(F;\hat{\mathcal{P}})=S(F;\hat{\mathcal{P}}_1)<8\delta_\varepsilon=\varepsilon$. Thus $F\in R[0,1]$ and $\int_0^1 F(x)\,dx=0$.

***

Example 5: Let $G:[0,1]\to\mathbb{R}$ be defined by $G(x)=x$, if $x=\frac{1}{n}$ for some $n\in\mathbb{N}$ and $G(x)=0$ elsewhere in $[0,1]$.

For a given $\varepsilon>0$, define $E_\varepsilon=\{x\in[0,1]\,|\,G(x)\ge\varepsilon\}$. Since there are only finitely many $n\in\mathbb{N}$ such that $\frac{1}{n}\ge\varepsilon$, E_ε is a finite set. Let n_ε be the number of points in E_ε. Choose $\delta=\frac{\varepsilon}{2n_\varepsilon}$. Consider any tagged partition $\hat{\mathcal{P}}$ of $[0,1]$ such that $\|\hat{\mathcal{P}}\|<\delta_\varepsilon$. Let $\hat{\mathcal{P}}_0$ be the subset $\hat{\mathcal{P}}$ of with tags outside of E_ε and let $\hat{\mathcal{P}}_1$ be the subset of $\hat{\mathcal{P}}$ with tags in E_ε. Then, as in Example (7), we have

$$0\le S(G;\hat{\mathcal{P}})=S(G;\hat{\mathcal{P}}_1)<(2n_\varepsilon)\delta_\varepsilon=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $G\in R[0,1]$ and $\int_0^1 G=0$.

***

So far we have considered examples which show that certain standard functions are Riemann integrable and what are its integrals. Infact the definition of the Riemann integral allows us to compute the value of the integral, if it exists, as a limit of Riemann sums.

We shall prove a theorem which is useful for computing the integral if a function f is Riemann integrable.

Theorem 2: Let $f=[0,b]\to\mathbb{R}$ be Riemann integrable. Then for any sequence of partitions $\{\mathcal{P}_n\}$ of $[a,b]$ with $\lim\limits_{n\to 0}\|\hat{\mathcal{P}}_n\|=0$ and for any associated sequence of of tags $\{t_n\}$, we have

$$\lim_{n\to\infty}S(f;\hat{\mathcal{P}}_n)=\int_a^b f(x)dx.$$

Proof: Let $\{\hat{\mathcal{P}}_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a,b]$ with $\lim\limits_{n\to\infty}\|\hat{\mathcal{P}}_n\|=0$, let $\{t_n\}_{n=1}^{\infty}$ be an associated sequence of tagged sets and let $\varepsilon>0$. Since f is Riemann integrable, there exists $\delta>0$ such that for all the partitions of $[a,b]$ with $\|\mathcal{P}\|<\delta$ and associated tagged set T we have

$$\left|S(f,\hat{\mathcal{P}})-\int_a^b f(x)dx\right|<\varepsilon \qquad \dots (9)$$

Also we are given that $\lim_{n\to\infty}\|\hat{\mathcal{P}}_n\| = 0.$ This implies that given $\delta > 0,$ there exists $N \in \mathbb{N}$ such that $\|\mathcal{P}_n\| < \delta$ for all $n \geq \mathbb{N}$. This together with the inequality given in (9) shows that for all $n \geq \mathbb{N},$

$$\left| S(f,\hat{\mathcal{P}}_n) - \int_a^b f(x)dx \right| < \varepsilon$$

Hence we get that $\lim_{n\to\infty} S(f,\hat{\mathcal{P}}_n) = \int_a^b f(x)dx.$ ■

We shall now give an example to illustrate the theorem above.

Example 6: If $f(x) = x$ is Riemann integrable on $[0,1],$ then show that $\int_0^1 x\,dx = \frac{1}{2}.$

Solution: Let us consider the partition $\mathcal{P}_n = \left\{\frac{i}{n} : i = 0,1,\ldots,n\right\}$ and tags $\left\{\frac{i}{n}, i = 1,\ldots,n\right\}$. Then we have

$$S(f,\hat{\mathcal{P}}_n) = \frac{n^2+2n}{2n^2}$$

$$\therefore \lim_{n\to\infty} S(f,\hat{\mathcal{P}}_n) = \lim_{n\to\infty}\frac{n^2+n}{2n^2} = \frac{1}{2}$$

Hence by Theorem 3 $\int_0^1 x\,dx = \frac{1}{2}$

[You recall that according to the integration formula, you already know that $\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$

***

Remark 3: Remember that the theorem will help you to compute the value of the Riemann integral of a function if only if we assume that f is Riemann integrable.

Why don't you try some exercises now.

E3) Show that the function $f:[0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } 1 \leq x \leq 2. \end{cases}$$

is Riemann integrable on $[0,2]$ and evaluate its integral.

E4) If the function $f(x) = x^2$ is Riemann integrable, then show that

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

E5) Let $g(x) = \begin{cases} 0 \text{ if } & x \in [0,1] \text{ is rational} \\ \frac{1}{x} \text{if } & x \in [0,1] \text{is irrational} \end{cases}$

Explain why $g \notin R[0,1]$. However, show that there exists a sequence $(\hat{\mathcal{P}}_n)$ of tagged partitions of $[0,1]$ such that $\|\hat{\mathcal{P}}_n\| \to 0$ and $\lim_n S(f;\hat{\mathcal{P}}_n)$ exists.

By now you must have realised that unlike continuity and differentiability, it is not easy to check the Riemann integrability of a function. We need to look for a criterion to prove that a function is Riemann integrable. In the next section we shall consider this.

14.4 CRITERIA FOR RIEMANN INTEGRABILITY

In this section we shall discuss two criteria for Riemann integrability. These criteria will help us to decide on the existence of Riemann integrability of a given function. It does not tell us the value of the integral. We shall first prove a theorem which is powerful tool for showing the existence.

It is reasonable to try to check the Riemann integrability of a given function without guessing the value of its integral which may or may not exist. The following theorem on the Cauchy Criterion is a powerful tool in this context.

Theorem 3 (Cauchy Criterion for Integrability): A function $f:[a,b] \to \mathbb{R}$ belongs to $R[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ are any tagged partitions of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\hat{\mathcal{Q}}\| < \eta_\varepsilon$, then $\left|S(f;\hat{\mathcal{P}}) - S(f:\hat{\mathcal{Q}})\right| < \varepsilon$.

Proof: Assume that $f \in \mathbb{R}[a,b]$ with integral L. Let $\varepsilon > 0$ be given. Since $f \in \mathbb{R}[a,b]$, there exists $\eta_\varepsilon > 0$ such that

$$\left|S(f;\hat{\mathcal{P}}) - L\right| < \frac{\varepsilon}{2}$$

for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \eta_\varepsilon$.

Hence if $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ are any two tagged partitions of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\hat{\mathcal{Q}}\| < \eta_\varepsilon$, then $\left|S(f;\hat{\mathcal{P}}) - L\right| < \varepsilon/2$ as well as $\left|S(f;\hat{\mathcal{Q}}) - L\right| < \varepsilon/2$. Therefore

$$\begin{aligned} \left|S(f;\hat{\mathcal{P}}) - S(f;\hat{\mathcal{Q}})\right| &\le \left|S(f;\hat{\mathcal{P}}) - L + L - S(f;\hat{\mathcal{Q}})\right| \\ &\le \left|S(f;\hat{\mathcal{P}}) - L\right| + \left|L - S(f;\hat{\mathcal{Q}})\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Conversely, assume that a given function $f:[a,b] \to \mathbb{R}$ has the property that for every $\varepsilon > 0$ there exists a corresponding $\eta_\varepsilon > 0$ such that whenever $\hat{p}$ and $\hat{\mathcal{Q}}$ are tagged partitions of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\hat{\mathcal{Q}}\| < \eta_\varepsilon$, then $|S(f;\hat{\mathcal{P}}) - S(f:\hat{\mathcal{Q}})| < \varepsilon$. Then by choosing $\varepsilon = 1/n$ for each $n \in \mathbb{N}$, we select $\delta_n > 0$ such that for any two tagged partitions $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ with norms less than δ_n, we have

$$|S(f;\hat{\mathcal{P}}) - S(f;\hat{\mathcal{Q}})| < 1/n.$$

The choice of δ_n can be done in such a way that $\delta_n \geq \delta_{n+1}$ for all $n \in \mathbb{N}$. For otherwise, we replace δ_n by $\delta'_n = \min\{\delta_1, ..., \delta_n\}$. Now, for each $n \in \mathbb{N}$, choose a tagged partition $\hat{\mathcal{P}}$ with $\|\hat{\mathcal{P}}_n\| < \delta_n$. Clearly, if $m > n$ then $\|\hat{\mathcal{P}}_m\| < \delta_m \leq \delta_n$. Hence both $\hat{\mathcal{P}}_m$ and $\hat{\mathcal{P}}_n$ have norms less than δ_n, so that

$$|S(f;\hat{\mathcal{P}}_n) - S(f;\hat{\mathcal{P}}_m| < \frac{1}{n} \qquad \text{… (10)}$$

for all $m > n$.

Consequently, the sequence $(S(f;\hat{\mathcal{P}}_m))_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, by Cauchy Convergence Criterion for sequence this sequence converges in $\mathbb{R}$. Take $A = \lim_m S(f;\hat{\mathcal{P}}_m)$. Fixing $n \in \mathbb{N}$ and passing to the limit in (10) as $m \to \infty$, we obtain

$$|S(f;\hat{\mathcal{P}}_n) - A| < \frac{1}{n}$$

for $n \in \mathbb{N}$.

We claim that $f \in R[a,b]$ and A is the Riemann integral of f. For this, consider any $\varepsilon > 0$. Choose $K \in \mathbb{N}$ satisfying $K > \frac{2}{\varepsilon}$ so that $1/K < \frac{\varepsilon}{2}$.
If $\hat{\mathcal{Q}}$ is any tagged partition with $\|\hat{\mathcal{Q}}\| < \delta_k$, then $|S(f;\hat{\mathcal{Q}}) - S(f;\hat{\mathcal{P}}_K)| < 1/K$, since $\|\hat{\mathcal{P}}_K\| < \delta_K$. Consequently,

$$|S(f;\hat{\mathcal{Q}}) - A| \leq |S(f;\hat{\mathcal{Q}}) - S(f;\hat{\mathcal{P}}_K)| + |S(f;\hat{\mathcal{P}}_k) - A|$$
$$\leq \frac{1}{K} + \frac{1}{K} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $f \in R[a,b]$ with integral A. This completes the proof. ■

The example below illustrates the Cauchy Criterion.

Example 7: Let $g : [0,3] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 2, & \text{for } 0 \leq x \leq 1, \\ 3, & \text{for } 1 < x \leq 3. \end{cases}$$

Using Cauchy criterion show that $\int_0^3 g(x)\,dx$ exists.

Solution: We have seen that $g \in R[0,3]$. Also, if $\hat{\mathcal{P}}$ is any tagged partition of $[0,3]$ with $\|\hat{\mathcal{P}}\| < \delta$, then as we have seen in Example 4,

$$8 - 5\delta \le S(g;\hat{\mathcal{P}}) \le 8 + 5\delta \qquad \dots (11)$$

Hence if $\hat{\mathcal{Q}}$ is another tagged partition of $[0,3]$ with $\|\hat{\mathcal{Q}}\| < \delta$, then

$$8 - 5\delta \le S(g;\hat{\mathcal{Q}}) \le 8 + 5\delta$$

and consequently,

$$-8 - 5\delta \le -S(g;\hat{\mathcal{Q}}) \le -8 + 5\delta \qquad \dots (12)$$

Adding the inequalities (11) and (12), we obtain

$$-10\delta \le S(g;\hat{\mathcal{P}}) - S(g;\hat{\mathcal{Q}}) \le 10\delta$$

Interchanging $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ in the above, we obtain

$$-10\delta \le S(g;\hat{\mathcal{Q}}) - S(g;\hat{\mathcal{P}}) \le 10\delta$$

and hence

$$\left|S(g;\hat{\mathcal{P}}) - S(g;\hat{\mathcal{Q}})\right| \le 10\delta.$$

Thus, by choosing $\eta_\varepsilon = \delta = \varepsilon/20$ corresponding to any given $\varepsilon > 0$, we see that the Cauchy Criterion is satisfied for g.

