8.1 INTRODUCTION

In this unit, we shall introduce you to the notion of Lebesgue measure which forms a basis for the study of integration theory developed by the French Mathematician L. Lebesgue.

As we already mentioned in the block introduction, the basic question with regard to integration is how to extend the notion of length to sets which are not necessarily intervals. That means, if possible, to every subset $E \subseteq \mathbb{R}$ we would like to associate a non-negative number $m(E)$, which has all its basic features of ‘length’. We shall call it the measure of $E$.

In this unit, we shall introduce you to the notion of measure of a set. We shall begin the discussion by defining $\sigma$-algebra of set which we shall cover in Sec. 8.2. Thus, we shall define the notion of outermeasure which can be applied to every subset of $\mathbb{R}$. Though the outermeasure has almost all reasonable properties that the length function have, it does not possess an important property of countable additivity which we shall explain in this section. This is taken care of by putting some extra conditions of outermeasure.

In Sec. 8.3, we shall define Lebesgue measure of a set using the idea of outmeasure and discuss some basic properties. You will see that the measure of an interval is equal to the length of the interval so that the definition of measure agrees with the length function.

In Sec. 8.4, we introduce you to the notion of Lebesgue measurable functions. You will see that the standard functions like continuous functions are Lebesgue measurable. Throughout this block we shall be considering the extended real line $\mathbb{R}_\infty$—the set of real number with $+\infty$ and $-\infty$ added to it. According to this the functions, we will be considering will be extended real-valued function.

Objectives

After studying this unit, you should be able to

- identify whether a class of subsets of set $X$ is a $\sigma$-algebra or not;
- find the outer measure of a set;
- check whether a set is measurable or not;
- check whether a function is measurable or not.
8.2 OUTER MEASURE

From your earlier courses on calculus you are already familiar with the fact that the area of an irregular closed shape can be measured by defining the concept of integral as the “Signed area of the region enclosed”. Later a more structured definition of integral was formulated by the concept of “Riemann integral”. You might have studied that the Riemann integral has certain limitations which makes it necessary to get the concept of integral redefined so that it will be applicable to more varied situations. You may note that the basic question while computing an integral of a function over a set is that how do we extend the idea of “length” to sets other than intervals. For that we first introduce to a collection of sets known as σ-algebra of subsets of \( \mathbb{R} \). In the next sub-section we shall discuss this.

8.2.1 Preliminaries

Here we make some definitions which hold for any non-empty set.

**Definition 1:** Let \( X \) be a non-empty set and \( \mathcal{F} \) be a collection of subsets of \( X \). The collection \( \mathcal{F} \) is called an algebra if it has the following property

\[
i) \quad A \cup B \in \mathcal{A} \text{ whenever } A, B \in \mathcal{A} \\
ii) \quad A^c \in \mathcal{A} \text{ whenever } A \text{ is in } \mathcal{A}
\]

Then it follows from the De Morgan’s law that \( A \cap B \in \mathcal{A} \) whenever \( A, B \in \mathcal{A} \) and \( \emptyset \) (the empty set), \( X \in \mathcal{A} \). Also by taking unions two at a time, we see that if \( A_1, A_2, \ldots, A_n \) are sets in \( \mathcal{A} \), then we have the following:

\[ A_1 \cup A_2, \ldots, \cup A_n \in \mathcal{A} \quad (1) \]

and

\[ A_1 \cap A_2, \ldots, \cap A_n \in \mathcal{A} \quad (2) \]

Now we shall prove some simple results regarding the collection \( \mathcal{F} \).

**Proposition 1:** Given any collection \( \mathcal{C} \) of subsets of \( X \), there is a smallest algebra \( \mathcal{A} \) which contains \( \mathcal{C} \), that is, there is an algebra \( \mathcal{A} \) containing \( \mathcal{C} \) and such that if \( \mathcal{B} \) is any algebra containing \( \mathcal{C} \), then \( \mathcal{B} \) contains \( \mathcal{A} \).

**Proof:** Let \( \mathcal{F} \) be the family of all algebras (of subsets of \( X \)) which contain \( \mathcal{C} \). Let \( \mathcal{A} = \cap \{ \mathcal{B} : \mathcal{B} \in \mathcal{F} \} \). Then \( \mathcal{C} \) is a subcollection of \( \mathcal{A} \), since each \( \mathcal{B} \) in \( \mathcal{F} \) contains \( \mathcal{C} \). Moreover, \( \mathcal{A} \) is an algebra. For if \( A \) and \( B \) are in \( \mathcal{A} \), then for each \( \mathcal{B} \in \mathcal{F} \) we have \( A \in \mathcal{B} \) and \( B \in \mathcal{B} \). Since \( \mathcal{B} \) is an algebra, \( A \cup B \) belongs to \( \mathcal{B} \). Since this is true for every \( \mathcal{B} \in \mathcal{F} \), we have \( A \cup B \) in \( \cap \{ \mathcal{B} : \mathcal{B} \in \mathcal{F} \} \). Similarly, we see that if \( A \in \mathcal{A} \), then \( A \in \mathcal{A} \). From the definition of \( \mathcal{A} \), it follows that if \( \mathcal{B} \) is an algebra containing \( \mathcal{C} \), then \( \mathcal{B} \supset \mathcal{A} \). \( \square \)

**Proposition 2:** Let \( \mathcal{A} \) be an algebra of subsets and \( \langle A_i \rangle \) a sequence of sets in \( \mathcal{A} \). Then there is a sequence \( \langle B_i \rangle \) of sets in \( \mathcal{A} \) such that \( B_n \cap B_m = \emptyset \) for \( n \neq m \) and

\[ \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i. \]
Proof: Since the proposition is trivial when \( (A_i) \) is finite, we assume \( (A_i) \) to be an infinite sequence. Set \( B_1 = A_1 \), and for each natural number \( n > 1 \) define

\[
B_n = A_n [A_1 \cup A_2 \cup \ldots \cup A_{n-1}]
= A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots \cap \tilde{A}_{n-1}.
\]

Since the complements and intersections of sets are in \( \mathcal{A} \), we have each \( B_n \in \mathcal{A} \). We also have \( B_n \subseteq A_n \). Let \( B_n \) and \( B_m \) be two such sets, and suppose \( m < n \). Then \( B_m \subseteq A_m \), and so

\[
B_m \cap B_n \subseteq A_m \cap B_n
= A_m \cap A_n \cap \ldots \cap \tilde{A}_m \cap \ldots
= (A_m \cap \tilde{A}_m) \cap \ldots
= \emptyset \cap \ldots
= \emptyset.
\]

Since \( B_i \subseteq A_i \), we have

\[
\bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} A_i.
\]

Let \( x \in \bigcup_{i=1}^{\infty} A_i \). Then \( x \) must belong to at least one of the \( A_i \)'s. Let \( n \) be the smallest value of \( i \) such that \( x \in A_i \). Then \( x \in B_n \), and so \( x \in \bigcup_{n=1}^{\infty} B_n \). Thus

\[
\bigcup_{n=1}^{\infty} B_n \supseteq \bigcup_{n=1}^{\infty} A_n,
\]

and we have

\[
\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.
\]

Now we make another definition.

