UNIT 10 FOURIER SERIES AND FOURIER TRANSFORM

Structure

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10.1 INTRODUCTION

In the last two units, we have introduced you to the theory of Lebesgue measure and Lebesgue integral. In this unit, we discuss Fourier series. We discuss how Lebesgue's integration theory helps to analyse certain functions by considering its Fourier series and also how we can get back the function from its Fourier series. In Sec. 10.2, we discuss the Fourier series of certain standard functions and also some methods for computing the Fourier coefficients. In Sec. 10.3, we discuss the Fourier sine and cosine series. Sec. 10.4, deals with the convergence of Fourier series. We have considered Cesaro summability and Abel summability and briefly touched upon pointwise convergence of Fourier series.

Objectives

After studying this unit, you should be able to

- define Fourier coefficients of periodic function;
- compute Fourier series of functions defined on $L^1[-\pi, \pi]$;
- explain Abel summability, Cesaro summability and point-wise convergence and establish the relationship between them;
- define Fourier transform for function in $L^1(\mathbf{R})$;
- state and prove Riemann Lebesgue lemma.

10.2 INTRODUCTION TO FOURIER SERIES

The theory of Fourier Series is concerned mainly with the analysis of periodic functions. We begin with some preliminary results on periodic function.

We first recall the definition of periodic function

Definition 1: Let $\alpha > 0$ be a real number. A function $f : \mathbf{R} \to \mathbf{R}$ is periodic with period α, or α-periodic, if we have $f(x + \alpha) = f(x)$ for every real number x.

For example the real valued functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are 2π-periodic, so also is the complex valued function e^{ix}. These functions are also

4π-periodic, 6π-periodic, etc. The function $f(x) = x$, however, is not periodic. The constant function $f(x) = 1$ is α-periodic for every α.

We observe that if a function f is α-periodic then $f(x + k\alpha) = f(x)$ for every integer k. This can be verified by using induction for the positive k, and then use a substitution to convert the positive k result to a negative k result. The case $k = 0$ is of course trivial. In particular, if a function f is 1-periodic, then we have $f(x + k) = f(x)$ for every $k \in \mathbf{Z}$. Because of this, 1-periodic functions are sometimes also called **Z**-periodic.

For example, for any integer n, the functions $\cos(2\pi nx); \sin(2\pi nx)$, are all **Z**-periodic.

The following is another example of a **Z**-periodic function. Let us consider the function f defined by

$$\begin{aligned} f(x) &= 1, \text{ when } x \in [n, n + \frac{1}{2}) \\ &= 0, \text{ when } x \in [n + \frac{1}{2}, n + 1) \end{aligned}$$

where n is any positive integer. Then f is **Z**-periodic function on **R**.

We also observe that

i) If f and g are **Z**-periodic functions, then $f + g, f - g$ and fg are periodic functions.

ii) If c is a real number, then cf is also periodic.

Now we are ready to discuss Fourier Series.

Definition 2: A Trigonometric series is a series of the form is

$$\frac{a_0}{2} + \sum_{n \in \mathbf{N}} a_n \cos nt + \sum_{n \in \mathbf{N}} b_n \sin nt, t \in \mathbf{R},$$

where the $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 1}$ are complex sequences.

It is easy to convert the nth partial sum

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt$$

of the trigonometric series into exponential form

$$s_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

by taking $b_0 = 0$, and for all $k \geq 0$,

$$c_k = \frac{a_k - ib_k}{2} \text{ and } c_{-k} = \frac{a_k + ib_k}{2} \tag{1}$$

To convert from exponential to trigonometric form, use $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$. We say that $\sum_{n \in \mathbf{Z}} c_n e^{int}$ converges (somehow) to s(t) if the symmetric partial sums $\sigma_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$ converges to s(t) as $n \to \infty$.

Definition 3: If $f \in L_1^r[-\pi, \pi]$, then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, n \in \mathbf{Z}, \tag{2}$$

is called the **nth Fourier Coefficient for f,** and $\sum_{n\in\mathbf{Z}} \widehat{f}(n)e^{int}$ is called the **Exponential (or Complex) form of the Fourier Series** for f.

Another name for $\widehat{f}(n)$ is the Finite Fourier Transform of f evaluated at n. Equivalently,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt, n \in \mathbf{N} \cup \{0\} \tag{3}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt, n \in \mathbf{N}, \tag{4}$$

are called the nth Fouricr Cosine and Sine Coefficients for f, respectively and

$$f(t) = \frac{a_0}{2} + \sum_{n\in\mathbf{N}} a_n \cos nt + \sum_{n\in\mathbf{N}} b_n \sin nt$$

is also called the Trigonometric form of the Fourier Series for f. The a_n and b_n are related to $\widehat{f}(n)$ by Eqns. (3) and (4).