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Since the Cauchy Criterion is necessary and sufficient for the Riemann integrability of a function, it can be used for concluding that if a given function is Riemann integrable or not. The following example yields a function which is not Riemann integrable.

Example 8: Let $f:[0,1] \to \mathbb{R}$ be the **Dirichlet function**, defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0,1] \text{ is rational} \\ 0 \text{ if } x \in [0,1] \text{ is irrational} \end{cases}$$

Using Cauchy criterion show that f is not Riemann integrals.

Solution: Let $\varepsilon_0 = \frac{1}{2}$. If $\hat{\mathcal{P}}$ is any partition of $[0,1]$ all of whose tags are rational numbers then $S(f;\hat{\mathcal{P}}) = 1$, while if $\hat{\mathcal{Q}}$ is any tagged partition of $[0,1]$ all of whose tags are irrational numbers then $S(f;\hat{\mathcal{Q}}) = 0$. Note that it is always possible to choose such tagged partitions $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ with $\|\hat{\mathcal{P}}\| < \delta$ as well as

$\|\hat{\mathcal{Q}}\| < \delta$ for any choice of $\delta > 0$. But, in all such cases, we obtain $\left|S(f;\hat{\mathcal{P}}) - S(f;\hat{\mathcal{Q}})\right| = 1 > \frac{1}{2} = \varepsilon_0$ (Note that the choice of ε_0 can be any real number in $(0,1)$). Thus the Cauchy Criterion is not satisfied by f. Hence the Dirichlet function f is not Riemann integrable.

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Thus we have established a criterion for checking the existence of Riemann integrability.

Next we shall prove result which will be used to establish the Riemann integrability of some important classes of functions.

Theorem (Squeeze Theorem) 4: Let $f:[a,b] \to \mathbb{R}$. Then $f \in R[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $R[a,b]$ with

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon \qquad \ldots (13)$$

such that for all $x \in [a,b]$,

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \qquad \ldots (14)$$

Proof: If $f \in R[a,b]$, then for any given $\varepsilon > 0$, it is enough to choose $\alpha_\varepsilon = \omega_\varepsilon = f$ so that both (13) and (14) follow obviously.

To see the converse, let $\varepsilon > 0$ and choose $\alpha_\varepsilon, \omega_\varepsilon \in R[a,b]$, satisfying (13) and (14). Since $\alpha_\varepsilon, \omega_\varepsilon \in R[a,b]$, there exists $\delta_\varepsilon > 0$ such that if $\hat{\mathcal{P}}$ is any tagged partition with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$ then

$$\left|S(\alpha_\varepsilon;\mathcal{P}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon \text{ as well as } \left|S(\omega_\varepsilon;\mathcal{P}) - \int_a^b \omega_\varepsilon\right| < \varepsilon$$

From these inequalities, it follows that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon;\mathcal{P}) \text{ and } S(\omega_\varepsilon;\mathcal{P}) < \int_a^b \omega_\varepsilon + \varepsilon$$

In view of inequality (13), we have

$$S(\alpha_\varepsilon;\mathcal{P}) \le S(f;\mathcal{P}) \le S(\omega_\varepsilon;\mathcal{P}).$$

Hence

$$\int_b^a \alpha_\varepsilon - \varepsilon < S(f;\mathcal{P}) < \int_a^b \omega_\varepsilon + \varepsilon$$

If $\hat{\mathcal{Q}}$ is another tagged partition with $\|\hat{\mathcal{Q}}\| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f;\mathcal{Q}) < \int_a^b \omega_\varepsilon + \varepsilon$$

Subtracting these two inequalities, we obtain

$$\int_a^b (\alpha_\varepsilon - \omega_\varepsilon) - 2\varepsilon < S(f;\mathcal{P}) - S(f;\mathcal{Q}) < \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon$$

and consequently,

$$\left|S(f;\mathcal{P}) - S(f;\mathcal{Q})\right| < \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon$$

But by (3), we have $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$. Hence $\left|S(f;\mathcal{P}) - S(f;\mathcal{Q})\right| < 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f \in R[a,b]$. This completes the proof. ■

By now you must have realized that the complexity involved in working with the Riemann sums is due to the condition that for a given $\varepsilon > 0$ we need to work with different partitions and associated tagged points.

The following theorem shows that every Riemann integrable function is bounded.

Before that we state an important property of Riemann integrability.

Theorem 5 (Bounded Theorem): If $f \in R[a,b]$, then f is bounded on $[a,b]$.

Proof: Assume that $f \in \mathbb{R}[a,b]$ is a unbounded function with integral L. Then for the choice $\varepsilon = 1$, there exists $\delta_1 > 0$ such that $\left|S(f;\hat{\mathcal{P}}) - L\right| < 1$ for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$ with $\left\|\hat{\mathcal{P}}\right\| < \delta_1$. Since the triangle inequality yields $\left|(\left|x\right|) - (\left|y\right|)\right| \le \left|x - y\right|$ for all $x, y \in \mathbb{R}$, this implies that

$$\left|S(f;\hat{\mathcal{P}})\right| < \left|L\right| + 1 \qquad \text{… (15)}$$

Let $\mathcal{Q} = \{[x_{i-1}, x_i]\}_{n=1}^{n}$ be a partition of $[a,b]$ with $\left\|\mathcal{Q}\right\| < \delta_1$. Since $\left|f\right|$ is not bounded on $[a,b]$, there is at least one subinterval in $\mathcal{Q}$, say $[x_{k-1}, x_k]$, on which $\left|f\right|$ is not bounded. If there is no such subinterval in $\mathcal{Q}$, then $\left|f\right|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, and hence it is bounded on $[a,b]$ itself by $\max\{M_1, ..., M_n\}$.

Now, we tag the partition $\mathcal{Q}$ as follows: Choose $t_i = x_i$ for all $i \ne k$. Since $\left|f\right|$ is not bounded on $[x_{k-1}, x_k]$, there is some $x \in [x_{k-1}, x_k]$ such that $\left|f(x)\right| > M$ where $M = (1/(x_k - x_{k-1})) \times \left(\left|L\right| + 1 + \left|\sum_{i \ne k} f(t_i)(x_i - x_{i-1})\right|\right)$. Choose $t_k = x$ for completing the tagging of $\mathcal{Q}$. Then

$$\left|f(t_k)(x_k - x_{k-1})\right| > \left|L\right| + 1 + \left|\sum_{t \ne k} f(t_i)(x_i - x_{i-1})\right|$$

and consequently,

$$\left|S(f;\hat{Q})\right| \geq \left|f(t_k)(x_k - x_{k-1})\right| - \left|\sum_{i \neq k} f(t_i)(x_i - x_{i+1})\right| > |L| + 1,$$

using the inequality $|x-y| \geq \left|\left(|x| - |y|\right)\right|$ for all $x, y \in \mathbb{R}$, as mentioned in the beginning. This is contradicts to the inequality in (1,5) for the tagged partition $\hat{Q}$ and hence f is not Riemann integrable on $[a,b]$. ■

In fact, we have proved the following.

Corollary 1: An unbounded function cannot be Riemann integrable.

The Boundedness theorem says that every Riemann integrable function is necessarily bounded.

Let us now briefly recall the theory of integration you learnt in the Calculus course.

We begin with a bounded function define on $[a,b]$.

Let $f:[a,b] \to \mathbb{R}$ be bounded and let $\hat{\mathcal{P}} = \{|x_{i-1}, x_i|\}_{i=1}^{n}$ be a partition of $[a,b]$. For $i = 1,2,...,n,$ define

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and}$$
$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Define the upper sum $U(\mathcal{P}, f)$ and the lower sum $L(\mathcal{P}, f)$ of f corresponding to the partition $\mathcal{P}$ by

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ and} \quad \ldots (16)$$

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad \ldots (17)$$

The upper and lower Riemann integrals of f denoted by $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ respectively are defined by

$$\text{Upper interval} = \overline{\int_a^b} f(x)\,dx = \inf U(\mathcal{P}, f) \text{ and}$$

$$\text{Lower interval} = \underline{\int_a^b} f(x)\,dx = \sup L(\mathcal{P}, f)$$

where the infimum and the supremum are taken over all possible partitions $\mathcal{P}$ of $[a,b]$. If the upper and lower integrals are equal, then the function f is Riemann integrable over $[a,b]$ and the common value is defined as the Riemann integral of f.

Theorem 6: Let $f:[a,b] \to \mathbb{R}$ be bounded and let $\mathcal{P} = \{a = x_0, ..., x_n = b\}$ be a partition of $[a,b]$. Then for all associated set of tagged points $T = \{t_i, ..., t_n\}$, inequalities $L(f,\hat{\mathcal{P}}) \le S(f;\hat{\mathcal{P}}) \le U(f,\hat{\mathcal{P}})$.

Proof: The inequality follows from the fact that if $t_i \in [x_{i-1}, x_i]$ for all $i \in \{1,2,..,n\}$, then $m_i \le f(t_i) \le M_i$ for all $i \in \{1,2,...,n\}$. Hence

$$\sum_{i=1}^{n} m_i \, \Delta x_i \le \sum_{i=1}^{n} f(t_i) \, \Delta x_i \le M_i \, \Delta x_i$$

Hence the result. ■

Remark 4: The above theorem says that the Riemann integration which we have defined in this unit will give the same value as the one that are obtained by considering the upper sum and the lower sum that is obtained by the process illustrated in the Calculus course. Note that the latter process assumes that the function is bounded.

The origin of the theory of integration claims that integral defined by considering the lower sums and upper sums is due to the Mathematician Gaston Darboux.

The integrals considered by Darboux sums(upper sums and lower sums) are called **Darboux integral**. Infact a function is Darboux-integrable if and only if it is Riemann integrable and the values of the two integrals, if they exist are equal.

Let us now go back to Theorem 5 above which says that if the Riemann sums are squeezed between upper sums and lower sums can get arbitrarily close to each other, then f in Riemann integrable. The only way this cannot happen is when the function oscillates too much, that is, if the function is highly discontinuous. The following is an example of a function which is not Riemann integrable. The function is called Dirichlet function.

Example 9: The Dirichlet function f defined by

$$f(x) = \begin{cases} 0, & \text{for } x \in [0,1] - \mathcal{Q} \\ 1, & \text{for } x \in \mathcal{Q} \cap [0,1] \end{cases}$$

is not Riemann integrable.

Solution: You can easily calculated that for any partition $\hat{\mathcal{P}}, U(\hat{\mathcal{P}}, f) = 1$ and $L(\hat{\mathcal{P}}, f) = 0$.

$\therefore$ The function is not Darboux integrable and therefore not Riemann integrable.

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You can try these exercises now.

E6) Suppose that $f:[a,b] \to \mathbb{R}$ and that $f(x) = 0$ except for a finite number of points $c_1,...,c_n$ in $[a,b]$. Prove that $f \in \mathbb{R}[a,b]$ and that $\int_a^b f = 0.$

This brings to the end of this unit. Let us summarize now the points discussed in this unit.

14.5 SUMMARY

In this unit we have covered the following points:

1. We have explained the concept of Riemann integral of a real-valued function f defined on an interval $[a,b]$.
2. We have discussed the computation of Riemann integral of some known functions.
3. We have given the Cauchy criterion of Riemann integrability.
4. We have explained the connection between Riemann integral and Darboux integral of a function.

14.6 SOLUTIONS AND ANSWERS

E1) Hints/Answers

a) $\|\mathcal{P}_1\| = 2$ b) $\|\mathcal{P}_2\| = 2$ c) $\|\mathcal{P}_3\| = 1.4$

c) $\|\mathcal{P}_4\| = 2$

E2) Hints/Answers

a) $0^2.1+1^2.1+2^2.2 = 0+1+8 = 9$

b) 37

c) 13

d) 33

E3) Consider any tagged partition $\hat{\mathcal{P}}$ of $[0,2]$ with $\|\hat{\mathcal{P}}\| < 1$. Let $\hat{\mathcal{P}}_1$ be the subset of $\hat{\mathcal{P}}$ having its tags in $[0,1[$ where $f(x)=2$ and $\hat{\mathcal{P}}_2$ be the subset of $\hat{\mathcal{P}}$ with its tags in $[1,2]$ where $f(x)=1$. Then

$$S(f;\hat{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= \sum_{t_i\in[0,1[,\, i=1}^{n} f(t_i)(x_i - x_{i-1}) + \sum_{t_i\in[1,2],\, i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Hence $S(f,\hat{\mathcal{P}}) = S(f,\hat{\mathcal{P}}_1) + S(f,\hat{\mathcal{P}}_2)$

Let $\|\hat{\mathcal{P}}\| < \beta$. Then $\beta < 1$, since by the assumption, $\|\hat{\mathcal{P}}\| < 1$.