**Definition 2:** Let \( X \) be a non-empty set. An algebra \( \mathcal{C} \) of subsets of \( X \) is called a \( \sigma \)-algebra if sequence \( \{A_i\}_{i \in \mathbb{N}} \) is a sequence of sets in \( \mathcal{C} \), then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{C} \).

A slight modification of the proof of Proposition 1 gives us the following proposition.

**Proposition 3:** Given any collection \( \mathcal{C} \) of subsets of \( X \), there is a smallest \( \sigma \)-algebra which contains \( \mathcal{C} \); that is, there is a \( \sigma \)-algebra \( \mathcal{A} \) containing \( \mathcal{C} \) such that if \( \mathcal{B} \) is any \( \sigma \)-algebra containing \( \mathcal{C} \) then \( \mathcal{A} \subseteq \mathcal{B} \).

You can note from Definitions 1 and 2, that every \( \sigma \)-algebra is an algebra. Can you think of an algebra which is not a \( \sigma \)-algebra?
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Let us consider some examples.

**Example 1**: Let \( X = \{a, b, c, d\} \). Consider the following

\[
\mathcal{G}_1 = \{X, \phi, \{d\}\}
\]

and

\[
\mathcal{G}_2 = \{X, \phi, \{d\}, \{a, b, c\}\}
\]

Then \( \mathcal{G}_1 \) is an algebra. It is also a \( \sigma \)-algebra. But \( \mathcal{G}_2 \) is not an algebra since \( \{a\}^C \notin \mathcal{G}_2 \).

**Example 2**: Let \( X \) be a non-empty set and let

\[
\mathcal{G}_1 = \mathcal{P}(X) = \{A : A \subset X\}, \text{ the power set of } X,
\]

\[
\mathcal{G}_2 = \{X, \phi\}
\]

Then you can verify that \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are \( \sigma \)-algebras. \( \mathcal{G}_2 \) is often called the trivial algebra.

**Example 3**: Let \( X \) be a non-empty set and let \( n(A) \) denotes the number of elements in the set \( A \). Now we define

\[
\mathcal{G}_3 = \{A \subset X : \text{ either } n(A) \text{ is finite or } n(A^C) \text{ is finite}\}
\]

Then we note that

i) \( X \in \mathcal{G}_3 \), since \( X^C = \phi \)

ii) \( A \in \mathcal{G}_3 \) implies that either \( n(A) \) is finite or \( n(A^C) \) is finite. If \( n(A) \) is finite, \( n((A^C)^C) = n(A) < \infty \). \( A^C \in \mathcal{G}_3 \). If \( n(A^C) < \infty \), then by definition \( A^C \in \mathcal{G}_3 \). This shows that in both cases \( A^C \in \mathcal{G}_3 \). That is if \( A \in \mathcal{G}_3, A^C \in \mathcal{G}_3 \).

iii) Let \( A, B \in \mathcal{G}_3 \). If either \( n(A) < \infty \) or \( n(B) < \infty \), then

\[
n(A \cap B) = \min\{n(A), n(B)\} < \infty.
\]

Thus, \( A \cap B \in \mathcal{G}_3 \). On the other hand if both \( n(A^C) < \infty \) and \( n(B^C) < \infty \) (note that here we are using the connector “and” not “or”), then

\[
n((A \cap B)^C) = |A^C \cup B^C| \leq |A^C| + |B^C| < \infty.
\]

which implies that \( (A \cap B)^C \in \mathcal{G}_3 \). Thus \( \mathcal{G}_3 \) is an algebra. Now we will show that \( \mathcal{G}_3 \) is not a \( \sigma \)-algebra. For that let us consider a set \( A \) with \( n(A) = \infty \). Let

\[
A = \{a_1, a_2, \ldots\} = \{a_i\}_{i=1}^{\infty}
\]

Then \( A \notin \mathcal{G}_3 \). For each \( i \), let \( A_i = \{a_i\} \). Then each \( A_i \in \mathcal{G}_3 \), but

\[
\bigcup_{i=1}^{\infty} A_i = A \notin \mathcal{G}_3.
\]

Therefore \( \mathcal{G}_3 \) is not a \( \sigma \)-algebra.
Now that we have seen some examples of $\sigma$-algebras, we want to see whether the intersection and union of two or more $\sigma$-algebras is a $\sigma$-algebra or not. From the definition it is clear that the intersection is a $\sigma$-algebra. But the union need not be a $\sigma$-algebra (Refer E?)

Next we shall see some methods for generating more $\sigma$-algebras.

For any given set $X$, we know that $\mathcal{P}(X)$-the power set of $X$ is a $\sigma$-algebra. This leads us to the following definition.

**Definition 3:** If $\mathcal{A}$ is a class of subsets of $X$, then the $\sigma$-algebra generated by $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is defined as

$$
\sigma(\mathcal{A}) = \bigcup_{\mathcal{C} \in \mathcal{I}(\mathcal{A})} \mathcal{C}
$$

where $\mathcal{I}(\mathcal{A}) = \{ \mathcal{C} : \mathcal{A} \subseteq \mathcal{C} \text{ and } \mathcal{C} \text{ is a } \sigma\text{-algebra} \}$ is the collection of all $\sigma$-algebras containing the class $\mathcal{A}$.

**Remark:** $\sigma(\mathcal{A})$ is the smallest $\sigma$-algebra containing $\mathcal{A}$

Note that, since the power set $\mathcal{P}(X)$ contains $\emptyset$ and is itself a $\sigma$-algebra, the collection $\mathcal{I}(\mathcal{A})$ is not empty and hence the intersection in the above definition is well-defined. For example, we can take $\mathcal{A}$ as the class of open sets in $\mathbb{R}$ or any in any metric space $X$. This $\sigma$-algebra was found very useful in the development of measure theory. In view of this, it is given a special name as given in the definition below.