Even though the exponential form provides the most symmetric, elegant formulation of Fourier series, we use mainly the trigonometric form.

Fourier series provide highly effective means to investigate periodic functions. Using them, you can dissect a sound into component parts called harmonics; the process is known as harmonic analysis. The $a_0/2$ is the neutral position, $a_1 \cos t + b_1 \sin t$ the fundamental tone, $a_2 \cos 2t + b_2 \sin 2t$ the first octave and so on. The amplitudes a_n and b_n, which go to 0 with increasing n, determine the importance of the overtones in toto. In essence, Fourier analysis of periodic phenomena enables reduction of the problem to determine the response to one harmonic at a time, then taking a limit of a linear combination (superposition) of these.

We compute some Fourier series for functions $f \in L_2^r[-\pi, \pi]$. Since

$$B = \left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{\sin nt}{\sqrt{\pi}}, \frac{\cos nt}{\sqrt{\pi}} : n \in \mathbf{N}\right\}$$

is an orthonormal basis for $L_2^r[-\pi, \pi]$, any $f \in L_2^r[-\pi, \pi]$ may be written as the infinite series of its projections,

$$f = \sum_{e\in B} \langle f, e\rangle e.$$

The terms of the series are of the form

$$\langle f, e\rangle e = \left\langle f, \frac{\cos nt}{\sqrt{\pi}}\right\rangle \frac{\cos nt}{\sqrt{\pi}} \text{ or } \left\langle f, \frac{\sin nt}{\sqrt{\pi}}\right\rangle \frac{\sin nt}{\sqrt{\pi}}$$

Example 1: Let us compute the Fourier series for the step function

$$f(t) = \begin{cases} 0, & -\pi \le t < 0 \\ 1, & 0 \le t < \pi \end{cases}$$

By Eqn. 3, its cosine coefficients are $a_0 = 1$ and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt = \frac{1}{\pi}\int_0^{\pi}\cos ntdt = 0, n \in \mathbf{N}$$

Its sine coefficients are

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nt\, dt = \frac{1 - \cot n\pi}{n\pi} = \frac{1}{n\pi}(1 - (-1)^n), n \in \mathbf{N}.$$

Therefore the Fourier series of f is

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n\in\mathbf{N}}(1 - (-1)^n)\frac{\sin nt}{n} = \frac{1}{2} + \frac{2}{\pi}\sum_{n\in\mathbf{N}}\frac{\sin(2n-1)t}{2n-1} \tag{5}$$

* * *

Example 2: Let us find the Fourier series for

$$g(t) = \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

Solution: Since g is odd, the cosine coefficients a_n are all 0, whereas

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin nt\, dt = \frac{2}{n\pi}(1 - \cos n\pi$$

$$= \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

The Fourier series for g is therefore $\frac{4}{\pi}\sum_{n\in\mathbf{N}}\frac{\sin(2n-1)t}{2n-1}$.

* * *

Remark: There is nothing special about $[-\pi, \pi]$. If $[-p, p]$, $p > 0$, is any closed interval and if $f \in L_2^r[-p, p]$, the Fourier coefficients for f are given by

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}dt, n \in \mathbf{N}\cup\{0\} \tag{6}$$

and

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}dt, n \in \mathbf{N} \tag{7}$$

The Fourier series for f is

$$\frac{a_0}{2} + \sum_{n\in\mathbf{N}}\left(a_n\cos\frac{n\pi t}{p} + b_n\sin\frac{n\pi t}{p}\right) \tag{8}$$

Example 3: Let us find that the Fourier series for the function given by

$$f(t) = \begin{cases} -6 - t, & -6 \le t \le -3, \\ t, & -3 \le t \le 3, \\ 6 - t, & 3 \le t \le 6, \end{cases}$$

on $[-6, 6]$

Solution: Since f is odd, we need only compute the sine coefficients b_n; since $f(t)\sin[(n\pi t)/6]$ is even, we can double the expression in Eqn. (7) and integrate from 0 to 6:

$$
\begin{aligned}
b_n &= \frac{2}{6}\int_0^6 f(t)\sin\frac{n\pi t}{6}dt \\
&= \frac{1}{3}\int_0^3 t\sin\frac{n\pi t}{6}dt + \frac{1}{3}\int_3^6 (6-t)\sin\frac{n\pi t}{6}dt, n \in \mathbf{N}
\end{aligned}
$$