We claim that the interval $[0,1-\beta)$ is contained in the union of subintervals in $\hat{\mathcal{P}}$ with tags $t_i \in [0,1[$.

To prove this, let $u \in [0,1-\beta]$. Obviously, $u \le 1-\beta$. Since $\hat{\mathcal{P}}_1$ is a subset of $\hat{\mathcal{P}}$ having its tags in $[0,1[$, u lies in some subinterval $[x_{i-1}, x_i]$ with tag $t_i \in [0,1[$. Then $x_{i-1} \le u$. This together with $u \le 1-\beta$ imply that $x_{i-1} \le 1-\beta$, so that $x_{i-1}+\beta \le 1$. But $x_i - x_{i-1} \le \|\hat{\mathcal{P}}\| = \beta$. Hence $x_i < x_{i-1}+\beta \le 1$, and in such a case $[x_{i-1}, x_i] < [0,1]$, so that the tag $t_i \in [0,1[$.

Thus each $u \in [0,1-\beta]$ lies in some subinterval of $\hat{\mathcal{P}}$ with tag $t_i \in [0,1[$. Consequently, the interval $[0,1-\beta]$ is contained in the union of all subintervals in $\hat{\mathcal{P}}$ having their tags $t_i \in [0,1[$. Hence the claim.

Further, let $v = u$, then $v \in [x_{i-1}, x_i]$ with tag $t_i \in [0,1[$ for some i. Then $x_{i-1} < 1$ and $v \le x_i$. Also, $x_i - x_{i-1} \le \left\|\hat{\mathcal{P}}\right\| = \beta$.

So, $v \le x_i \le x_{i-1} + \beta$

$< 1 + \beta$ [since $x_{i-1} < 1$]

$\therefore v \in [0,1+\beta]$

Thus $U \subset [0,1+\beta]$.

Since $f(t_i) = 2$ for all the tags $t_i \in [0,1[$, we have

$$2 \le 1 - \beta \le S(f,\hat{\mathcal{P}}) \le 2\ (1+\beta) \quad \text{… (18)}$$

A similar argument shows that the union of all subintervals with tags $t_i \in [1,2]$ contains the interval $[1+\beta, 2]$ of length $1-\beta$ and is contained in $[1-\beta, 2]$ of length $1-\beta$.

Therefore

$$1 \times (1-\beta) \le S(f,\hat{\mathcal{P}}_2) \le 1 \times (1+\beta) \quad \text{… (19)}$$

Adding the inequalities (18) and (19), we get

$$3(1-\beta) \le S(f,\hat{\mathcal{P}}) = S(f,\hat{\mathcal{P}}_1) + S(f,\hat{\mathcal{P}}_2)$$
$$\le 3(1+\beta)$$

Thus $-3\beta \le S(f,\hat{\mathcal{P}}) - 3 \le 3\beta$ or equivalently

$$\left|S(f,\hat{\mathcal{P}}) - 3\right| \le 3\beta = 3\left\|\hat{\mathcal{P}}\right\|$$

Now let $\varepsilon > 0$ be given. Choose δ_ε to be any positive number less than min $\left(1, \frac{\varepsilon}{3}\right)$. Hence for all partitions $\hat{\mathcal{P}}$ of $[0,2]$ with $\left\|\hat{\mathcal{P}}\right\| < \delta_\varepsilon$,

$$\left|S(f,\hat{\mathcal{P}}) - 3\right| \le \varepsilon$$

Therefore by the definition of Riemann integrability, $f(x)$ is integrable and $\int_0^2 f(x)dx = 3.$

E4) Let $\{\mathcal{P}_n\}$ where $\mathcal{P}_n = \left\{\frac{i}{n}, i = 0,1,...,n\right\}$ be a sequence of partitions and $\left\{\frac{i}{n}, i = 1,...,n\right\}$ be the associated sequence of tags

Now, $S(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(t_i)\Delta x_i$

$$= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \text{ [As } \Delta x_i = \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{(n+1)(2n+1)}{6n^2} \left[\text{since } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \right]$$

Therefore, $\lim_{n\to\infty} S(f, \mathcal{P}_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$

$$= \lim_{n\to\infty} \frac{1}{6}\left\{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right\}$$

$$= \frac{1}{6} \times 2 = \frac{1}{3}$$

E5) Hint: Same as Example

Here there are n points $i_1, i_2, ..., i_n$ where f is not zero, each of which can belong to the subintervals in a given tagged partition. Choose $\delta_\varepsilon = \frac{\varepsilon}{2n}$ (proceed same as in the example to get the desired result).

E6) Hints/Answers

If $u \in [x_{i-1}, x_i]$, then $x_{i-1} \le u$ so that $c_1 \le t_i \le x_i \le x_{i-1} + \left\|\hat{\mathcal{P}}\right\|$ and hence $c_1 - \left\|\hat{\mathcal{P}}\right\| \le x_{i-1} \le u$. Also $u \le x_i$ so that $x_i - \left\|\hat{\mathcal{P}}\right\| \le x_{i-1} \le t_i \le c_2$ and hence $u \le x_i \le c_2 + \left\|\hat{\mathcal{P}}\right\|$.

UNIT 15

PROPERTIES OF RIEMANN INTEGRABLE FUNCTIONS

Structure **Page No.**

15.1 INTRODUCTION

In the last unit you have learnt the definition and some basic facts about Riemann integrability. You studied that certain standard functions like constant function, identity function and polynomial function etc are integrable on any interval $[a,b]$. We want to know about the integrability of many more functions. In this unit we discuss some properties of Riemann integrals. This will help us indentify some more functions in $R[a,b]$. You will see that $R[a,b]$ includes all continuous functions.

In Sections 15.2 you will study that the sum and difference of two integrable functions are integrable. So also is multiplication of an integrable function by a fixed real number (called a scalar multiplication).

In Section 15.3 we discuss the Riemann integrability of continuous functions, monotone functions and step functions. Lastly we discuss another useful theorem known as additivity Theorem.

Objectives

After reading this unit you should be able to

- check the Riemann integrability of large number of functions that are expressed as the sum of simple functions;
- identify a step function and find its integral
- apply the result that both the class of continuous functions and the class of

monotonone functions are included in the class of Riemann integrable functions.

15.2 Basic Properties (or Algebra) of Riemann Integrable functions

In this section we state and prove some basic results on algebra of Riemann integrable functions.

The following theorem tells us that when the basis algebraic operations, namely sum and multiplication by a constant, are applied on Riemann integrable functions, the resulting functions are also Riemann integrable.

Theorem 1: Suppose that f and g are in $R[a,b]$. Then the following holds:

i) The function kf is in $R[a,b]$ and for each $k \in \mathbb{R}$, and $\int_a^b k\, f = k \int_a^b f$

ii) The function $f+g$ is in $R[a,b]$, and $\int_a^b f + g = \int_a^b f + \int_a^b g.$

Proof: We shall first prove (i)

Let $\hat{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be any tagged partition of $[a,b]$. Then

$$S(kf;\hat{\mathcal{P}}) = \sum_{i=1}^n (kf)(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n k \cdot f(t_i)(x_i - x_{i-1})$$

$$= k\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = kS(f;\hat{\mathcal{P}}) \qquad \ldots (1)$$

Let $\varepsilon > 0$ be given. If $k = 0$, then by Eqn. (1), $S(kf;\hat{\mathcal{P}}) = 0$ for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$. So, in this case, kf is Riemann integrable, and $\int_a^b k\, f = 0 = k\int_a^b f$. Now, assume that $k \neq 0$. Since $f \in R[a,b]$, corresponding to $\frac{\varepsilon}{|k|}$, there exists δ such that for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$ with $\|\hat{\mathcal{P}}\| < \delta$ we have

$$\left| S(f;\hat{\mathcal{P}} - \int_a^b f \right| < \frac{\varepsilon}{|k|} \qquad \ldots (2)$$

Using the triangle inequality, Eqn. (1) and (2), we obtain

$$\left| S(kf;\hat{\mathcal{P}}) - k\int_a^b f \right| = \left| S(kf;\hat{\mathcal{P}}) - kS(f;\hat{\mathcal{P}}) + kS(f;\hat{\mathcal{P}}) - k\int_a^b f \right|$$

$$\leq \left| S(kf;\hat{\mathcal{P}}) - kS(f;\hat{\mathcal{P}}) \right| + \left| kS(f;\hat{\mathcal{P}}) - k\int_a^b f \right|$$

$$= |k| \left| S(f;\hat{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$$

This is true for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$ with $|\hat{\mathcal{P}}| < \delta$. Hence (i) follows.

To prove (ii), we note that

$$S(f+g;\hat{\mathcal{P}}) = \sum_{i=1}^{n}(f+g)(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(f(t_i)+g(t_i))(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= S(f;\hat{\mathcal{P}}) + S(g;\hat{\mathcal{P}}).$$

for every tagged partition $\hat{\mathcal{P}}$ of $[a,b]$.

Let $\varepsilon > 0$. As in the proof of the uniqueness Theorem 3 in unit 14, we obtain a number $\delta_\varepsilon > 0$ such that if $\hat{\mathcal{P}}$ is any tagged partition with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$, then both

$$\left|S(f;\hat{\mathcal{P}}) - \int_a^b f\right| < \varepsilon/2 \text{ and } \left|S(g;\hat{\mathcal{P}}) - \int_a^b g\right| < \frac{\varepsilon}{2} \qquad \ldots (3)$$

Hence

$$\left|S(f+g;\hat{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right| = \left|S(f;\hat{\mathcal{P}}) + S(g;\hat{\mathcal{P}}) - \int_a^b f - \int_a^b g\right|$$

$$\le \left|S(f;\hat{\mathcal{P}}) - \int_a^b f\right| + \left|S(g;\hat{\mathcal{P}}) - \int_a^b g\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f+g \in R[a,b]$ and that its integral is the sum of the integrals of f and g. ■

Corollary 1: If $f_1,\ldots, f_n$ are in $R[a,b]$ and if $k_1,\ldots,k_n \in \mathbb{R}$, then the linear combination $f = \sum_{i=1}^{n} k_i f_i$ belongs to $R[a,b]$ and

$$\int_a^b f = \sum_{i=1}^{n} k_i \int_a^b f_i$$

We leave the proof of this as an exercise for you to try, (see E1).

Next we shall prove another theorem.

Theorem 2: Suppose that f and g are in $R[a,b]$ such that $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.

Proof: Since $f(x) \le g(x)$ for all $x \in [a,b]$, we note that $f(t_i) \le g(t_i)$ for all the tags t_i of any given tagged partition $\hat{\mathcal{P}}$ of $[a,b]$. Then

$$S(f;\hat{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = S(g;\hat{\mathcal{P}}) \qquad \ldots (4)$$

Let $\varepsilon > 0$. As in the proof of the uniqueness Theorem 3 in unit 14, we obtain a number $\delta_\varepsilon > 0$ such that if $\hat{\mathcal{P}}$ is any tagged partition with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$, then both $\left|S(f;\hat{\mathcal{P}}) - \int_a^b f\right| < \frac{\varepsilon}{2}$ and $\left|S(g;\hat{\mathcal{P}}) - \int_a^b g\right| < \frac{\varepsilon}{2}$. These respectively yield

$$\int_a^b f - \frac{\varepsilon}{2} < S(f;\hat{\mathcal{P}}) \text{ and } S(g;\hat{\mathcal{P}}) < \int_a^b g + \frac{\varepsilon}{2}.$$

From Eqn. (4), we have $S(f;\hat{\mathcal{P}}) \le S(g;\hat{\mathcal{P}})$, so that $\int_a^b f \le \int_a^b g + \varepsilon$. But since $\varepsilon > 0$ is arbitrary, we conclude that

$$\int_a^b f \le \int_a^b g .$$

■

Now look at the following corollary.

Corollary 2: If $f \in R[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, then

$$\left|\int_a^b f\right| \le M(b-a).$$

We leave the proof of this as an exercise for you to try, see E2).