**Example 4:** Let $\mathcal{A}$ be the class of open sets in a metric space $X$. Then $\sigma(\mathcal{A})$ is the $\sigma$-algebra generated by the open sets. This $\sigma$-algebra is called **Borel $\sigma$-algebra** and the elements of this $\sigma$-algebra are called **Borel sets**.

---

You can try some exercises now.

**E1)** Let $X = \{1, 2, 3\}$ and

$\mathcal{C}_1 = \{\emptyset, \{1\}, \{1, 2, 3\}, X, \emptyset\}, \mathcal{C}_2 = \{\{1, 2\}, \{3\}, X, \emptyset\}$. Check whether $\mathcal{C}_1$ and $\mathcal{C}_2$ are both algebras or not. Also check whether $\mathcal{C}_1 \cup \mathcal{C}_2$ is an algebra or not.

**E2)** Let $X$ be a non-empty set and let

$$
\mathcal{C} = \{A \subseteq X : A \text{ is countable or } A^c \text{ is countable} \}
$$

Then show that $\mathcal{C}$ is a $\sigma$-algebra.

---

Henceforth we shall consider the so-called extended real number system. This is obtained by adjoining to $\mathbb{R}$ the real number system, two additional elements $-\infty$ and $+\infty$ with the following calculus, i.e. $\mathbb{R}^\infty = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ This enlarged set is called the extended real number system and is denoted by $\mathbb{R}^\infty$.

For any real number $x$, we assume $-\infty < x < \infty$ and define the following

$$
x + \infty = \infty, \quad x - \infty = -\infty
$$
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\[ x \cdot \infty = \infty, \text{ if } x > 0. \]
\[ x \cdot (-\infty) = -\infty \text{ if } x > 0. \]

We also set
\[ \infty + \infty = \infty, -\infty - \infty = \infty \]
\[ \infty \cdot (\pm \infty) \cdot \pm \infty, -\infty \cdot (\pm \infty) = \mp. \]

The operation \( \infty - \infty \) is left undefined, but we shall adopt the apparently arbitrary convention that \( 0 \cdot \infty = 0 \). (This serves our purpose while defining integration.)

A function whose values are in the set of extended real numbers is called an extended real-valued function.

Now we are ready to define outer measure for subsets of \( \mathbb{R}^\infty \). In the next subsection we shall discuss this.

### 8.2.2 Outer Measure of a Set

We already mentioned earlier that the first step in extending the notion of integral is to extend the notion of length as a function to a large class of sets. For instance, we could define the "length" of an open set to be the sum of the lengths of the open intervals of which it is composed. Since the class of open sets is still too restricted for our purposes, we would like to construct a set function \( m \) which assigns to each set \( E \) in some collection \( \mathcal{C} \) of sets of real numbers a nonnegative extended real number \( m(E) \) called the measure of \( E \).

Ideally, we would like \( m \) to have the following properties:

1. \( m \) is defined for each set \( E \) of real numbers;
2. For an interval \( I \), \( m(I) = \ell(I) \) where \( \ell \) stands for length;
3. If \( \{E_n\} \) is a sequence of disjoint sets (for which \( m \) is defined) \( m(\cup E_n) = \sum mE_n \);
4. \( m \) is translation invariant; that is, if \( E \) is a set for which \( m \) defined and if \( E + y \) is the set \( \{x + y : x \in E\} \), obtained by replacing each point \( x \) in \( E \) by the point \( x + y \), then

\[ m(E + y) = mE. \]

Unfortunately, if we choose \( \mathcal{C} = \mathcal{P}(\mathbb{R}) \), the power set on \( \mathbb{R} \), as we shall see later, it is impossible to construct a set function having all four of these properties; and it is not known whether there is a set function satisfying the first three properties. Consequently, one of these properties must be weakened, and it is most useful to retain the last three properties and to weaken the first condition so that \( mE \) need not be defined for all sets \( E \) of real numbers. We want \( mE \) to be defined on as large a collection of sets as possible and will find it convenient to require the family \( \mathcal{C} \) of sets for which \( m \) is defined to be a \( \sigma \)-algebra.

**Definition 4:** Thus we shall say that \( m \) is a **countably additive measure** if it is a nonnegative extended real-valued function whose domain of definition is a \( \sigma \)-algebra \( \mathcal{C} \) of sets (of real numbers) and we have \( m(\cup E_n) = \sum mE_n \) for each sequence \( E_n \) of disjoint sets in \( \mathcal{C} \).
In the next section, we shall discuss the construction of \(\sigma\)-algebra of subsets of \(\mathbb{R}\) with a countably additive measure which is translation invariant and has the property that \(m(I) = \ell(I)\) for each interval \(I\).

We shall begin by introducing an outer measure for every subset of \(\mathbb{R}\). For each set \(A\) of real numbers consider the countable collections \(\{I_n\}\) of open intervals which cover \(A\), that is, collections for which \(A \subseteq \bigcup I_n\), and for each such collection consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, this sum is uniquely defined independently of the order of the terms. We define the outer measure \(m^* A\) of a set \(A\) to be the infimum of all such sums.

**Definition 5:** For any set \(A \subseteq \mathbb{R}\), we define \(m^*(A)\) as

\[
m^* A = \inf_{\mathcal{A} \subseteq \mathcal{I}_n} \sum \ell(I_n).
\]

where \(\{I_n\}\) is a collection of open intervals which covers \(A\).

It follows immediately from the definition of \(m^*\) that \(m^* \phi = 0\) and that if \(A \subset B\), then \(m^* A \leq m^* B\). Also each set consisting of a single point has outer measure zero. We establish two propositions concerning outer measure:

**Proposition 4:** The outer measure of an interval is its length.

**Proof:** We begin with the case in which we have a closed finite interval, say \([a, b]\). Since the open interval \((a - \epsilon, b + \epsilon)\) contains \([a, b]\) for each positive \(\epsilon\), we have \(m^*[a, b] \leq \ell(a - \epsilon, b + \epsilon) = b - a + 2\epsilon\). Since \(m^*[a, b] \leq b - a + 2\epsilon\) for each positive \(\epsilon\), we must have \(m^*[a, b] \leq b - a\). Thus we have only to show that \(\lambda^*[a, b] \geq b - a\). But this is equivalent to showing that if \(\{I_n\}\) is any countable collection of open intervals covering \([a, b]\), then

\[
\sum \ell(I_n) \geq b - a.
\]

By the Heine-Borel theorem (see Unit 3, Block 1), any collection of open intervals covering \([a, b]\) contains a finite subcollection which also covers \([a, b]\), and since the sum of the lengths of the finite subcollection is no greater than the sum of the lengths of the original collection, it suffices to prove the inequality (1) for finite collection \(\{I_n\}\) which cover \([a, b]\). Since \(a\) is contained in \(\bigcup I_n\), there must be one of the \(I_n\)'s which contains \(a\). Let this be the interval \((a_1, b_1)\). We have \(a_1 < a < b_1\). If \(b_1 \leq b\), then \(b_1 \in [a, b]\), and since \(b_1 \notin (a_1, b_1)\), there must be an interval \((a_2, b_2)\) in the collection \(\{I_n\}\) such that \(b_1 \notin (a_2, b_2)\); that is, \(a_2 < b_1 < b_2\). Continuing in this fashion we obtain a sequence \((a_1, b_1), \ldots, (a_k, b_k)\) from the collection \(\{I_n\}\) such that \(a_i < b_{i+1} < b_i\).