With $w = \pi t/6$,

$$b_n = \frac{12}{\pi^2}\int_0^{\pi/2} w\sin nwdw + \frac{12}{\pi^2}\int_{\pi/2}^{\pi}(\pi - w)\sin nwdw.$$

Integrating by parts and simplifying gives

$$b_n = \frac{24}{\pi^2 n^2}\sin\frac{n\pi}{2},$$

Therefore, the Fourier series is

$$\frac{24}{\pi^2}\left(\sin\frac{\pi t}{6} - \frac{1}{3^2}\sin\frac{3\pi t}{6} + \frac{1}{5^2}\sin\frac{5\pi t}{6} - \dots\right).$$

* * *

E1) Find the Fourier series for $f(t) = |t|$ on $[-\pi, \pi]$.

E2) Show that the Fourier series for $f(t) = t^2$ on $[-\pi, \pi]$ is
$$\frac{\pi^2}{3} + 4\sum_{n\in\mathbf{N}}\frac{(-1)^n\cos nt}{n^2}.$$

10.3 FOURIER SINE SERIES AND COSINE SERIES

As noted in earlier, if a square-summable function f is defined on an interval $[-p, p]$, its Fourier series is given by

$$\frac{a_0}{2} + \sum_{n\in\mathbf{N}}\left(a_n\cos\frac{n\pi t}{p} + b_n\sin\frac{n\pi t}{p}\right), \tag{9}$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}dt, n \in \mathbf{N}\cup\{0\}, \tag{10}$$

and

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}dt, n \in \mathbf{N} \tag{11}$$

If f is defined only on $[0, p]$, we can extend f to $[-p, p]$ as an even or as an odd function or, depending on f, neither. If we consider the odd extension f_0– taking f_0 on $[-p, 0]$ to be the negative of what f is on $[0, p]$ and taking its 2p-periodic extension–then all the Fourier cosine coefficients a_n are $0, n \in \mathbf{N} \cup \{0\}$, and Eqn.9 is a series consisting only of sines, a Fourier sine series for f:

$$\sum_{n \in N} \left(b_n \sin \frac{n\pi t}{p} \right)$$

Likewise, if f had been extended as an even function, then all the sine coefficients drop out, and we get a Fourier cosine series representation for f. We illustrate these half-range or half-interval expansions next.

Example 4: (Sine and Cosine Series): Expand the function

$$f(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ 0, & \frac{1}{2} < t < 1, \end{cases}$$

as (a) a Fourier sine series; (b) a Fourier cosine series; and (c) a Fourier series.

Solution:

a) **Fourier sine series**. The odd extension of f is

$$f_0(t) = \begin{cases} 0, & -1 < t < -\frac{1}{2} \\ -1, & -\frac{1}{2} < t < 0, \\ 1, & 0 < t < \frac{1}{2}, \\ 0, & \frac{1}{2} < t < 1, \end{cases}$$

on $(-1, 1)$. Consequently, all the cosine coefficients a_n are 0. Since $f_0(t) \sin nt$ is even,

$$b_n = \int_{-1}^{1} f_0(t) \sin n\pi t \, dt = 2 \int_0^{1/2} \sin n\pi t \, dt = \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right).$$

We get the following pattern

$$b_1 = \frac{2}{\pi}, b_2 = \frac{2}{\pi}, b_3 = \frac{2}{3\pi}, b_4 = 0, \ldots$$

Therefore, the Fourier sine series f is

$$\frac{2}{\pi} \left(\sin \pi t + \frac{2 \sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} + \frac{\sin 5\pi t}{5} + \frac{2 \sin 6\pi t}{6} + \ldots \right)$$

b) **Fourier cosine series**. The even extension of f is

$$f_e(t) = \begin{cases} 1, & t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ 0, & t \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \end{cases}$$

The sine coefficients b_n are all 0; $a_0 = 2\int_0^1 f_e(t)dt = 2\int_0^{1/2} dt = 1$ and

$$a_n = 2\int_0^1 f_e(t)\cos n\pi t\, dt = 2\int_0^{1/2} \cos n\pi t\, dt = \frac{2}{n\pi}\sin\frac{n\pi}{2}, n \in \mathbf{N}$$