Here are some exercises for you.

E1) Prove Corollary 1

E2) Prove Corollary 2

E3) If f and g are two integrable functions, then how that the product fg is integrable.

In the next section we shall consider the integrability of certain clause of functions.

15.3 CLASSES OF RIEMANN INTEGRABLE FUNCTIONS

In this section we discuss the relationship of Riemann integrability with continuous and monotone functions.

We shall first discuss the Riemann integral of a step function. Let us now look at the definition of a step function.

Definition: A function $\varphi : [a,b] \to \mathbb{R}$ is called a **step function** if it has only a finite number of distinct values, each being assumed on one or more subintervals of $[a,b]$.

For instance, assume that $\varphi : [a,b] \to \mathbb{R}$ is a step function and $k_1, k_2, ..., k_n$ are the distinct values assumed by φ on the subintervals $I_1 = [d_0, d_1)$, $I_2 = [d_1, d_2)$, $I_3 = [d_2, d_3), .., I_n = [d_{n-1}, d_n]$ respectively, where $a = d_0 < d_1 < d_2 < ... < d_n = b$. Define $\varphi_j : [a,b] \to \mathbb{R}$ by

$$\varphi_j(x) = \begin{cases} 1 \text{ if } x \in I_j \\ 0 \text{ if } x \notin I_j \end{cases}.$$

Then each φ_j is a step function and is called an "elementary step function" and every step function φ can be expressed as a linear combination of elementary step functions φ_j as $\varphi = \sum_{j=1}^{n} k_j \varphi_j$.

We shall illustrate this with an example.

Example 1: Let $\varphi : [-1,3] \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} 2, \text{ for } -1 \le x < 1, \\ -1, \text{ for } 1 \le x < 2, \\ 1, \text{for } 2 \le x \le 3. \end{cases}$$

Show that φ is a step function and φ can be expressed as $\varphi = \sum k_j \varphi_j$.

Solution: We note that φ is a function which assumes only 3 values, namely $-1,1$ and 2. Take $I_1 = [-1,1[, I_2 = [1,2[, I_3 = [2,3]$. Clearly, for all $x \in [-1,3]$,

$$\varphi(x) = (2 \times \varphi_1(x)) + (-1 \times \varphi_2(x)) + (1 \times \varphi_3(x))$$

where φ_1, φ_2 and φ_3 are elementary step functions corresponding to I_1, I_2 and I_3 respectively. .

Thus $\varphi = (2 \times \varphi_1) + (-1 \times \varphi_2) + (1 \times \varphi_3) = \sum_{j=1}^{3} k_j \varphi_j$ where $k_1 = 2, k_2 = -1$ and $k_3 = 1$.

***

Next we shall prove a theorem.

Theorem 3: Suppose that $c \le d$ are points in $[a,b]$. If $\varphi : [a,b] \to \mathbb{R}$ is the step function defined by

$$\varphi(x) = \begin{cases} \alpha, \text{ if } x \in [c,d] \\ 0, \text{ if } x \notin [c,d] \end{cases}$$

then $\varphi \in R[a,b]$ and that $\int_a^b \varphi = \alpha(d-c)$.

Proof: We first assume that $\alpha > 0$. The Riemann sum $S(\varphi, \hat{\mathcal{P}})$ corresponding to a tagged partition $\hat{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ is given by $S(\varphi, \hat{\mathcal{P}}) = \sum_{i=1}^{n} \varphi(t_i)(x_i - x_{i-1})$.

For $\varepsilon > 0$, choose $\delta_\varepsilon = \min\left\{\frac{\varepsilon}{3\alpha}, \frac{d-c}{3}\right\}$. If $\hat{\mathcal{P}}$ is any tagged partition of $[a,b]$ such that $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$, then $\|\hat{\mathcal{P}}\| < \varepsilon / 4\alpha$, so that the union of subintervals of the form $[x_{i-1}, x_i]$ in $\hat{\mathcal{P}}$ with tags in $[c,d]$ contains the interval $[c+\delta_\varepsilon, d-\delta_\varepsilon]$ and is contained in $[c-\delta_\varepsilon, d+\delta_\varepsilon]$. Therefore

$$\alpha(d-c-2\delta_\varepsilon) \le S(\varphi; \hat{\mathcal{P}}) \le \alpha(d-c+2\delta_\varepsilon)$$

and hence

$$\alpha(d-c)-2\alpha\delta_{\varepsilon} \leq S(\varphi;\hat{\mathcal{P}}) \leq \alpha(d-c)+2\alpha\delta_{\varepsilon},$$

Consequently,

$$\left|S(\varphi,\hat{\mathcal{P}})-\alpha(d-c)\right| \leq 2\alpha\delta_{\varepsilon} < 3\alpha\delta_{\varepsilon} = \varepsilon$$

for all tagged partitions $\hat{\mathcal{P}}$ of $[a,b]$ such that $\|\hat{\mathcal{P}}\| < \delta_{\varepsilon}$. Hence $\varphi \in R[a,b]$ and that $\int_a^b \varphi = \alpha(d-c)$. If $\alpha = 0$, then the function $\varphi = 0$ so that $\varphi \in R[a,b]$ and $\int_a^b \varphi = \alpha(d-c)$ obviuosly. If $\alpha < 0$, then we write $\alpha = -\beta$, where $\beta = |\alpha| > 0$. In this case, the function $\varphi : [a,b] \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \beta \ \text{if} \ \ x \in [c,d] \\ 0 \ \text{if} \ \ x \notin [c,d]. \end{cases}$$

is Riemann integrable on $[a,b]$ and that $\int_a^b \varphi = \beta(d-c)$. But $\varphi = -1 \cdot \varphi$.

Hence $\varphi \in R[a,b]$ and $\int_a^b \varphi = \int_a^b -1 \cdot \varphi = -1.\int_a^b \varphi = -1.\beta(d-c) = \alpha(d-c)$. ■

Using Theorem 1 we can easily establish the following theorem.

Theorem 4: If J is a subinterval of $[a,b]$ having endpoints $c < d$ and if $\varphi_J : [a,b] \to \mathbb{R}$ defined by

$$\varphi_J(x) = \begin{cases} 1 \ \ \text{if} \ \ x \in J \\ 0 \ \ \text{if} \ \ x \notin J. \end{cases}$$

Then $\varphi_J \in R[a,b]$ and $\int_a^b \varphi_J = d - c$.

The theorem follows by taking $\alpha = 1$ in Theorem 1. ■

The next theorem shows that the step functions are Riemann integrable.

Theorem 5: If $\varphi : [a,b] \to \mathbb{R}$ is a step function, then $\varphi \in R[a,b]$.

Proof: Since $\varphi : [a,b] \to \mathbb{R}$ is a step function, it will assume only finite number of distinct values, say $k_1, k_2, ..., k_m$ on some subintervals $J_j, J_2, ..., J_m$ respectively with each J_j having endpoints $c_j < d_j$. As seen earlier, φ can be expressed in the form $\varphi = \sum_{j=1}^{m} k_j \varphi_{J_j}$ as a linear combination of elementary step functions φ_{J_j}. Then by Theorem 2, each $\varphi_{J_j} \in \mathbb{R}[a,b]$ and consequently, each $k_j.\varphi_{J_j} \in R[a,b]$, by Theorem 1 of previous section. Again by Theorem 1 of previous section, it follows that $\sum_{j=1}^{m} k_j.\varphi_{J_j} \in R[a,b]$. Thus $\varphi \in R[a,b]$. Also,

$$\int_a^b \varphi = \int_a^b \left(\sum_{j=1}^{m} k_j \varphi_{J_j} \right)$$

$$= \sum_{j=1}^{n} \left(\int_a^b k_j \varphi_{J_j} \right)$$, using part (ii) of Theorem 1 of previous section

$$= \sum_{j=1}^{n} k_j \left(\int_a^b \varphi_{J_j} \right)$$, using Part (i) of Theorem 2 of previous section.

$$= \sum_{j=1}^{n} k_j (d_j - c_j),$$ by Theorem 2. ■

Now we will show that the continuous function are Riemann integrable.

Theorem 6: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then $f \in R[a,b]$.

Proof: Since the function f is a continuous function on a closed and bounded interval, it follows that f is uniformly continuous on $[a,b]$. [Please see Theorem? in Unit? Block 4]. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that whenever $u, v \in [a,b]$ with $|u - v| < \delta_\varepsilon$, then we have

$$|f(u) - f(v)| < \frac{\varepsilon}{b-a}.$$

Again since f is continuous, then f attains its maximum and minimum values on each partition $\hat{\mathcal{P}}$ of $[a,b]$.

Let $\hat{\mathcal{P}} = \{I_i\}_{i=1}^{n}$ be a partition of $[a,b]$ such that $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$. Let u_i and v_i be points of I_i where f attains its minimum and maximum values respectively, on I_i. Let α_ε be the step function defined by

$$\alpha_\varepsilon(x) = \begin{cases} f(u_i) \text{ for } x \in [x_{i-1}, x_i) \; i = 1, \ldots, n-1 \\ f(u_n) \text{ for } x \in [x_{n-1}, x_n] \end{cases}$$

Let ω_ε be defined similarly using the points v_i instead of the u_i. i.e.,

$$\omega_\varepsilon(x) = \begin{cases} f(v_i) \text{ for } x \in [x_{i-1}, x_i) \, (i = 1, \ldots, n-1) \\ f(v_n) \text{ for } x \in [x_{n-1}, x_n] \end{cases}$$

Obviously, for all $x \in [a,b]$, we obtain

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x).$$

Consequently,

$$0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^{n} ((f(v_i) - f(u_i)) \, (x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a} \right) (x_i - x_{i-1})$$

$$= \left(\frac{\varepsilon}{b-a} \right) \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \left(\frac{\varepsilon}{b-a} \right)(b-a) = \varepsilon .$$

Hence by the Squeeze Theorem (Theorem 7 of Unit 14), $f \in R[a,b]$. ■

Next we shall consider the Riemann Integrability of Monotone Functions. The following theorem shows that monotone functions are Riemann integrable though they are not necessarily continuous.

Theorem 7: If $f:[a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in R[a,b]$.

Proof: Suppose that f is increasing on the interval $[a,b], a < b$. If $\varepsilon > 0$ is given, we choose $q \in \mathbb{N}$ such that

$$h = \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{b-a} \qquad \dots (5)$$

Let $y_k = f(a) + kh,$ for $k = 0,1,\dots q$.

Take $A_k = f^{-1}([y_{k-1}, y_k[)$ for $k = 0,1,\dots,q-1$ and $A_q = f^{-1}([y_{q-1}, y_q])$. These sets $\{A_k\}$ are pair wise disjoint and have union $[a,b]$.

The Characterization Theorem implies that each A_k is either (i) empty, (ii) contains a single point, or (iii) is a nondegenerate interval (not necessarily closed) in $[a,b]$. We discard the sets for which (i) holds and relabel the remaining ones. If we adjoin the endpoints to the remaining intervals $\{A_k\}$, we obtain closed intervals. These relabeled intervals $\{A_k\}_{k=1}^{q}$ are pairwise disjoint, satisfy $[a,b] = \bigcup_{k=1}^{q} A_k$ and that $f(x) \in [y_{k-1}, y_k]$ for $x \in A_k$. We now define step functions α_ε and ω_ε on $[a,b]$ by setting

$$\alpha_\varepsilon(x) = y_{k-1} \text{ and } \omega_\varepsilon(x) = y_k$$

for $x \in A_k$. It is clear that

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$$

for $x \in [a,b]$ and that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{k=1}^{q} (y_k - y_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{q} h \cdot (x_k - x_{k-1}) = h \cdot (b-a) < \varepsilon, \text{ by Eqn. (5).}$$

Since $\varepsilon > 0$ is arbitrary, the Squeeze Theorem (Theorem 4) implies that $f \in R[a,b]$. ■

By now you must have realised that the squeeze theorem is very important for proving many other important theorems. Thus we have learnt that the class of Riemann integrable function include continuous functions, monotone functions and step functions. The following theorem is also useful when we discuss Riemann integrability on the union of closed intervals.

Theorem 8 (Additivity Theorem): Let $f:[a,b] \to \mathbb{R}$ and let $c \in]a,b[$. Then

$f \in R[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both Riemann integrable. In this case $\int_a^b f = \int_a^c f + \int_c^b f$.

The proof of the theorem is omitted.