Since \(\{I_n\}\) is a finite collection, our process must terminate with some interval \((a_k, b_k)\). But it terminates only if \(b \in (a_k, b_k)\), that is, if \(a_k < b < b_k\). Thus

\[
\sum \ell(I_n) \geq \ell(a_k, b_k) = (b_k - a_k) + \ell(b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1) = b_k - a_k - (a_k - b_{k-1}) - (a_{k-1} - b_k) + \cdots + (a_1 - b_{k-1}) + a_{k-1} - a_1.
\]

since \(a_i < b_{i+1}\). But \(b_k - b_{i+1} = a_{i+1} - a_i\), and since we have \(b_k - a_1 > b - a\), whence

\[
\sum \ell(I_n) \geq \ell(a_k, b_k) = (b_k - a_k) - \ell(a_k - b_{k-1}) - \ell(a_{k-1} - b_k) + \cdots + \ell(a_1 - b_{k-1}) + a_{k-1} - a_1.
\]

This shows that \(m^* [a, b] \geq b - a\).
If $\ell$ is any finite interval, then, given $\epsilon > 0$, there is a closed interval $J \subseteq I$ such that $\ell(J) > \ell(I) - \epsilon$. Hence

$$\ell(I) - \epsilon < \ell(J) = m^*J \leq m^*I \leq \lambda^*I = \ell(I) = \ell(I).$$

Thus for each $\epsilon > 0$

$$\ell(I) - \epsilon < m^*I \leq \ell(I),$$

and so $m^*I = \ell(I)$.

If $I$ is an infinite interval, then given any real number $\Delta$, there is a closed interval $J \subseteq I$ with $\ell(J) = \Delta$. Hence $m^*I \geq m^*J = \ell(J) = \Delta$. Since $m^*I \geq \Delta$ for each $\Delta, m^*I = \infty = \ell(I)$. \qed

**Proposition 5:** The outer measure of a countable set is zero.

**Proof:** Let $E$ be a countable set.

Let $E = \{x_n\}_{n=1}^{\infty}$. For any $\epsilon > 0$, put $I_n = \left(x_n - \frac{\epsilon}{2n+1}, x_n + \frac{\epsilon}{2n+1}\right)$.

Then $\{I_n\}$ covers $E$. We have

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2n} = \epsilon \times \frac{1}{1 - \frac{1}{2}} = 2\epsilon \geq m^*(A) \text{ by definition of } m^*(A).$$

This is possible only if $m^*(A) = 0$.

Hence the claim. \qed

**Proposition 6:** Outer measure is countably subadditive, i.e. for any sequence of sets $\{E_n\}$,

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

(Note that both sides may be infinite here.)

**Proof:** If one of $E_n$ has an infinite outer measure $m^*$ then the inequality is trivial. So assume $m^*(E_n) < \infty, \forall n$.

**Step 1:** Let us prove first a simpler statement:

$$m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2).$$

Take an $\epsilon > 0$ and we show an even easier inequality

$$m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2) + \epsilon$$

This is however sufficient because taking $\epsilon = \frac{1}{n}$ and letting $n \to \infty$ we get what we need.
So for any $\epsilon > 0$ we find covering sequences $(I_k^n)$ for $E_1$ and $(I_k^m)$ for $E_2$ such that

$$\sum_{k=1}^{\infty} \ell(I_k^n) \leq m^*(E_1) + \frac{\epsilon}{2},$$

$$\sum_{k=1}^{\infty} \ell(I_k^m) \leq m^*(E_2) + \frac{\epsilon}{2},$$

hence, adding up,

$$\sum_{k=1}^{\infty} \ell(I_k^n) + \sum_{k=1}^{\infty} \ell(I_k^m) \leq m^*(E_1) + m^*(E_2) + \epsilon.$$

The sequence of intervals $(I_1^n, I_2^n, I_3^n, I_4^n, \ldots)$ covers $E_1 \cup E_2$, hence

$$m^*(E_1 \cup E_2) \leq \sum_{k=1}^{\infty} \ell(I_k^n) + \sum_{k=1}^{\infty} \ell(I_k^m),$$

which combined with the previous inequality gives the result.

**Step 2:** If the right-hand side is infinite, then the inequality is of course true. So, suppose that $\sum_{n=1}^{\infty} m^*(E_n) < \infty$. For each given $\epsilon > 0$ and $n \geq 1$ find a covering sequence $(I_k^n)_{k \geq 1}$ of $E_n$ with

$$\sum_{k=1}^{\infty} \ell(I_k^n) \leq m^*(E_n) + \frac{\epsilon}{2^n}.$$

The iterated series converges:

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \ell(I_k^n) \right) \leq \sum_{n=1}^{\infty} m^*(E_n) + \epsilon < \infty$$

and since all its terms are non-negative,

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \ell(I_k^n) \right) = \sum_{n,k=1}^{\infty} 1(I_k^n)$$

The system of intervals $(I_k^n)_{k,n \geq 1}$ covers $\bigcup_{n=1}^{\infty} E_n$, hence

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n,k=1}^{\infty} \ell(I_k^n) \leq \sum_{n=1}^{\infty} m^*(E_n) + \epsilon.$$
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**Example 6:** Let us compute the outer measure of \( \mathbb{R} \). Note that for any real number \( a > 0 \), \( (a,a) \subseteq \mathbb{R} \). \( m^*(\mathbb{R}) \geq m^*\{(a,a)\} = 2a \). If we let \( a \to \infty \), we get that \( m^*(\mathbb{R}) = \infty \).