The Fourier cosine series for f is therefore

$$\frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos \pi t}{1} - \frac{\cos 3\pi t}{3} + \frac{\cos 5\pi t}{5} - \ldots\right).$$

c) **Fourier series**. The periodic extension of f happens to be neither even nor odd with period $2p = 1$, so

$$a_n = \frac{1}{1/2}\int_{-1/2}^{1/2} f(t)\cos\frac{2n\pi t}{1}dt = 2\int_0^{1/2}\cos 2n\pi t dt, n \in \mathbf{N} \cup \{0\},$$

and

$$b_n \frac{1}{1/2}\int_{-1/2}^{1/2} f(t)\sin\frac{2n\pi t}{1}dt = 2\int_0^{1/2}\sin 2n\pi t dt, n \in \mathbf{N} =$$

This yields the Fourier series

$$1 + \frac{2}{\pi}\left(\sin 2\pi t + \frac{\sin 6\pi t}{3} + \frac{\sin 10\pi t}{5} + \frac{\sin 14\pi t}{7} + \ldots\right)$$

* * *

Next we shall prove the following result.

Theorem 1: (Integral Shift): If f is 2p-periodic and integrable over $[-p, p]$, then for any real numbers a and b,

$$\int_a^b f(t)dt = \int_{a+2p}^{b+2p} f(t)dt \text{ and } \int_{-p}^{p} f(t)dt = \int_a^{a+2p} f(t)dt.$$

Proof: For any a and b, with $s = t + 2p$ and the fact that $f(s - 2p) = f(s)$,

$$\int_a^b f(t)dt = \int_{a+2p}^{b+2p} f(s-2p)ds = \int_{a+2p}^{b+2p} f(s)ds,$$

which proves the first equality. Now replace a by $a - 2p$ and b by a to get

$$\int_{a+2p}^{b+2p} = \int_a^{a+2p}$$

then let $a = -p$. □

Note: The theorem above shows that if we integrate a 2p-periodic function over any interval of length 2p, we get the same value as over any other such interval. If f is continuous, it is easy to prove this by differentiating:

$$\frac{d}{dx}\int_x^{x+2p} f(t)dt = f(x + 2p) - f(x) = 0 \text{ for any } x,$$

from which it follows that $\int_x^{x+2p} f(t)dt$ is constant.

E3) Find the Fourier

a) cosine series for $f(t) = \sin t, 0 \leq t \leq \pi$.

b) cosine series for $f(t) = t^2, 0 \leq t \leq p$.

c) sine series for $f(t) = c$, a constant, for $0 \leq t \leq p$.

d) sine and cosine series for $f(t) = t, 0 < t < 2$.

Thus, we have seen that if a function is integrable (Lebesgue) on a compact space, say $[-p, p]$, then we can find the corresponding Fourier coefficients which gives rise to a Fourier series. Now the question is how we can get back the function f from Fourier series. This is essentially related to the convergence of Fourier Series. In the next section, we shall discuss this.

10.4 CONVERGENCE OF FOURIER SERIES

In this section, we shall discuss the convergence of Fourier series. From your Real Analysis course you already know that there are different types of convergence for a series of function (Refer Block 5, MTE-09). Here we shall consider the pointwise convergence of Fourier series. Recall that a series of functions is pointwise convergent if the sequence of partial sums is convergent.

So we are interested in the points at which the sequence of partial sums converge.

We have defined the Fourier coefficients of f as

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta. \tag{12}$$

These are well defined for each continuous function on $[-\pi, \pi]$, or more generally. The Fourier series of f is the series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}. \tag{13}$$

Associated with the series is the sequence of its partial sums

$$S_N(f; \theta) = \sum_{n=-N}^{N} \widehat{f}(n) e^{in\theta}, \tag{14}$$

$N = 0, 1, 2, \ldots$. If at a point θ of $[-\pi, \pi]$ the sequence (11) converges we say that the Fourier series (10) converges at θ (in usual sense). It would have been nice if such convergence did take place at every point θ. Unfortunately, this is not the case. There are continuous functions f for which the series (10) diverges for uncountably many θ. Now we can proceed in two directions:

1) Weaken the notion of convergence, or

2) Strengthen the conditions of f.

In the first direction, we will see that the Fourier series of every continuous function converges in the sense of Abel summability and Cesaro summability, both of which are weaker notions than pointwise convergence of the sequence (11). In the second direction we will see that if f is not only continuous but differentiable then the series (10) is convergent at every point θ to the limit $f(\theta)$.