Proof: ($\Leftarrow$) Suppose that the restriction f_1 of f to $[a,c]$ and the restriction f_2 of f to $[c,b]$ are Riemann integrable functions with Riemann integrals L_1 and L_2 respectively. Then, given $\varepsilon > 0$ there exists $\delta' > 0$ such that for any tagged partition $\mathcal{P}_1$ of $[a,c]$ with $\|\mathcal{P}_1\| < \delta'$, we have

$$|S(f_1;P_1) - L_1| < \frac{\varepsilon}{3} \qquad \ldots (6)$$

Also for the same $\varepsilon > 0$, there exists $\delta'' > 0$ such that for any tagged partition $\hat{\mathcal{P}}_2$ of $[c,b]$ with $\|\hat{\mathcal{P}}\| < \delta''$, we have

$$|S(f_2;\hat{\mathcal{P}}_2) - L_2| < \frac{\varepsilon}{3} \qquad \ldots (7)$$

Let M be a bound for $|f|$. Define $\delta_\varepsilon = \min\left\{\delta', \delta'', \frac{\varepsilon}{6M}\right\}$. For any tagged partition $\hat{\mathcal{Q}}$ of $[a,b]$ with $\|\hat{\mathcal{Q}}\| < \delta_\varepsilon$, we have to prove that

$$|S(f;\hat{\mathcal{Q}}) - (L_1 + L_2)| < \varepsilon \qquad \ldots (8)$$

We shall consider two cases here.

Case (i): If c is a partition point of $\hat{\mathcal{Q}}$, we split $\hat{\mathcal{Q}}$ into a partition $\hat{\mathcal{Q}}_1$ of $[a,c]$ and a partition $\hat{\mathcal{Q}}_2$ of $[c,b]$. Then

$$S(f;\hat{\mathcal{Q}}_2) = S(f;\hat{\mathcal{Q}}_1) + S(f;\hat{\mathcal{Q}}_2)$$

Since $\|\hat{\mathcal{Q}}_1\| < \delta'$ and $\|\hat{\mathcal{Q}}_2\| < \delta''$, using Eqns. 5 and 6, we see that

$$|S(f;\hat{\mathcal{Q}}) - (L_1 + L_2)| = |S(f;\hat{\mathcal{Q}}_1) + S(f;\hat{\mathcal{Q}}_2) - (L_1 + L_2)|$$
$$= |(S(f;\hat{\mathcal{Q}}_1) - L_1) + (S(f;\hat{\mathcal{Q}}_2) - L_2|$$
$$\le |(S(f;\hat{\mathcal{Q}}_1) - L_1)| + |(S(f;\hat{\mathcal{Q}}_2) - L_2)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$$

Hence we have proved (8) in this case.

Case (ii): If c is not a partition point in $\hat{\mathcal{Q}} = \{I_j, t_j)\}_{j=1}^m$, there exists $k \le m$ such that $c \in (x_{k-1}, x_k)$. Let $\mathcal{Q}_1$ be the tagged partition of $[a,c]$ defined by

$$\mathcal{Q}_1 = \{(I_1, t_1), \ldots, (I_{k-1}, t_{k-1}), ([x_{k-1}, c], c)\}$$

and $\mathcal{Q}_2$ be the tagged partition of $[c,b]$ defined by

$$\mathcal{Q}_2 = \{([c, x_k], c),\ (I_{k+1}, t_{k+1}), \ldots, (I_m, t_m)\}$$

A straightforward calculation yields

$$S(f;\mathcal{Q}) = \sum_{j=1}^{m} f(t_j)(x_j - x_{j-1})$$
$$= f(t_k)(x_k - x_{k-1}) + \sum_{j=1}^{k-1} f(t_j)(x_j - x_{j-1}) + \sum_{j=k+1}^{m} f(t_j)(x_j - x_{j-1})$$
$$= f(t_k)(x_k - x_{k-1}) + S(f;\mathcal{Q}_1) + S(f;\mathcal{Q}_2) - f(c)\cdot(x_k - x_{k-1})$$
$$= (f(t_k) - f(c))\cdot(x_k - x_{k-1}) + S(f;Q_1) + S(f;Q_2).$$

Hence

$$S(f;\mathcal{Q}) - S(f;\mathcal{Q}_1) - S(f;\mathcal{Q}_2) = (f(t_k) - f(c))\cdot(x_k - x_{k-1})$$

so that

$$\left|S(f;\mathcal{Q}) - S(f;\mathcal{Q}_1) - S(f;\mathcal{Q}_2)\right| \le \left|(f(t_k) - f(c))\right|\cdot\left|x_k - x_{k-1})\right|$$
$$\le (\left|f(t_k)\right| + \left|f(c))\right|)\cdot \delta_\varepsilon$$
$$\le 2M.\varepsilon/6M = \frac{\varepsilon}{3}. \qquad \ldots (9)$$

Since $\|\mathcal{Q}_1\| < \delta_\varepsilon < \delta'$ and $\|\mathcal{Q}_2\| < \delta_\varepsilon < \delta''$, we have

$$\left|S(f;\mathcal{Q}_1) - L_1\right| < \frac{\varepsilon}{3} \text{ as well as } \left|S(f;\mathcal{Q}_2) - L_2\right| < \frac{\varepsilon}{3}.$$

Hence $\left|S(f;\mathcal{Q}) - L_1 - L_2\right|$

$$= \left|S(f;\mathcal{Q}) - S(f;\mathcal{Q}_1) - S(f;\mathcal{Q}_2) + S(f;\mathcal{Q}_1) + S(f;\mathcal{Q}_2) - L_1 - L_2\right|$$
$$\le \left|S(f;\mathcal{Q}) - S(f;\mathcal{Q}_1) - S(f;\mathcal{Q}_2)\right| + \left|S(f;\mathcal{Q}_1) - L_1\right| + \left|S(f;\mathcal{Q}_2) - L_2\right|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus we have proved (8) in this case.

Since $\varepsilon > 0$ is arbitrary, $f \in R[a,b]$ and also,

$$\int_a^b f = L_1 + L_2 = \int_a^c f + \int_c^b f$$

($\Rightarrow$) Conversely, we assume that $f \in R[a,b]$. Given $\varepsilon > 0$, choose $\eta_\varepsilon > 0$ satisfying the Cauchy Criterion 1. Let f_1 be the restriction of f to $[a,c]$ and let $\hat{\mathcal{P}}_1$, $\hat{\mathcal{Q}}_1$ be tagged partitions of $[a,c]$ with $\left\|\hat{\mathcal{P}}_1\right\| < \eta_\varepsilon$ and $\left\|a\,i\;\hat{\mathcal{Q}}_1\right\| < \eta_\varepsilon$. By adding same additional partition points and tags from $[c,b]$, we can extend both $\hat{\mathcal{P}}_1$ and $\hat{Q}_1$ to tagged partitions $\hat{\mathcal{P}}$ and $\hat{Q}$ and $\hat{Q}$ of $[a,b]$ satisfying $\left\|\hat{\mathcal{P}}\right\| < \eta_\varepsilon$ and $\left\|\hat{\mathcal{Q}}\right\| < \eta_\varepsilon$. Since the same additional points and tags in $[c,b]$ are used for both $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ we obtain

$$S(f_1;\mathcal{P}_1) - S(f_1;\mathcal{Q}_1) = S(f;\mathcal{P}) - S(f;\mathcal{Q}).$$

Then $\left|S(f_1;\mathcal{P}_1) - S(f_1;\mathcal{Q}_1)\right| = \left|S(f;\mathcal{P}) - S(f;\mathcal{Q})\right| < \varepsilon$ since by Cauchy Condition 1, $f \in R[a,b]$, $\left\|\hat{\mathcal{P}}\right\| < \eta_\varepsilon$ and $\left\|\hat{\mathcal{Q}}\right\| < \eta_\varepsilon$ together yield $\left|S(f;\mathcal{P}) - S(f;\mathcal{Q})\right| < \varepsilon.$

Therefore again, by the Cauchy criteria, $f_1 \in R[a,c]$. In the same way, the restriction f_2 of f to $[c,b]$ is in $R[c,b]$. The equality (7) now follows from the first part of the theorem. This completes the proof. ■

We shall now make some corollaries.

Corollary 3: If $f \in R[a,b]$, and if $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is in $R[c,d]$.

Proof: Since $f \in R[a,b]$ and $c \in [a,b]$, it follows from Theorem 8 that its restriction to $[c,b]$ is in $R[c,b]$. But if $d \in [c,b]$, then another application of Theorem 8 shows that the restriction of f to $[c,d]$ is in $R[c,d]$. This completes the proof. ■

In fact we have the following result the proof of which is omitted.

Corollary 4: If $f \in R[a,b]$ and if $a = c_0 < c_1 < ... < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f.$$

Before we state another property, we make a definition.

Definition: If $f \in R[a,b]$ and if $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f = -\int_\alpha^\beta f, \text{ and, } \int_\alpha^\alpha f = 0.$$

Theorem 9: If $f \in R[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f, \qquad \text{… (10)}$$

for any permutations of α, β and γ, in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (10).

Proof: If any two of the numbers α, β, γ are equal, then Eqn. (10) holds. Thus we may suppose that all the three of these numbers are distinct. For the sake of symmetry, we introduce the expression

$$L(\alpha, \beta, \gamma) = \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f.$$

First, verify that Eqn. (10) holds if and only if $L(\alpha, \beta, \gamma) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments $\alpha, \beta,$ and γ. We note that the Additivity Theorem 7 implies that $L(\alpha, \beta, \gamma) = 0$ when $\alpha < \beta < \gamma$. But it is easily seen that both $L(\beta, \gamma, \alpha)$ and $L(\gamma, \alpha, \beta)$ are equal to $L(\alpha, \beta, \gamma)$. Moreover, the numbers

$$L(\beta, \alpha, \gamma), L(\alpha, \gamma, \beta), \text{and } L(\gamma, \beta, \alpha)$$

are all equal to $-L(\alpha,\beta,\gamma)$. Therefore, L vanishes for all possible configurations of these three points. This completes the proof. ■

Before ending this unit, we give an example of a function that is discontinuous at every rational number and is not monotone, but is Riemann integrable.

Example 2: Consider the **Thomae's function** $h:[0,1]\to\mathbb{R}$ where

$$h(x)=\begin{cases}0, \text{ if } x\in[0,1] \text{ is irrational}\\ \dfrac{1}{n}, \text{ if } x=\dfrac{m}{n}\in[0,1], m,n\in\mathbb{N}, (m,n)=1.\end{cases}$$

Show that $h\in R[0,1]$.

Solution: We note that h is discontinuous at every rational number and is not a monotonic function. Let $\varepsilon>0$. Then the set $E_\varepsilon=\left\{x\in[0,1]:h(x)\geq\dfrac{\varepsilon}{2}\right\}$ is finite. Let n_ε be the number of elements in E_ε. Choose $\delta_\varepsilon=\dfrac{\varepsilon}{4n_\varepsilon}$. Consider a tagged partition $\hat{\mathcal{P}}$ of $[0,1]$ with $\|\hat{\mathcal{P}}\|<\delta_\varepsilon$. Let $\hat{\mathcal{P}}$ be the subset of $\hat{\mathcal{P}}$ having tags in E_ε and $\hat{\mathcal{P}}_2$ be the subset of $\hat{\mathcal{P}}$ having tags elsewhere in $[0,1]$. We observe that $\hat{\mathcal{P}}_1$ has at most $2n_\varepsilon$ intervals and the sum of the lengths of these intervals is less than $2n_\varepsilon\delta_\varepsilon=\dfrac{\varepsilon}{2}$. Also $0<h(t_i)\leq 1$ for every tag t_i in $\hat{\mathcal{P}}_1$. The sum of the lengths of the subintervals in $\hat{\mathcal{P}}_2$ is less than or equal to 1 and $h(t_i)<\dfrac{\varepsilon}{2}$ for every tag t_i in $\hat{\mathcal{P}}_2$. Therefore we have

$$\left|S(h;\hat{\mathcal{P}})\right|=S(h;\hat{\mathcal{P}}_1)+S(h;\hat{\mathcal{P}}P_2)<1\cdot 2n_\varepsilon\delta_\varepsilon+\left(\frac{\varepsilon}{2}\right)\cdot 1=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we infer that $h\in R[0,1]$ with integral 0.

***

You can try some exercises now.