Now you can think of \( m^*(\mathbb{R} \setminus \mathbb{Q}) \), where \( \mathbb{R}, \mathbb{Q} \) denotes the set of irrationals (see E3).

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**E3) Compute \( m^*(\mathbb{R} \setminus \mathbb{Q}) \).**

The outer measure we have defined agrees with the length function on intervals. One of the important properties which makes the length function a good measure is that it is countably additive that is \( m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n) \). But we have seen that in general the outer measure is not countably additive. It is only subadditive. To have this property, we identify the class of "nice sets" for which the outer measure is countably additive for pairwise disjoint sequence of sets \( \{E_n\} \) we have

\[
m^* \left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} m^*(E_n)
\]

This is achieved by identifying those sets \( E \) which split every other set \( A \) additively. We shall explain this in the next section.

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**8.3 LEBESGUE MEASURABLE SETS**

In this section, we shall define a Lebesgue measure and discuss its basic properties.

**Definition 6:** A set \( E \subseteq \mathbb{R} \) is (Lebesgue-) measurable if for every set \( A \subseteq \mathbb{R} \) we have

\[
m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)
\]

where \( E^c = \mathbb{R} \setminus E \). If \( E \) is measurable, then we define the Lebesgue measure of \( E \) to be \( m(E) = m^*(E) \); if \( E \) is not measurable, then we say that \( m(E) \) is undefined.

(E is measurable if it splits any subset \( A \) in such a way that the outer measure of \( A \) is the sum of the outer measures of the breakups.)

We obviously have \( A = (A \cap E) \cup (A \cap E^c) \), hence by Proposition 2 we have

\[
m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)
\]

for any sets \( A \) and \( E \). This implies that \( E \) is Lebesgue measurable if and only if the following inequality holds:

\[
m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \text{for all } A \subseteq \mathbb{R}
\]
You can verify that \( \mathbb{R} \) and \( \phi \) are measurable in the above sense! (see E 1).

We shall not explore the class of Measurable sets. We denote by \( m \) the class of measurable sets. Then we have \( \phi \in m \) and \( \mathbb{R} \in m \).

We make a definition now.

**Definition 7:** If \( E \) is a subset of \( \mathbb{R} \) such that \( m^* E = 0 \) then \( E \) is called a **null set**. Note that the empty set \( \phi \) is a Null set but a Null set need not be empty.

**Theorem 1:** The following results hold for \( m \).

i) Any null set is measurable.

ii) Any interval is measurable.

**Proof:** i) If \( N \) is a null set, then we have \( m^* (N) = 0 \). So for any \( A \subset \mathbb{R} \) we have

\[
m^* (A \cap N) \leq m^* (N) = 0 \quad \text{since} \quad A \cap N \subseteq N
\]

and adding together we have

\[
m^* (A) \geq m^* (A \cap N^c) + m^* (A \cap N).
\]

This shows that \( N \) is measurable.

ii) Let \( E = I \) be an interval. Suppose, for example, that \( I = [a, b] \). Take any \( A \subset \mathbb{R} \) and \( \varepsilon > 0 \). Find a covering of \( A \) with

\[
m^* (A) \leq \sum_{n=1}^{\infty} l(I_n) \leq m^* (A) + \varepsilon.
\]

Clearly the intervals \( I'_n = I_n \cap [a, b] \) cover \( A \cap [a, b] \) hence

\[
m^* (A \cap [a, b]) \leq \sum_{n=1}^{\infty} l(I'_n).
\]

The intervals \( I''_n = I_n \cap (-\infty, a) \) and \( I'''_n = I_n \cap (b, +\infty) \) cover \( A \cap [a, b]^c \) so

\[
m^* (A \cap [a, b]^c) \leq \sum_{n=1}^{\infty} \ell(I''_n) + \sum_{n=1}^{\infty} \ell(I'''_n).
\]

Putting the above three inequalities together we obtain that the inequality in (5) is true.

If \( I \) is unbounded, \( I = [a, \infty) \) say, then the proof is even simpler since it is sufficient to consider \( I'_n = I_n \cap [a, \infty) \) and \( I''_n = I_n \cap (-\infty, a) \).

**Theorem 2:** The following holds for \( \mathcal{M} \)

i) if \( E \in \mathcal{M} \) then \( E^c \in \mathcal{M} \),

ii) if \( E_1 \) and \( E_2 \) are measurable and \( E_1 \cap E_2 = \phi \), then \( E_1 \cup E_2 \) is measurable.
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iii) if $E_1$ and $E_2$ are measurable, then $E_1 \cap E_2$ is measurable.

**Proof:** i) Suppose $E \in \mathcal{M}$ and take any $A \subseteq \mathbb{R}$. We have to show that

$$m^*(A) = m^*(A \cap E^c) + m^*(A \cap (E^c)^c),$$

$$= m^*(A \cap E^c) + m^*(A \cap E), \text{ since } (E^c)^c = E$$

Hence the claim.

ii) Let $A \subseteq \mathbb{R}$. We have the condition for $E_1$:

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2^c) \quad (6)$$

Now, apply Eqn. 6 for $E_2$ with $A \cap E_1^c$ in place of $A$:

$$m^*(A \cap E_1^c) = m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c)$$

$$= m^*(A \cap (E_1^c \cap E_2)) + m^*(A \cap (E_1^c \cap E_2^c))$$

Since $E_1$ and $E_2$ are disjoint, $E_1^c \cap E_2 = E_2$. By De Morgan's law $E_1^c \cap E_2^c = (E_1 \cup E_2)^c$. We substitute and get

$$m^*(A \cap E_1^c) = m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c).$$

Substituting this into (Eqn.6) we get

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c). \quad (7)$$

Now by the subadditivity property of $m^*$ we have

$$m^*(A \cap E_1) + m^*(A \cap E_2) \geq m^*((A \cap E_1) \cup (A \cap E_2))$$

$$= m^*(A \cap (E_1 \cup E_2))$$

so Eqn.(6) gives

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

which is sufficient for $E_1 \cup E_2$ to belong to $\mathcal{M}$.