Abel Summability and Cesaro Summability

Consider any series, with real or complex terms $x_n, n = 1, 2, \ldots$:

$$\sum_{n=1}^{\infty} x_n. \tag{15}$$

If for every real number $0 < r < 1$ the series $\sum_{n=1}^{\infty} r^n x_n$ converges and if $\sum_{n=1}^{\infty} r^n x_n$ approaches a limit L as $r \to 1$, then we say that the series (13) is **Abel summable and its Abel limit is L.**

For example the series Σx_n where x_n is alternatively $+1$ and -1 is abel summable. Then it can be noted that if the series (12) converges in the usual sense to L, then it is also Abel summable to L. In the light of this we can state the following theorem:

Theorem 2: If the Fourier series of a continuous function defined on $[-\pi, \pi]$ converges in the usual sense to L, then it is Abel summable to L.

Now we shall state the following theorem, the proof of which is omitted.

Theorem 3: If f is a continuous function on $[-\pi, \pi]$, then its Fourier series is Abel summable and has Abel limit $f(\theta)$ at every θ.

Next we shall consider Cesaro summability. We define it as follows.

For the series in (12) let

$$s_N = \sum_{n=1}^{N} x_n, N = 1, 2, \ldots$$

be the sequence of its partial sums. Consider the averages of these partial sums:

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n}.$$

If the sequence $\{\sigma_n\}$ converges to a limit L as $n \to \infty$, then we say that the series (15) is **summable to L in the sense of Cesaro, or is Cesaro summable to L.** This is sometimes also called **(C, 1) summability** or summability by the method of the first arithmetic mean.

Why don't you look at the connection between Abel and Cesaro summability? We leave it as exercises for you to try.

As in the earlier case, we have the following theorem, the proof of which is omitted.

Theorem 4: If f is continuous on $[-\pi, \pi]$, then its Fourier Series is $(C, 1)$ summable.

Try these exercises now.

E4) If the Fourier series of a function f is Cesaro summable, then show that is is abel summable.

Next we shall consider pointwise convergence. Before we discuss the convergence theorem, we shall discuss an important theorem called Riemann Lebesgue Lemma. The proof of the theorem is based on another basic result called Weierstrass approximation which you must have studied at undergraduate level.

By the Weierstrass approximation theorem every continuous function on $[-\pi, \pi]$ is a uniform limit of exponential polynomials. In other words if f is a continuous function on $[-\pi, \pi]$ then for every $\epsilon > 0$ there exists an exponential polynomial $P_N(t)$ given by

$$P_N(t) = \sum_{n=-N}^{N} a_n e^{int} \tag{16}$$

such that

$$\sup_{-\pi \le t \le \pi} |f(t) - P_N(t)| \le \epsilon. \tag{17}$$

This can be used to prove:

Theorem 5: (The Riemann-Lebesgue Lemma): If f is a continuous function on $[-\pi, \pi]$ then

$$\sum_{|n| \to \infty} \widehat{f}(n) = 0.$$

Proof: We want to show that given an $\epsilon > 0$ we can find an N such that for all $|n| > N$ we have $|\widehat{f}(n)| < \epsilon$. Choose p_N to satisfy (13) and (14). Note that for $|n| > N, \widehat{p}(n) = 0$. hence for $|n| > N$ we have

$$\widehat{f}(n) = \widehat{f}(n) - \widehat{p}(n) = (f - p)\widehat{\ }(n).$$

But from (13) we get

$$|(f - p)\widehat{\ }(n)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} [f(t) - p(t)] e^{-int} dt \right| \qquad \le \frac{1}{2\pi} \epsilon 2\pi = \epsilon.$$

□

Pointwise Convergence of Fourier Series

There are several theorems that ensure convergence of the Fourier series of f when f satisfies some conditions stronger than continuity.

Before we go further we look at the convergence criteria. We note that the Fourier series is pointwise convergent to θ if the sequence of its partial sums given in Eqn.(11)

$$S_N(f; \theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}, N = 0, 1, 2, \ldots$$

converge to θ.

Now we obtain an integral representation of each S_N.

For each integer n let $e_n(t) = e^{int}$. If f is a continuous function on $[-\pi, \pi]$ then

$$\begin{aligned}(f * e_n)(\theta) &= \int_{-pi}^{\pi} f(t)e_n(\theta - t)dt \\ &= e^{in\theta} \int_{-\pi}^{\pi} f(t)e^{-int}dt \\ &= 2\pi\hat{f}(n)e^{in\theta}\end{aligned}$$

Hence, we can write the partial sums as

$$S_N(f;\theta) = \frac{1}{2\pi} \sum_{n=-N}^{N} (f * e_n)(\theta) = (f * D_N)(\theta), \qquad (18)$$

where

$$D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}. \qquad (19)$$

The expression $D_N(t)$ is called the Dirichlet kernel. Recall that we have already defined the convolution operator* in Unit 9. Accordingly, we have

$$S_N(f;\theta) = \int_{-\pi}^{\pi} f(\theta - t)D_n(t)dt. \qquad (20)$$

The Dirichlet kernel has the following properties

i) $D_N(-t) = D_N(t)$

ii) $\int_{-\pi}^{\pi} D_N(t)dt = 1$

iii) $D_N(t) = \frac{1}{2\pi} \frac{\sin\left(N + \frac{1}{2}\right)t}{\sin t/2}.$

Now we state the following theorem.