E4) Consider the function h defined by

$$h(x)=\begin{cases}x+1, \text{ if } x\in[0,1] \text{ is rational,}\\ 0 \quad, \text{ if } x\in[0,1] \text{ is irrational.}\end{cases}$$

Show that h is not Riemann integrable.

E5) If $S(f;\mathcal{P})$ is any Riemann sum of $f:[a,b]\to\mathbb{R}$, show that there exists a step function $\varphi:[a,b]\to\mathbb{R}$ such that $\int_a^b\varphi=S(f;\mathcal{P})$.

E6) We have shown in Unit 10, Block 4 that the function $f:]0,1[\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}$ is continuous, but not uniformly continuous. Does this contradict Theorem 4? Justify your answer.

With this we come to an end of this unit.
Let us summarise the points we have discussed in this unit.

15.5 SUMMARY

In this unit we have covered the following points.

1. We have shown that if f and g are in $R[a,b]$, then $f+g \in R[a,b]$. Also for any function $f \in R[a,b]$, for any$\in \mathbb{R}$.
2. We have proved that the step functions, continuous functions and monotone functions are Riemann integrable.
3. We have stated and proved additivity theorem.

15.6 SOLUTIONS AND ANSWER

E1) Since $f_i's$ are in $R[a,b]$ for $i=1,...,n$, then using theorem 1, we have $k_i f_i \in R[a,b]$.

Now, we will use mathematical induction theory. Let $p(n)$ be the statement that $\sum_{i=1}^{n} k_i f_i$ belongs to $R[a,b]$ for any $n \in \mathbb{N}$.

So, $p(1)$ is true as $k_i f_i \in R[a,b]$.

Now, let $p(m-1)$ its true & $g = \sum_{i=1}^{m-1} k_i f_i$.

So, g & $k_m f_m$ both are integrable.

Therefore, using property (ii) of theorem 1. We have $g + k_m f_m = \sum_{i=1}^{m} k_i f_i$ belongs to $R[a,b]$. Therefore, $p(m)$ is true.
So, by mathematical induction, $p(n)$ is true for all n.

E2) Let $g_1(x) = -M$ for all $x \in [a,b]$ and $g_2(x) = M$ for all $x \in [a,b]$.

Given that $|f(x)| \le M$ for all $x \in [a,b]$

$$\Rightarrow -M \le f(x) \le M$$
$$\Rightarrow g_1(x) \le f(x) \le g_2(x)$$

Therefore, from theorem 2, we have

$$\int_a^b g_1(n) \le \int_a^b f(n) \le \int_a^b g_2(n)$$

$$\Rightarrow \int_a^b (-M) \le \int_a^b f(x) \le \int_a^b M$$

$$\Rightarrow -M(b-a) \le \int_a^b f(n) \le M(b-a)$$

$$\Rightarrow \left| \int_a^b f \right| \le M(b-a)$$

E3) Notice that $fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$. Now, since f & g are integrable. So, if you can only show that f^2 is integrable whenever f is integrable then your work is done.

E4) Let p_n be the partition of $[0,1]$ defined by $p_n = (x_0, x_1, x_2, ..., x_n)$, where $x_r = \frac{r}{n}, 0 \le r \le n$. Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$ for $r = 1, 2, ..n$

Then $S(\lambda, (p_n, \alpha)) = \frac{1}{n}(x_1 + x_2 + .. + x_n + n)$

$$= \frac{1}{n}\left(\frac{1+2+...+n}{n} + n\right)$$

Let us choose β_r in $[x_{n-1}, x_n]$ by $\beta_r = x_r - \frac{1}{\sqrt{5}n}$

Then $S(h, (p_n, \beta)) = \frac{1}{n}[0 + 0 + ... + 0]$

$$= 0$$

Let us consider the sequence of partitions $(p_n), \|p_n\| = \frac{1}{n}, \lim_{n\to\infty}\|p_n\| = 0.$

$$\lim_{n\to\infty} S(h, (p_n, \alpha)) = \frac{3}{2}, \lim_{n\to\infty} S(h, (p_n), \beta)) = 0$$

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(h, (P, \xi_r))$ converges to different limits, f is not integrable on $[0,1]$.

E5) Let $\hat{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$

So, $s(f, \hat{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

Let ϕ be a step function such that

$$\phi(x) = \begin{cases} f_1, & x \in [a = x_0, x_1] \\ f_2, & x \in [x_1, x_2] \\ \vdots & \\ f_x, & x \in [x_{n-1}, x_n = b] \end{cases}$$

$$\int_a^b \phi = \int_{a=x_0}^{x_1} f_1 + \int_{x_1}^{x_2} f_2 + ... + \int_{x_{n-1}}^{x_{n=b}} f_n$$

$$= f_1(x_1 - x_0) + f_2(x_2 - x_1) + ... + f_n(x_1 - x_{n-1})$$

$$= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= S(f, \hat{\mathcal{P}})$$

E6) This does not contradict theorem 9, as the does domain f is an open interval.

UNIT 16

IMPORATANT THEOREMS

Structure ___________________________ Page No.

16.1 INTRODUCTION

In the previous two units we discussed how to evaluate the integral of a function as a limit of Riemann sums. Some important classes of Riemann integrable functions were also discussed. You have observed that the evaluation of an integral is a tedious work. In this unit we look at some powerful and easier techniques for evaluating the definite integral in certain situations. This is done by introducing the idea of primitive of a function.

You have already studied that if a function $F:[a,b] \to \mathbb{R}$ is such that $F'(x) = f(x)$ for all $x \in [a,b]$, then F is called an **antiderivative** or a **primitive** of f on $[a,b]$ [Refer Block-5 Calculus course BMTC-131]. There you have learnt that whenever f has an antiderivative F, then all the functions $F + c, c \in \mathbb{R}$ are also antiderivatives of f. However, the theory of Riemann integral is independent of the concept of the antiderivative. In this unit, we will explore the connection between the notions of derivative and integral.

The relationship between integral and derivative is established by an important theorem known as fundamental theorem of calculus which is due to two famous mathematicians cum physicists Sir Issac Newton and Gottfried Wilhelm Leibniz. In Sec 16.2 we first define the term "Primitive" of a function. Then we state and prove the fundamental theorem of calculus. We shall state and prove two forms of this theorem. We explain how the theorem helps to evaluate a definite integral for continuous function without using the Riemann sums.

In Sec. 16.3 we state and prove mean value theorem for integrals. We illustrate the theorem with examples.

In Sec. 16.4, we shall explain a method of checking convergence and divergence of an infinite series by associating the terms of the series to the values of a function f where f is a monotonic, decreasing and integrable function defined over an interval $[a,b]$. We introduce improper integrals and state and prove an integral test for checking the convergence of a series.

Objectives

After studying this unit, you should be able to:

- find the primitive of certain standard functions,
- state, prove and apply the fundamental theorem of Calculus for integrals and explain its importance,
- state and prove the mean value theorems for integrals and explain their importance,
- use Cauchy integral test to check the convergence of a series.

16.2 FUNDAMENTAL THEOREM OF CALCULUS

In this section we shall discuss one of the important theorems in Calculus known as 'Fundamental Theorem of Calculus. This theorem gives the connection between derivative and integral.

Before we discuss the theorem we shall recall the definition of term 'primitive' of a function which you have studied in the 1st semester Calculus course, Block 5 of BMTC-131.

We start with the definition.

Definition 1: Let f be a function defined on an interval $I=[a,b]$. Then a function F is a primitive of f on I if F is differentiable on I and $F'=f$ for all $x \in I$.

For example it follows from the formulas of integration that if $f(x)=x^3$, then the function $F(x)=\frac{x^4}{4}$ is the primitive of the function f.

Let us consider another function $f:[-1,1] \to \mathbb{R}$ defined by

$$f(x)=\begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 & \text{if } \quad 0 \le x < 1 \end{cases}$$

This function is not the derivative of any function $F:[-1,1] \to \mathbb{R}$. Indeed if f is the derivative of a function $F:[-1,1] \to \mathbb{R}$ then by the intermediate value property of derivatives. f must have an intermediate value property. But clearly, the function f given above does not have the intermediate value property. Hence f cannot be the derivative of any function $F:[-1,1] \to \mathbb{R}$.

However if $f:[-1,1] \to \mathbb{R}$ is continuous, then f is the derivative of a function $F:[-1,1] \to \mathbb{R}$. This leads us to following general theorem.

However if f is continuous, then we have the following theorem.

Theorem 1: Let f be integrable on $[a,b]$. Define a function F on $[a,b]$ as

$$F(x)=\int_a^x f(t)\,dt,\ \forall\ x\in[a,b].$$

Then F is continuous on $[a,b]$. Furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0 and $F'(x_0)=f(x_0)$.

Proof: Since f is integrable on $[a,b]$, it is bounded. In other words, there exists a positive number M such that $|f(x)|\le M,\ \forall\ x\in[a,b]$.

Let $\varepsilon>0$ be any number. Choose $x, y\in[a,b], x<y$, such that $|x-y|<\dfrac{\varepsilon}{M}$.

Then $$|F(y)-F(x)|=\left|\int_a^y f(t)\,dt-\int_a^x f(t)\,dt\right|$$

$$=\left|\int_a^x f(t)\,dt+\int_x^y f(t)\,dt-\int_a^x f(t)\,dt\right|$$

$$=\left|\int_x^y f(t)\,dt\right|$$

$$\le\int_x^y |f(t)|dt$$

$$\le\int_x^y M dt=M(y-x)<\varepsilon$$

Similarly we can discuss the case when $y<x$. This shows that F is continuous on $[a,b]$. Infact this proves the uniform continuity of F.

Now, suppose f is continuous at a point x_0 of $[a,b]$.

We can choose some suitable $h\neq 0$ such that $x_0+h\in[a,b]$.

Then $$F(x_0+h)-F(x_0)=\int_a^{x_0+h} f(t)\,dt-\int_a^{x_0} f(t)dt$$

$$=\int_a^{x_0} f(t)dt+\int_{x_0}^{x_0+h} f(t)\,dt-\int_a^{x_0} f(t)\,dt$$

Thus $$F(x_0+h)-F(x_0)=\int_{x_0}^{x_0+h} f(t)\,dt \qquad \ldots (1)$$

Now $$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right|=\left|\frac{1}{h}\int_{x_0}^{x+h} f(t)\,dt-\frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt\right|$$

$$=\frac{1}{|h|}\left|\int_{x_0}^{x_0+h}[f(t)-f(x_0)]dt\right|.$$

Since f is continuous at x_0, given a number $\varepsilon>0, \exists$ a number $\delta>0$ such that $|f(x)-f(x_0)|<\varepsilon/2,$ whenever $|x-x_0|<\delta$ and $x\in[a,b]$. So, if $|h|<\delta,$ then $|(f(t)-f(x_0)|<\varepsilon/2,$ for $t\in[x_0,x_0+h],$ and consequently $\left|\int_{x_0}^{x_0+h}[f(t)-f(x_0)]dt\right|\le\frac{\varepsilon}{2}|h|$. Therefore $\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right|\le\frac{\varepsilon}{2}<\varepsilon,$ if $|h|<\delta.$

Therefore, $\lim_{h\to 0}\frac{F(x_0+h)-F(x_0)}{h}=f(x_0),$ i.e., $F'(x_0)=f(x_0).$

Which shows that F is differentiable at x_0 and $F'(x_0)=f(x_0)$ from Theorem 1, you can easily deduce the following theorem: ■

Theorem 2: Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Let $F:[a,b]\to\mathbb{R}$ be a function defined by

$$F(x)=\int_a^x f(t)dt,x\in[a,b].$$

Then $F'(x)=f(x),a\le x\le b.$

This is the first result which links the concepts of integral and derivative. It says that, if f is continuous on $[a,b]$ then there is a function F on $[a,b]$ such that $F'(x)=f(x),\ \forall\ x\in[a,b].$

You have seen that if $f:[a,b]\to\mathbb{R}$ is continuous, then f has a primitive F. Is such a function F unique? Clearly the answer is 'no'.

Infact, for any given function f, a primitive of f is not unique. For example, the functions $x\to\sin^{-1}x$ and $x\to-\cos^{-1}x,\ x\in[-1,1]$ are both primitives of the function $x\to\frac{1}{\sqrt{1-x^2}}$. Indeed, $\sin^{-1}x=\frac{\pi}{2}+(-\cos^{-1}x).$

However, two primitives of a given function are related they can only differ by a constant. That means we have the following proposition.