Finally, put $A = E_1 \cup E_2$ in (Eqn. 7) to get

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2),$$

which completes the proof.

iii) The proof is left as an exercise(see $E_2$).

\[ \square \]

**Theorem 3:** If $E_n \in \mathcal{M}$, $n = 1, 2, \ldots$ and $E_j \cap E_k = \emptyset$ for $j \neq k$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{M} \text{ and we have}$$

$$m^* \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m^*(E_n). \quad (8)$$

**Proof:** We begin as in the proof of Theorem 2(ii). Then we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2^c)$$

$$+ m^*(A \cap (E_1 \cup E_2)^c)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c)$$
(see Eqn. 7) and after n steps we expect

\[ m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^* \left( A \cap \left( \bigcup_{k=1}^{n} E_k \right)^c \right) \]  

(9)

Let us demonstrate this by induction. The case \( n = 1 \) is the first line above. Suppose that

\[ m^*(A) = \sum_{k=1}^{n-1} m^*(A \cap E_k) + m^* \left( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \right). \]  

(10)

Since \( E_n \in \mathcal{M} \), we may apply Eqn.(6) with \( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \) in place of A:

\[ m^* \left( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \right) = m^* \left( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \cap E_n \right) + m^* \left( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \cap E_n^c \right). \]  

(11)

Now we make the same observations

\[ \left( \bigcup_{k=1}^{n-1} E_k \right)^c \cap E_n = E_n \ (E_i \text{ are pairwise disjoint}), \]

\[ \left( \bigcup_{k=1}^{n-1} E_k \right)^c \cap E_n^c = \left( \bigcup_{k=1}^{n} E_k \right)^c \ (\text{by De Morgan's law}) \]

Inserting these into (11) we get

\[ m^* \left( A \cap \left( \bigcup_{k=1}^{n-1} E_k \right)^c \right) = m^* \left( A \cap E_n \right) + m^* \left( A \cap \left( \bigcup_{k=1}^{n} E_k \right)^c \right), \]

and inserting this into the induction hypothesis gives

\[ m^*(A) = \sum_{k=1}^{n-1} m^*(A \cap E_k) + m^*(A \cap E_n) + m^* \left( A \cap \left( \bigcup_{k=1}^{n} E_k \right)^c \right) \]

as required to complete the induction step. Thus Eqn.(9) holds for all \( n \) by induction. Since this holds for all \( n \), letting \( n \to \infty \) gives the desired result. \( \square \)

Next we shall show that the fact that \( E_n \)'s are pairwise disjoint is not necessary in order to ensure that their union belongs to \( \mathcal{M} \). However, to get the equality in (8), we need assumption that \( E_n \)'s are pairwise disjoint, which does not hold otherwise. We shall establish this in a number of steps. Let us prove the following theorem.

**Theorem 4:** If \( E_1, E_2 \in \mathcal{M} \), then \( E_1 \cup E_2 \in \mathcal{M} \) (not necessarily disjoint).

**Proof:** Here we give a sketch of the proof.

\[ m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c). \]  

(12)

Next, applying (12) with \( A \cap E_1^c \) in place of A, we get

\[ m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c). \]
We insert this into (12) to get
\[ m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c). \] (13)

By de Morgan's law, \( E_1^c \cap E_2^c = (E_1 \cup E_2)^c \) so (as before)
\[ m^*(A \cap E_1^c \cap E_2^c) = m^*(A \cap (E_1 \cup E_2)^c). \] (14)

By subadditivity of \( m^* \) we have
\[ m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) \geq m^*(A \cap (E_1 \cup E_2)). \] (15)

Inserting (15) and (14) into (13) we get
\[ m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \]
as required.

\[ \square \]

**Theorem 5:** If \( E_k \in \mathcal{M}, k = 1, \ldots, n \) then \( E_1 \cup \ldots \cup E_n \in \mathcal{M} \) (not necessarily disjoint).

This can be proved by applying induction method. There is nothing to prove for \( n = 1 \). Suppose the claim is true for \( n - 1 \). Then
\[ E_1 \cup \ldots \cup E_n = (E_1 \cup \ldots \cup E_{n-1}) \cup E_n \]
so that the result follows from Theorem 4.

**Theorem 6:** If \( E_1, E_2 \in \mathcal{M} \), then \( E_1 \cap E_2 \in \mathcal{M} \).

We have \( E_1^c, E_2^c \in \mathcal{M} \) by (ii), \( E_1^c \cup E_2^c \in \mathcal{M} \) by Step 2, \( (E_1^c \cup E_2^c)^c \in \mathcal{M} \) by (ii) again, but by de Morgan's law the last set is equal to \( E_1 \cap E_2 \).

**Theorem 7:** Let \( E_k \in \mathcal{M}, k = 1, 2, \ldots \). We define an auxiliary sequence of pairwise disjoint sets \( F_k \) with the same union as \( E_k \):
\[ \begin{align*}
F_1 &= E_1 \\
F_2 &= E_2 \setminus E_1 = E_2 \cap E_1^c \\
F_3 &= E_3 \setminus (E_1 \cup E_2) = E_3 \cap (E_1 \cup E_2)^c \\
&\quad \ldots \\
F_k &= E_k \setminus (E_1 \cup \ldots \cup E_{k-1}) = E_k \cap (E_1 \cup \ldots \cup E_{k-1})^c.
\end{align*} \]

By Steps 3 and 4 we know that all \( F_k \) are in \( \mathcal{M} \). By the very construction they are pairwise disjoint so by Step 1 their union is in \( \mathcal{M} \). We shall show that
\[ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k. \]

This will complete the proof since the latter is now in \( \mathcal{M} \). The inclusion
\[ \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k. \]
is obvious since for each \( k, F_k \subseteq E_k \) by definition. For the inverse let
\[ a \in \bigcup_{k=1}^{\infty} E_k. \]
Put \( S = \{n \in \mathbb{N} : a \in E_n\} \) which is non-empty since \( a \) belongs to the union. Let \( n_0 = \min S \in S \). If \( n_0 = 1 \), then \( a \in E_1 = F_1 \). Suppose \( n_0 > 1 \). So \( a \in E_{n_0} \) and, by the definition of \( n_0 \), \( a \notin E_1, \ldots, a \notin E_{n_0-1} \). By the definition of \( F_{n_0} \) this means that \( a \in F_{n_0} \) so \( a \) is in \( \bigcup_{k=1}^{\infty} F_k \).

**Proposition 7:** Suppose that \( A, B \in \mathcal{M} \).
i) If $A \subset B$ then $m(A) \leq m(B)$.

ii) If $A \subset B$ and $m(B) < \infty$ is finite then $m(B \setminus A) = m(B) - m(A)$.

iii) $m$ is translation-invariant.

The proof is left as an exercise for you (see E4).

**Proposition 8:** If $A \in \mathcal{M}$, $m^*(A \Delta B) = 0$ then $B \in \mathcal{M}$ and $m^*(A) = m^*(B)$.

You can prove this by yourself (see E5).