Definition 4: Let f be a continuous function defined on $[-\pi, \pi]$. Then we say that f satisfies Lipschitz continuous at θ if there exists a constant M and a $\delta > 0$ such that

$$|f(\theta) - f(t)| < M|\theta - t| \quad \text{if } |\theta - t| < \delta. \qquad (21)$$

This condition is stronger than continuity but weaker than differentiability of f at θ.

Theorem 6: Let f be an integrable function on $[-\pi, \pi]$. If f is Lipschitz continuous at θ then

$$\lim_{n\to\infty} S_n(f;\theta) = f(\theta)$$

Proof: We want to prove

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(\theta - t)D_n(t)dt = f(\theta).$$

Using the property (ii) of the Dirichlet kernel, this amounts to showing

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} [f(\theta - t) - f(\theta)]D_n(t)dt = 0.$$

Choose δ and M to satisfy (21). Using property (iii) of $D_n(t)$, we observe that if $|t| < \delta$ then for all n

$$|[f(\theta - t) - f(\theta)]D_n(t)| \le \frac{1}{\pi} M \left|\frac{t/2}{\sin t/2}\right|$$

Hence for $0 < \epsilon < \delta$ we have

$$\left|\int_{-\epsilon}^{\epsilon} [f(\theta - t) - f(\theta)]D_n(t)dt\right| \le C\epsilon$$

for some constant C. Then applying the fact that f is integrable and using the properties of $D_n(t)$ we get the desired result. □

So far we have been dealing with Fourier series for functions defined on $[-\pi, \pi]$. In the next we shall consider functions defined on **R**. More precisely we will consider functions in $L^1(\mathbf{R})$ and discuss the Fourier's theory in this case.

E5) Let f be a continuous function on $[-\pi, \pi]$. Show that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos nt = 0.$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin nt = 0.$$

10.5 FOURIER TRANSFORM

In this section, we shall introduced you to notion of Fourier transform.

Recall from Unit 9 that $L^1(\mathbf{R})$ consists of all functions (measurable) such that

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

We begin with a definition.

Definition 5: Let $f \in L^1(\mathbf{R})$. **The Fourier transform of f** denoted by $\hat{f}$ is a function defined on **R** and is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \quad \omega \in \mathbf{R} \tag{22}$$

Note that $\hat{f}$ is well-defined since $f \in L^1(\mathbf{R})$.

We also note that the variable t and ω are used because of the wide applications of Fourier transform theory in electrical engineering. In electrical engineering t is referred as time and ω as frequency. To study the relationship between f and $\hat{f}$ under various operation we use the notation

$$f(t) \longrightarrow \hat{f}(\omega)$$

We first note that $\hat{f}$ is well defined for every $\omega \in \mathbf{R}$.

Remark: You recall that on the interval $[-\pi, \pi]$, $\int_{-\pi}^{\pi} f(x)e^{-inx}dx_2$ called Fourier coefficients denoted by $\hat{f}(n)$.

Now we discuss some properties of Fourier transform.

Theorem 7: Let $f \in L^1(\mathbf{R}), \hat{f}$ is continuous on $\mathbf{R}$.

Proof: We shall use the sequential definition of continuity. Let $\{\omega_n\}$ be a sequence in $\mathbf{R}$ such that $\omega_n \longrightarrow w$ in $\mathbf{R}$. Then

$$\begin{aligned}|\hat{f}(\omega_n) - \hat{f}(\omega)| &= \left|\int_{-\infty}^{\infty} f(t)(e^{-it\omega n} - e^{it\omega})dt(t)\right| \\ &\leq {}^{\infty}_{-\infty}|f(t)||e^{-t\omega n} - e^{it\omega}|dt(t)\end{aligned}$$

Now $|f(t)||e^{-i\omega nt} - e^{i\omega t}| \leq 2|f(t)|$

$$\begin{aligned}(\because |e^{i\omega nt} - e^{i\omega t}| &\leq |e^{i\omega nt}| + |e^{i\omega t}| \\ &= 1 + 1 = 2)\end{aligned}$$

Also as $n \longrightarrow \infty, \omega_n \longrightarrow \omega$

Hence $|f(t)||e^{i\omega nt} - e^{i\omega t}| \longrightarrow 0 \forall x$.