Proposition 1: Let F_1 and F_2 be primitives of a function f on an interval $I=[a,b]$. Then there exists some constant c such that

$$F_2(x)=F_1(x)+c,\ \text{ for some } x\in I.$$

Proof: Let F_1 and F_2 be two primitives of f on I, then $F_1'(x)=f(x)$ and $F_2'(x)=f(x),x\in I$, so that $F_2'(x)=F_1'(x),x\in I.$ Therefore it follows that there exists some constant c such that $F_2(x)=F_1(x)+c.$ ■

Now we shall state an important theorem which gives the connection between primitives of a function and the integral of f on an interval I. The theorem is known as the fundamental theorem of calculus.

Theorem 3 (Extension of the First form of Fundamental Theorem of

Calculus): If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: Since f is assumed to be continuous on $[a,b]$, we have $f \in \mathbb{R}[a,b]$. Let $\varepsilon > 0$ be given. Then there exists $\delta_\varepsilon > 0$ such that

$$\left| S(f;\mathcal{P}) - \int_a^b f \right| < \varepsilon \qquad \dots (2)$$

for every tagged partition $\hat{\mathcal{P}}$ with $\|\hat{\mathcal{P}}\| < \delta_\varepsilon$. If $[x_{i-1}, x_i], i = 1,\dots,n$ are the subintervals corresponding to $\hat{\mathcal{P}}$, then

$$\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

But, the Mean value Theorem applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ for $i = 1,\dots,n$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i)\cdot(x_i - x_{i-1}).$$

Substituting these in the previous sum yields

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} F'(u_i)(x_i - x_{i-1}).$$

Let $\hat{\mathcal{P}}_u$ denote the tagged partition $\{([x_{i-1}, x_i], u_i)\}_{i=1}^n$. Then $\|\hat{\mathcal{P}}_u\| < \delta_\varepsilon$ since $\|\hat{\mathcal{P}}_u\| < \delta_\varepsilon$. Also, $S(f;\hat{\mathcal{P}}_u) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} F'(u_i)(x_i - x_{i-1})$ since $F'(u_i) = f(u_i)$. Consequently, $F(b) - F(a) = S(f;\hat{\mathcal{P}}_u)$. Substituting this in (12), we obtain

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\int_a^b f(x)\,dx = F(b) - F(a)$. ■

Remark 1: The assertion of the Fundamental Theorem of Calculus holds even if there are some exceptional points c where $F'(c)$ does not exist, or where it does not equal to $f(c)$.

Example 1: Evaluate $\int_a^b x\,dx$.

Solution: Define $F(x) = \frac{1}{2}x^2$ and $f(x) = x$ for all $x \in [a,b]$. Then $F'(x) = f(x)$ for all $x \in [a,b]$. Since f is continuous, we have $f \in R[a,b]$.

Therefore the Fundamental Theorem 2 (with $E = \emptyset$, implies that

$$\int_a^b x\,dx = F(b) - F(a) = \frac{1}{2}(b^2 - a^2).$$

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Example 2: Use Fundamental Theorem to evaluate the integral $\int_0^1 2^x\,dx.$

Solution: Define $F(x) = \frac{2^x}{\ln 2}$. Then $F'(x) = 2^x$. Also since $f(x) = 2^x$ is continuous on $[0,1]$, it is Riemann integrable in $[0,1]$. Therefore by the fundamental theorem (Theorem 3).

$$\int_0^1 2^x\,dx = F(1) - F(0)$$

$$= \frac{2}{\ln_e 2} - \frac{1}{\ln_e 2} = \frac{1}{\ln_e 2}.$$

***

We next show that the derivative of a differentiable function need not be integrable.

Example 3: Let $F : [0,1] \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} x^2 \cos(1/x^2) & \text{for } x \in (0,1], \\ 0 & \text{for } x = 0. \end{cases}$$

Show that F' exists, but F' is not Riemann integrable.

Solution: For all $x \in (0,1]$, we have

$$F'(x) = 2x\cos(1/x^2) + (2/x)\sin(1/x^2).$$

Further,

$$F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \cos\left(\frac{1}{x^2}\right)}{x} = \lim_{x\to 0} x \cdot \cos\left(\frac{1}{x^2}\right) = 0.$$

Thus F is differentiable at every point of $[0,1]$.

Since the first term $f_1(x) = 2x\cos(1/x^2)$ in F' is continuous on $[0,1]$, it belongs to $R[0,1]$. However, the second term $f_2(x) = (2/x)\sin(1/x^2)$ in F' is not bounded, so it does not belong to $R[0,1]$.

If $F' \in R[0,1]$, then $f_2 = F' - f_1 \in R[0,1]$, leads to a contradiction. Hence F' is not Riemann integrable.

Example 4: Check whether the conclusion of Theorem 2 holds for the function $f(x) = \operatorname{sgn} x$ on $[-1,1]$.

Solution: We note that the signum function sgn is defined by

$$\text{sgn}(x) = \begin{cases} +1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

i.e. $\text{sgn}(x) = \dfrac{x}{|x|}, \text{for } x \neq 0$

$= 0, \quad \text{for } x = 0$

and this function belongs to $R[-1,1]$.

We also note that $f(x)$ is not continuous at $x = 0$, therefore we cannot apply Theorem 2. Now let us check whether the conclusion of theorem 2 holds or not. Now, if $z < 0$, then

$$F(z) = \int_{-1}^{z} f(x)dx = \int_{-1}^{z} -1.\,dx = -z - 1 = |z| - 1.$$

If $z > 0$, then

$$F(z) = \int_{-1}^{z} f = \int_{-1}^{0} f(x)dx + \int_{0}^{z} f(x)dx = \int_{-1}^{0} -1.\,dx + \int_{0}^{z} (+1).dx$$

$$= -1 + z = |z| + 1$$

If $z = 0$, then

$$F(0) = \int_{-1}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} (-1)\,dx$$

$$= 1$$

Thus, $F(x) = \begin{cases} |x| + 1, & x \neq 0 \\ 1, & x = 0 \end{cases}$

But $F'(0)$ does not exist. Hence F is not an antiderivative of f on $[-1,1]$.

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Here are some exercises for you.

E1) Use Fundamental theorem to evaluate the integral $\int_a^b x^n\, dx.$

E2) Let $G(x) = 2\sqrt{x}$ for $x \in [0,b]$. Then G is continuous on $[0,b]$ and $G'(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [0,b]$. Does there exist a $g \in [0,b]$, such that $\int_0^b g(x)dx = G(b) - G(0)$? Justify your answer.

In the section we shall discuss Mean value theorem for integrals.

16.3 MEAN VALUE THEOREM FOR INTEGRALS

Analogues to the mean value theorem for differentiable functions, there are mean value theorems for certain Riemann integrable functions also.

Theorem 3: Let $f:[a,b]\to\mathbb{R}$ be continuous. Then $f\in\mathbb{R}[a,b]$ and there exists $\xi\in[a,b]$ such that $\int_a^b f = f(\xi).(b-a)$.

Proof: Since $f:[a,b]\to\mathbb{R}$ is continuous, as observed earlier, f is bounded and Riemann integrable on $[a,b]$.

Let $m=\inf\{f(x):x\in[a,b]\}$ and $M=\sup\{f(x):x\in[a,b]\}$. Then

$m\le f(x)\le M$ for all $x\in[a,b]$ and hence $m.(b-a)\le\int_a^b f\le M.(b-a)$.

Take $r=\dfrac{\int_a^b f(x)dx}{(b-a)}$ so that $m\le r\le M$ from the above.

Since f is continuous on $[a,b]$, the values m and M are attained by f on $[a,b]$. Thus $m,M\in f([a,b])$ and since $m\le r\le M$, by intermediate value theorem, there exists $\xi\in[a,b]$ such that $f(\xi)=r$.

Hence $f(\xi)=\dfrac{\int_a^b f(x)dx}{(b-a)}$ so that $\int_a^b f = f(\xi).(b-a)$. ■

Example 5: Illustrate the Theorem 3 for the function $f:[a,b]\to\mathbb{R}$ be defined by $f(x)=x$.

Solution: We note that f is continuous, $f\in\mathbb{R}[a,b]$ and

$$\int_a^b f(x)dx=\frac{b^2-a^2}{2}=\left(\frac{b+a}{2}\right).(b-a) \qquad \ldots (1)$$

Let ξ be the mid point of $[a,b]$. Then $\xi\in[a,b]$ and $\xi=\dfrac{b+a}{2}$.

Further $f(\xi)=\xi=\dfrac{b+a}{2}$.

Consequently, from Eqn. (1) we get that $\int_a^b f(x)dx=f(\xi).(b-a)$. Hence the theorem 3 holds for the $\xi=\dfrac{b+a}{2}$, the middle point of the interval $[a,b]$.

***

Remark 2: Note that the condition that f is continuous is necessary for the theorem to hold. For example let us consider the function f defined on $[3,7]$ as follows:

$$f(x) = \begin{cases} 2 & \text{if } 3 \le x < 5 \\ 5 & \text{if } 5 \le x \le 7 \end{cases}$$

$$\int_3^7 f(x)dx = \int_3^5 f(x)dx + \int_5^7 f(x)dx$$
$$= 4 + 10 = 14$$

But $\frac{1}{b-a}\int_3^7 f(x)dx = \frac{1}{4} \times 14 = \frac{14}{4}$

The number $\frac{14}{4}$ can never be assumed by the function f on any $\xi \in [a,b]$ due to the choice of f. This shows that continuity is a necessary condition for the theorem to hold.

You can now try some exercises now.

E3) Let $f \in \mathbb{R}[a,b]$ and define $F(x) = \int_a^x f$ for $x \in [a,b]$.

a) Evaluate $G(x) = \int_c^x f$ in terms of F, where $c \in [a,b]$.

b) Evaluate $H(x) = \int_a^b f$ in terms of F.

c) Evaluate $S(x) = \int_x^{\sin x} f$ in terms of F.

E4) Let $f : [0,3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1, \\ 1 & \text{for } 1 \le x < 2, \\ x & \text{for } 2 \le x \le 3. \end{cases}$$

Obtain formulas for $F(x) = \int_0^x f$.

E5) If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0,1]$, show that $f(x) = 0$ for all $x \in [0,1]$.

In this section we shall discuss infinite series revisted.

16.4 INFINITE SERIES REVISTED

In this section we apply the techniques of integration learnt in Unit 14 and 15 to obtain useful information about the convergence of certain type of infinite series of positive terms.

We introduce a method for determining the convergence and divergence of certain series of the form $\sum_{n=1}^{\infty} a_n$, where the terms a_n, for each $n \in \mathbb{N}$, is such

that $a_n = f(n)$ where f is a non-negative monotonic, decreasing integrable function defined on $[1,\infty]$. Before that we extend the concept of integral to an unbounded interval and to an unbounded function.
We begin with a definition.

Definition 2: If a function $f:[a,\infty] \to \mathbb{R}$ is integrable on $[a,b]$ for every $b > a$ and $\lim_{b\to\infty}\int_a^b f$ exists, either as a real number, or as $\pm\infty$, we denote the limit by $\int_a^\infty f$ and call it the improper integral of f over $[a,\infty)$. In case $\lim_{b\to\infty}\int_a^b f$ is real number, we speak of the improper integral $\int_a^\infty f$ as being convergent; otherwise divergent.

Note that when the limit does not exist, the integral symbol does not represent a real number, for example $\int_1^\infty \frac{1}{x}dx$ is an improper integral with $\int_1^\infty \frac{1}{x}dx = \infty$, because $\lim_{b\to\infty}\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}(\ln b) = \infty$. The improper integral is divergent. Where as $\int_1^\infty \frac{1}{x^2}dx$ is an improper integral with $\int_1^\infty \frac{1}{x^2}dx = 1$. This is because $\lim_{b\to\infty}\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$. The improper integral is convergent.

Now we state and prove a test known as "integral test" to check the convergence and divergence of certain series.

Theorem 4 (Cauchy's Integral Test): Let f be a real valued function with domain $[1,\infty[$ such that

i) $f(x) \geq 0, \forall\, x \geq 1$ (f is non-negative)

ii) $x < y \Rightarrow f(x) > f(y)$, ($f$ is a monotonically decreasing function)

iii) $f(x)$ be integrable for $x > 1$ such that $f(n) = u_n$ i.e. $f(n)$ is associated with series $\sum u_n$.