**Theorem 8:** Let $A$ be a subset of $\mathbb{R}$. Then the following holds:

i) For a given $\epsilon > 0$ there exists an open set $O$ such that

$$A \subset O, \quad m(O) \leq m^*(A) + \epsilon.$$  

Consequently, for any $E \in \mathcal{M}$ there exists an open set $O$ containing $E$ such that $m(O \setminus E) < \epsilon$.

ii) There exists a sequence of open sets $O_n$ such that

$$A \subset \bigcap_n O_n, \quad m\left(\bigcap_n O_n\right) = m^*(A).$$

**Proof:**

i) By definition of $m^*(A)$ we can find a sequence $(I_n)$ of intervals with $A \subset \bigcup_n I_n$ and

$$\sum_{n=1}^{\infty} l(I_n) - \frac{\epsilon}{2} \leq m^*(A).$$

Each $I_n$ is contained in an open interval whose length is very close to that of $I_n$; if the left and right endpoints of $I_n$ are $a_n$ and $b_n$ respectively let $J_n = a_n - \frac{\epsilon}{2^{n+2}}, b_n + \frac{\epsilon}{2^{n+2}}$. Let $O = \bigcup_n J_n$, which is open. Then $A \subset O$ and

$$m(O) \leq \sum_{n=1}^{\infty} l(J_n) \leq \sum_{n=1}^{\infty} l(I_n) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon.$$  

When $m(E) < \infty$ the final statement follows at once, since then $m(O \setminus E) = m(O) - m(E) \leq \epsilon$. When $m(E) = \infty$ we first write $\mathbb{R}$ as a countable union of finite intervals: $\mathbb{R} = \bigcup (-n, n)$. Now $E_n = E \cap (-n, n)$ has finite measure, so we can find an open $O_n \supset E_n$ with $m(O_n \setminus E_n) < \frac{\epsilon}{2^n}$.

The set $O = \bigcup_n O_n$ is open and contains $E$. Now

$$O \setminus E = \left(\bigcup_n O_n\right) \setminus \left(\bigcup_n E_n\right) \subset \bigcup_n (O_n \setminus E_n),$$

so that $m(O \setminus E) \leq \sum_n m(O_n \setminus E_n) \leq \epsilon$ as required.

ii) In (i) use $\epsilon = \frac{1}{n}$ and let $O_n$ be the open set so obtained. With $E = \bigcap_n O_n$ we obtain a measurable set containing $A$ such that $m(E) < m(O_n) \leq m^*(A) + \frac{1}{n}$ for each $n$, hence the result follows. 

$\square$
Remark: Theorem 8 shows how the freedom of movement allowed by the closure properties of the σ-field \( \mathcal{M} \) can be exploited by producing, for any set \( A \subset \mathbb{R} \), a measurable set \( O \supset A \) which is obtained from open intervals with two operations (countable unions followed by countable intersections) and whose measure equals the outer measure of A.

**Theorem 9:** Every Borel set is measurable. In particular each open set and each closed set is measurable.

If \( E \) is measurable set, we define the Lebesgue measure \( mE \) to be the outer measure of \( E \). Thus, \( m \) is the set function obtained by restricting the set function \( m^* \) to the family \( m \) of measurable sets. Two important properties of Lebesgue measure are summarized by the following propositions:

**Proposition 9:** Let \( \langle E_i \rangle \) be a sequence of measurable sets. Then

\[
m(\bigcup E_i) \leq \sum mE_i.
\]

If the sets \( E_n \) are pairwise disjoint, then

\[
m(\bigcup E_i) = \sum mE_i.
\]

**Proof:** The inequality is simply a restatement of the subadditivity of \( m^* \) given by Proposition 2. If \( \langle E_i \rangle \) is a finite sequence of disjoint measurable sets, then Lemma 9 with \( A = \mathbb{R} \) implies that

\[
m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} mE_i,
\]

and so \( m \) is finitely additive. Let \( \langle E_i \rangle \) be an infinite sequence of pairwise disjoint measurable sets. Then

\[
\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i,
\]

and so

\[
m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{n} mE_i.
\]

Since the left side of this inequality is independent of \( n \), we have

\[
m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} mE_i.
\]

The reverse inequality follows from countable subadditivity, and we have

\[
m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} mE_i.
\]

**Proposition 10:** Let \( \langle E_n \rangle \) be an infinite decreasing sequence of measurable sets, that is, a sequence with \( E_{n+1} \subset E_n \) for each \( n \). Let \( m(E_1) \) be finite. Then

\[
m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} mE_n.
\]
Proof: Let \( E = \bigcap_{i=1}^{\infty} E_i \), and let \( F_i = E_i \setminus E_{i+1} \). Then

\[
E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i,
\]

and the sets \( F_i \) are pairwise disjoint. Hence

\[
m(E_1 \setminus E) = \sum_{i=1}^{\infty} m(F_i) = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1}).
\]

But \( mE_i = mE + m(E_i \setminus E) \), and \( mE_i = mE_{i+1} + m(E_i \setminus E_{i+1}) \), since \( E \subseteq E_i \) and \( E_{i+1} \subseteq E_i \). Since \( mE_i \leq mE_1 < \infty \), we have \( m(E_i \setminus E) = mE_i - mE \) and \( m(E_i \setminus E_{i+1}) = mE_i - mE_{i+1} \). Thus

\[
mE_1 - mE = \sum_{i=1}^{\infty} (mE_i - mE_{i+1})
= \lim_{n \to \infty} \sum_{i=1}^{n} (mE_i - mE_{i+1})
= \lim_{n \to \infty} (mE_1 - mE_n)
= mE_1 - \lim_{n \to \infty} mE_n.
\]

Since \( mE_1 < \infty \), we have

\[
mE = \lim_{n \to \infty} mE_n.
\]

The following proposition expresses a number of ways in which a measurable set is very nearly a nice set. The proof is omitted.

**Proposition 11:** Let \( E \) be a given set. Then the following five statements are equivalent:

i) \( E \) is measurable;

ii) given \( \varepsilon > 0 \), there is an open set \( O \supset E \) with \( m^*(O \setminus E) < \varepsilon \);

iii) given \( \varepsilon > 0 \), there is a closed set \( F \subset E \) with \( m^*(E \setminus F) < \varepsilon \);

iv) there is a \( G \) in \( \mathcal{G}_6 \) with \( E \subset G \), \( m^*(G \setminus E) = 0 \);

(a set is a \( \mathcal{G}_6 \) if it is the intersection of a countable collection of open sets).

v) there is an \( F \) in \( \mathcal{F}_6 \) with \( F \subset E, m^*(E \setminus F) = 0 \);

(a set is in \( \mathcal{F}_6 \) if it is a countable union of closed sets).

vi) given \( \varepsilon > 0 \), there is a finite disjoint union \( U \) of open intervals such that \( m^*(U \Delta E) < \varepsilon \).