$\therefore$ By dominated convergence theorem.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x)||e^{i\omega nt} - e^{i\omega t}|dt(t) = \int_{-infty}^{\infty} \lim_{n\to\infty} |f(t)||e^{i\omega nt} - e^{i\omega x}|dt(t)$$

$$0$$

$\therefore \lim_{n\to\infty} |\hat{f}(\omega_n) - \hat{f}(\omega)| = 0$

$\therefore \hat{f}(\omega n) \longrightarrow \hat{f}(\omega)$ as $\omega_n \longrightarrow \omega$

$\therefore \hat{f}$ is continuous. □

Theorem 8: Suppose $f \in L^1(\mathbf{R}), \alpha, \lambda \in \mathbf{R}$.

a) If $g(x) = f(x)e^{i\alpha x}$, then $\hat{g}(\omega) = \hat{f}(\omega - \alpha)$

b) If $g(x) = f(x - \alpha)$, then $\hat{g}(\omega) = \hat{f}(\omega)e^{-i\alpha\omega}$

c) If $g \in L^1(\mathbf{R})$ and $h = f * g$, then $\hat{h}(\omega) = \hat{f}(\omega).g(\omega)$

d) If $g(x) = \overline{f(-x)}$ then $\hat{g}(\omega) = \overline{\hat{f}(\omega)}$

e) If $g(x) = f(x/\lambda)$ and $\lambda > 0$ then $\hat{g}(\omega) = \lambda\hat{f}(\lambda\omega)$.

Theorem 9: **(Inversion Theorem):** Let $\hat{f} \in L^1(\mathbf{R})$. Then for almost all $x \in \mathbf{R}$,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f} e^{-i\omega t} d\omega$$

The proof of the theorem is omitted. This is one of the important theorems in Fourier Analysis. This explains how we can recover f from its Fourier transform.

You can try an exercise now.

E6) Prove Theorem 8.

Fourier series has lot of applications to many branches of mathematics. One such branch is Number Theory which the German Mathematician, Hermann Weyl used Fejer's convergence theorem for Fourier series to prove a beautiful theorem in number theory, called the Weyl Equidistribution Theorem. Every real number x can be written as $x = [x] + \tilde{x}$, where $[x]$ is the integral part of x and is an integer, $\tilde{x}$ is the fractional part of x and is a real number lying in the interval
$[0, 1)$. Weyl's theorem says that if x is an irrational number then, for large N, terms of the sequence $\tilde{x}, (2x)^{\sim}, \ldots, (Nx)^{\sim}$ are scattered uniformly over $(0, 1)$.

One of the areas where Fourier series and transforms have major applications, is crystallography. In 1985 the Nobel Prize in Chemistry was given to Hauptman and Karle who developed a new method for calculating some crystallographic constants from their Fourier coefficients which can be inferred from measurements. Two crucial ingredients of their analysis are Weyl's equidistribution theorem and theorems of Toeplitz on Fourier series of nonnegative functions.

Fast computers and computations have changed human life in the last few decades. One of the major tools in these computations in the Fast Fourier Transform introduced in 1965 by J. Cooley and J. Tukey in a short paper titled an algorithm for the machine calculation of complex Fourier series. Their idea reduced the number of arithmetic operations required in calculating a Fourier transform. In 1993 it was estimated that nearly half of all supercomputer central processing unit time was used in calculating Fast Fourier Transform – used even for ordinary multiplication of large numbers.

Another major advance in the last few years is the introduction of wavelets. Here functions are expanded not in terms of Fourier series, but in terms of some other orthonormal bases that are suited to faster computations. This has led to new algorithms for signal processing and for numerical solutions of equations.

If conduction of heat is related to the theory of numbers, and if theorems about numbers are found useful in chemistry, this story has a moral. The boundary between deep and shallow may be sharper than that between pure and applied. [Extract from the book "Fourier Series" by Rajendra Bhatia.]

With this we come to an end of this unit.