Then $\sum f(n)$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent and $\sum f(n)$ is divergent if and only if $\int_1^\infty f(x)\,dx$ is divergent.

Proof: Since f is a decreasing function on $[1,\infty[$, we have
$f(n) \leq f(x) \leq f(n-1)$ for $n-1 \leq x \leq n, n = 2,3.....$

Consequently, $\int_{n-1}^{n} f(n)\,dx \leq \int_{n-1}^{n} f(x)dx \leq \int_{n-1}^{n} f(n-1)dx$

i.e. $f(n) \le \int_{n-1}^{n} f(x)dx \le f(n-1)$ for $n = 2,3.....$

Thus, $\sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx \le \sum_{k=2}^{n} f(k-1),$

But, $\sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx = \int_{1}^{n} f(x)dx$ and $\sum_{k=2}^{n} f(k-1) = \sum_{k=1}^{n-1} f(k).$

Therefore, for $n \ge 2,$

$\sum_{k=2}^{n} u_k \le \int_{1}^{n} f(x)dx \le \sum_{k=1}^{n-1} u_k$ i.e. $s_n - u_1 \le \int_{1}^{n} f(x)dx \le s_n - u_n$ where (s_n) denotes the sequence of partial sums of the series $\sum u_n$. Therefore,

$$u_n \le s_n - \int_{1}^{n} f(x)dx \le u_1$$

If we write $A_n = s_n - \int_{1}^{n} f(x)dx,$ we have

$$A_{n+1} - A_n = (s_{n+1} - s_n) - \left(\int_{1}^{n+1} f(x)dx - \int_{1}^{n} f(x)dx \right)$$

$$= u_{n+1} - \int_{n}^{n+1} f(x)dx \le 0$$

Therefore, $A_{n+1} \le A_n \forall n.$ Thus the sequence (A_n) is monotonically decreasing sequence. Also $A_n \ge u_n \ge 0 \forall n,$ therefore the sequence (A_n) is bounded below. Consequently (A_n) is convergent.

Now $s_n = A_n + \int_{1}^{n} f(x)dx$

The convergence of (A_n) implies that (s_n) and $\left(\int_{1}^{n} f(x)dx \right)$ converge or diverge together. Hence $\sum u_n$ and $\int_{1}^{\infty} f(x)dx$ converge or diverge together. ■

Remark 3: You may note that if the conditions of Cauchy's Integral Test are satisfied for $x \ge k$ (a positive integer), then $\sum_{n=k}^{\infty} u_n$ and $\int_{k}^{\infty} f(x)dx$ converge or diverge together. This can be seen from the following example:

Example 6: Discuss the convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ by using the Integral Test.

Solution: Here $u_n = \frac{1}{n^p}$

Let $f(x) = \frac{1}{x^p}$

For $p > 0, f$ is decreasing, positive integrable function. So by Cauchy's Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} f(x)dx$ converge or diverge together.

$$\int_1^x f(x)dx = \int_1^x \frac{dx}{x^p}$$

$$= \begin{cases} \log x & \text{if} \quad p = 1 \\ \dfrac{x^{1-p} - 1}{1-p} & \text{if} \quad p \neq 1 \end{cases}$$

$$\to \begin{cases} \infty & \text{if} \quad 0 < p \leq 1 \\ \dfrac{1}{p-1} & \text{if} \quad p > 1 \end{cases}$$

as $x \to \infty$

Therefore $\int_1^{\infty} f(x)\,dx$ converges for $p > 1$ and diverges for $0 < p \leq 1$ and hence the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p < 1$.

***

Example 7: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, where $p > 0$.

Solution: Let $f(x) = \frac{1}{x(\log x)^p}$ for $x > 2$.

If $p > 0$, then f is a positive, decreasing, integrable function on $[2, \infty]$. Hence by Cauchy's Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ and $\int_2^{\infty} \frac{1}{x(\log x)^p}$ converge or diverge together.

We have, for $p > 0$

$$\int_2^x \frac{dx}{x(\log x)^p} = \begin{cases} \log(\log x) - \log(\log 2) & \text{if} \quad p = 1 \\ \dfrac{(\log x)^{1-p} - (\log 2)^{1-p}}{1-p} & \text{if} \quad p \neq 1 \end{cases}$$

$$\to \begin{cases} \infty & \text{if} \quad p \leq 1 \\ \dfrac{(\log 2)^{1-p}}{p-1} & \text{if} \quad p > 1 \end{cases}$$

as $x \to \infty$

This shows that $\int_2^{\infty} f(x)\,dx$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Therefore the given series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

***

Example 8: Evaluate $\lim_{n\to\infty} \sum_{r=1}^{2n} \frac{n^2}{(2n+r)^3}$.

Solution: Let $S_{2n} = \sum_{r=1}^{2n} \frac{n^2}{(2n+r)} = \frac{1}{n}\sum_{r=1}^{2n} \left(\frac{1}{2+\frac{r}{n}} \right)^3$

Taken $f(x) = \frac{1}{(x+2)^3}$ where $x = \frac{r}{n}$

By Cauchy's integral test, we have

$$\lim_{n\to\infty} \sum_{r=1}^{2n} \frac{n^2}{(2n+r)^3} = \lim_{n\to\infty} \int_0^2 \frac{dx}{(x+2)^3}$$

$$= \left. \frac{-3}{(x+2)^2} \right|_0^2$$

$$= \frac{-3}{16}$$

***

Here are some exercises.

E7) Discuss the convergence of the series

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^p} (p > 0).$$

E8) Find $\lim_{n\to\infty} \sum_{r=1}^{3n} \frac{n}{(3n+r)^2}$.

With this we come to an end of this unit.

Let us summarise the points we have discussed in this unit.

16.4 SUMMARY

In this unit we have covered the following points.

1. We have covered the idea of primitive of a function. A function F is a primitive of f on an interval I if F is differentiable on I and $F' = f$.

2. We have explained how to find the primitive of a function for some standard function: polynomial functions, trigonometric functions, exponential functions and so on.

3. We stated and proved the fundamental Theorem of Calculus (Form I) and the statement.

4. We explained how to use FTC of Calculus to evaluate the integral.

5. We have defined the indefinite integral of a function defined on $[a,b]$ with respect to the point a. It is the function F defined by

$$F(z) = \int_a^a f(x)\,dx, \quad \forall\, z \in [a,b].$$

6. We stated the second form of the fundamental theorem of calculus which says that the indefinite F is differentiable at any point where f is continuous.

7. We state and proved the Mean Value Theorem for integrals of continuous functions. Give the statement here.

8. We have explained an integral test for testing the convergence of a series by associating an integral to the series.

16.5 SOLUTIONS/ANSWERS

E1) Define $F(x) = x\ln_e x - x.$ Then

$$F'(x) = x + \frac{1}{x} + \ln_e x - 1$$
$$= \ln_e x$$

Since $f(x) = \ln_e x$ is continuous on $[1, e]$, it is Riemann integrable. Therefore by the Fundamental Theorem, we have

$$\int_1^e \ln_e x\,dx = [x\ln_e x - x]_1^e$$
$$= (e - e) - (0 - 1) = 1.$$

E2) Define $G(x) = 2\sqrt{x}$ for $x \in [0, b]$, then G is continuous on $[0, b]$ and $G'(x) = 1\sqrt{x}$ for $x \in [0, b]$. Since $g = G'$ is not bounded on $[0, b]$, it does not belong to $\mathbb{R}[0, b]$ no matter how we define $g(0)$. Therefore the Fundamental Theorem 2 does not apply in this case.

E3) a) $F(x) = \int_a^x f$

$$= \int_a^c f + \int_c^x f$$

$= F(c) + G(x)$ [As by definition $F(c) = \int_a^c f$ and $G(x) = \int_c^x f$]

$\therefore G(x) = F(x) - F(c$

b) $H(x) = \int_a^b f = F(b)$ [As by definition of $F(x)$]

c) $F(\sin x) = \int_a^{\sin x} f$

$$= \int_a^x f + \int_x^{\sin x} f$$
$$= F(x) + S(x)$$

$\therefore S(x) = F(\sin x) - F(x).$

E4) $F(x) = \int_0^x f$

$$= \begin{cases} \dfrac{x^2}{2} & for\ \ 0 \le x < 1, \\ x & for\ \ 1 \le x < 2, \\ \dfrac{x^2}{2} & for\ \ 2 \le x \le 3 \end{cases}$$

It is clear that $F(x)$ is not continuous at $x = 1$ and

$$\lim_{x \to 2^-} \frac{F(x) - F(2)}{x-2} = \lim_{x \to 2^-} \frac{x-2}{x-2} = 1$$

$$\lim_{x \to 2^+} \frac{F(x) - F(2)}{x-2} = \lim_{x \to 2^+} \frac{\frac{x^2}{2} - 2}{x-2} = \lim_{x \to 2^+} \frac{\frac{1}{2}(x^2 - 4)}{x-2}$$

$$= \lim_{x+2^+} \frac{1}{2}(x+2) = \frac{1}{2} \times 4 = 2$$

So, $F'(x)$ does not exist at $x = 2$.

Therefore, $F(x)$ is differentiable in the whole interval $[0, 3]$ except at $x = 1, 2$.

$$F'(x) = \begin{cases} x & 0 \le x < 1 \\ 1 & 1 < x < 2 \\ x & 2 < x \le 3. \end{cases}$$

E5) $F_a() = \int_a f$

$= \int_a^c f + \int_c f, \ c \in [a, b]$

$= f(\xi).(c-a) + F_c()$ [From Theorem 6, we have $\int^c f = f(\xi)(c-a)$ where, $\xi \in [a, c]$

E6) Since, $f : [0,1] \to \mathbb{R}$ is continuous, the antiderivative of $f(x)$ exists. Let $F(x)$ be the antiderivative of $f(x)$. So, $f(x) = F(x)$ for all $x \in [0,1]$.

Now, $\int_0^x f = \int_x^1 f$ for all $x \in [0,1]$

$\Rightarrow F(x) - F(0) = F(1) - F(x)$ [Using Fundamental Theorem of Calculus]

$\Rightarrow 2F(x) = F(1) + F(0)$

So, $F(x) = \dfrac{F(1) + F(0)}{2}$ for all $x \in [0,1]$.

Therefore, $F(x)$ is a constant function for all $x \in [0,1]$

So, $F(x) = F'(x)$

$= 0$ for all $x \in [0,1]$.

E7) Let $f(x) = \dfrac{1}{x \log x (\log \log x)^p}, x \ge 3, \ p > 0$

If $p > 0$, then $f(x)$ is a positive, decreasing, integrable function on $[3, \infty]$. Hence by Cauchy's Integral Test,

$\sum_{n=3}^{\infty} \dfrac{1}{n \log n (\log \log n)^p}$ and $\int_3^{\infty} \dfrac{dx}{x \log x (\log \log x)^p}$ converge or diverge together.

We have, for $p > 0$,

$$\int_3^x \frac{dx}{x\log x(\log\log x)^p} = \begin{cases} \log(\log\log x) - \log(\log\log 3) & if \quad p = 1 \\ \dfrac{(\log\log x)^{1-p} - (\log(\log 3)^{1-p}}{1-p} & if \quad p \neq 1 \end{cases}$$

$$\to \begin{cases} \infty & if \quad p \leq 1 \\ \dfrac{(\log(\log 3)^{1-p}}{p-1} & if \quad p > 1 \end{cases}$$

This shows that $\int_3^\infty \frac{1}{x\log x(\log\log x)^p}$ $if \quad p > 1$ and diverges if $0 < p \leq 1$

Therefore, the given series $\sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^p}$ conveges if $p > 1$ and diverges if $0 < p \leq 1$.

E8) $\sum_{r=1}^{3n} \frac{n}{(3n-r)^2} = \frac{1}{n}\sum_{r=1}^{3n} \frac{1}{\left(3-\frac{r}{n}\right)^2}$

Take $f(x) = \frac{1}{(3-x)^2}$ where $x = \frac{r}{n}$

By Cauchy's integral test

$$\lim_{n\to\infty} \sum \frac{n}{(3n-r)^2} = \int_0^3 \frac{dx}{(3-x)^2}$$

$$= \frac{2}{(3-x)}$$