10.6 SUMMARY

In this unit, we have covered the following points:

1. We have defined Fourier coefficient for functions in $L^1[-\pi, \pi]$ and discussed the corresponding Fourier series.
2. We have introduced Able summability, Cesaro summability and point-wise convergence of a Fourier series and discussed their interrelationship.
3. We have defined Fourier transform for functions in $L^1\mathbf{R}$ and discussed some properties of it.
4. We have stated Fourier inversion formula.

10.7 HINTS/SOLUTIONS

E1) Since f is even, its sine coefficients b_n are 0 for every $n \in \mathbf{N}$. The cosine coefficients are

$$a_n = \frac{2}{\pi}\int_0^\pi t\cos nt\, dt.$$

Integrating by parts, we obtain

$$a_n = \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} \frac{-4}{\pi n^2}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

and $a_0 = (2/\pi)\int_0^\pi t\, dt = \pi$.

Therefore, the Fourier series for $|t|$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2}\right).$$

E2) The Fourier coefficients are given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2\pi^2}{3} \text{ and } a_n = \frac{1}{\pi}\int_{-pi}^{\pi} t^2\cos nt\, dt \text{ for } n \in \mathbf{N}.$$

Integrating the latter by parts gives

$$a_n = \frac{-2}{n\pi}\int_{-\pi}^{\pi} t\sin nt\, dt.$$

Another integration by parts yields

$$a_n = \frac{2}{n^2\pi} t\cos nt|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos nt\, dt = \frac{4(-1)^n}{n^2}.$$

Since f is even, each b_n is 0. Thus, the Fourier series for f(t) is

$$\frac{\pi^2}{3} + 4\sum_{n\in\mathbf{N}}\frac{(-1)^n\cos nt}{n^2}$$

E3) **Hint:** The cosine series can be computed as

a) $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2t}{3}+\frac{\cos 4t}{15}+\frac{\cos 6t}{35}+\dots\right)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\in N}\frac{\cos 2nt}{4n^2-1}$.

Note that this returns $|\sin t|$.

b) $\frac{p^2}{3}-\frac{4p^2}{\pi^2}\left(\cos\frac{\pi t}{p}-\frac{1}{4}\cos\frac{2\pi t}{p}+\frac{1}{9}\cos\frac{3\pi t}{p}-\dots\right)$.

c) $\frac{4c}{\pi}\left(\sin\frac{\pi t}{t}+\frac{1}{3}\sin\frac{3\pi t}{p}+\frac{1}{5}\sin\frac{5\pi t}{p}-\dots\right)$.

d) cosine series : $1-\frac{8}{\pi^2}\left(\cos\frac{\pi t}{2}+\frac{1}{3^2}\cos\frac{3\pi t}{2}+\frac{1}{5^2}\cos\frac{5\pi t}{2}+\dots\right)$.

E4) **Hint:** Apply the definition.

E5) **Hint:** Apply the definition.

E6) **Hint:** Apply Riemann Lebesgue lemma.

E7) **Hint:** We shall prove this by applying sequential convergence.

E8) a) **Hint:** Follows directly from the definition of the Fourier transform.

b) **Hint:** Apply the definition of Fourier transform and the relevant property of Lebesgue integral.

c) Let $g\in L^1(\mathbf{R}), h=f*g$

$$\begin{aligned}\hat{h}(\omega) &= \int_{-\infty}^{\infty}(f*g)(x)e^{ix\omega}dm(x)\\ &= \int_{-\infty}^{\infty}e^{-ix\omega}dm(x)\int_{-\infty}^{\infty}f(x-y)g(y)dm(y)\\ &= \int_{-\infty}^{\infty}g(y)e^{i\omega y}dm(y)\int_{-\infty}^{\infty}f(x-y)e^{i\omega(x-y)}dm(x)\\ &= \hat{g}(\omega)\hat{f}(\omega)\end{aligned}$$

d) If $g(x)=\overline{f(-x)}$ then $\hat{g}(\omega)=\overline{\hat{f}(\omega)}$

$$\begin{aligned}\text{For } \hat{g}(\omega) &= \int_{-\infty}^{\infty}g(x)e^{\cdot\, ix\omega}dm(x) \quad (\omega\in\mathbf{R})\\ &= \int_{-\infty}^{\infty}\overline{f(-x)}e^{\ ix\omega}dm(x)\\ &= \int_{-\infty}^{\infty}\overline{f(-x)e^{ix\omega}}dm(x)\\ &= \int_{-\infty}^{\infty}\overline{f(y)e^{-iy\omega}}dm(y) \quad (\text{where } y=-x)\\ &= \overline{\hat{f}(\omega)}\end{aligned}$$

e) **Hint:** Follows directly from the definition of the Fourier transform.