**A Nonmeasurable Set**

We are going to show the existence of a nonmeasurable set. If \( x \) and \( y \) are real numbers in \( [0, 1) \), we define the sum modulo 1 of \( x \) and \( y \) to be \( x + y \), if \( x + y < 1 \), and to be \( x + y - 1 \) if \( x + y \geq 1 \). Let us denote the sum modulo 1 of \( x \) and \( y \) by \( x + y \). Then \( + \) is a commutative and associative operation taking pairs of numbers in \( [0, 1) \) into numbers in \( [0, 1) \). If we assign to each \( x \in [0, 1) \)
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the angle $2\pi x$, then addition modulo 1 corresponds to the addition of angles. If $E$ is a subset of $[0, 1)$, we define the translate modulo 1 of $E$ to be the set $E + y = \{ z : z = x + y \text{ for some } x \in E \}$. If we consider addition modulo 1 as addition of angles, translation modulo 1 by $y$ corresponds to rotation through an angle of $2\pi y$. The following lemma shows that Lebesgue measure is invariant under translation modulo 1.

Lemma: Let $E \subset [0, 1)$ be a measurable set. Then for each $y \in [0, 1)$ the set $E + y$ is measurable and $m(E + y) = mE$.

Proof: Now $E_1 + y = E_1 + y$, and so $E_1 + y$ is measurable and we have $m(E_1 + y) = mE_1$, since $m$ is translation invariant. Also $E_2 + y = E_2 + (y - 1)$, and so $E_2 + y$ is measurable and $m(E_2 + y) = mE_2$. But $E + y = (E_1 + y) \cup (E_2 + y)$ and the sets $(E + y)$ and $(E_2 + y)$ are disjoint measurable sets. Hence $E + y$ is measurable and

$$m(E + y) = m(E_1 + y) + m(E_2 + y)$$
$$= mE_1 + mE_2$$
$$= mE.$$

We are now in a position to define a nonmeasurable set. If $x - y$ is a rational number, we say that $x$ and $y$ are equivalent and write $x \sim y$. This is an equivalence relation and hence partitions $[0, 1)$ into equivalence classes, that is, classes such that any two elements of one class differ by a rational number, while any two elements of different classes differ by an irrational number. By the axiom of choice there is a set $P$ containing exactly one element from each equivalence class. Let $(\langle r_i \rangle)_{i=0}^\infty$ be an enumeration of the rational numbers in $[0, 1)$ with $r_0 = 0$, and define $P_i = P + r_i$. Then $P_0 = P$. Let $x \in P_i \cap P_j$. Then $x = p_i + r_i = p_j + r_j$ with $p_i$ and $p_j$ belonging to $P$. But $p_i - p_j = r_j - r_i$ is a rational number, whence $p_i = p_j$. Since $P$ has only one element from each equivalence class, we must have $i = j$. This implies that if $i \neq j$, $P_i \cap P_j = \phi$, that is, that $(\langle P_i \rangle)$ is a pairwise disjoint sequence of sets. On the other hand, each real number $x$ in $[0, 1)$ is in some equivalence class and so is equivalent to an element in $P$. But if $x$ differs from an element in $P$ by the rational number $r_i$, then $x \in P_i$. Thus $\bigcup P_i = [0, 1)$. Since each $P_i$ is a translation modulo 1 of $P$, each $P_i$ will be measurable if $P$ is and will have the same measure. But if this were the case,

$$m(0, 1) = \sum_{i=1}^\infty mP_i = mP,$$

and the right side is either zero or infinite, depending on whether $mP$ is zero or positive. But this is impossible since $m(0, 1) = 1$, and consequently $P$ cannot be measurable.

While the above proof that $P$ is not measurable is a proof by contradiction, it should be noted that (until the last sentence) we have made no use of properties of Lebesgue measure other than translation invariance and countable additivity. Hence the foregoing argument gives a direct proof of the following theorem:

Theorem 10: If $m$ is a countably additive, translation invariant measure defined on a $\sigma$-algebra containing the set $P$, then $m(0, 1)$ is either zero or infinite.

The nonmeasurability of $P$ with respect to any translation invariant countably additive measure $m$ for which $m(0, 1)$ is 1 follows by contraposition.
Finally we show that monotone sequences of measurable sets behave as one would expect with respect to $m$.

**Theorem 11:** Suppose that $A_n \in \mathcal{M}$ for all $n \geq 1$. Then we have:

i) if $A_n \subseteq A_{n+1}$ for all $n$, then

$$m \left( \bigcup_{n} A_n \right) = \lim_{n \to \infty} m(A_n),$$

ii) if $A_n \supsetneq A_{n+1}$ for all $n$ and $m(A_1) < \infty$, then

$$m \left( \bigcup_{n} A_n \right) = \lim_{n \to \infty} m(A_n).$$

**Proof:** i) Let $B_1 = A_1, B_i = A_i - A_{i-1}$ for $i > 1$. Then $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ and the $B_i \in \mathcal{M}$ are pairwise disjoint, so that

$$m \left( \bigcup_{i} A_i \right) = \sum_{i=1}^{\infty} m(B_i) \text{ by countable additivity}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} m(B_i)$$

$$= \lim_{n \to \infty} m \left( \bigcup_{i=1}^{n} B_i \right) \text{ by additivity}$$

$$= \lim_{n \to \infty} m(A_n),$$

since $A_n = \bigcup_{i=1}^{n} B_i$ by construction.

ii) $A_1 \setminus A_1 = \phi \subset A_1 \setminus A_2 \subset \cdots \subset A_1 \setminus \cdots$ for all $n$, so that by (i)

$$m \left( \bigcup_{n} (A_1 \setminus A_n) \right) = \lim_{n \to \infty} m(A_1 \setminus A_n)$$

and since $m(A_1)$ is finite, $m(A_1 \setminus A_n) = m(A_1) - m(A_n)$. On the other hand, $\bigcup_{n} (A_1 \setminus A_n) = A_1 \setminus \cap_n A_n$, so that

$$m \left( \bigcup_{n} (A_1 \setminus A_n) \right) = m(A_1) - m \left( \bigcap_{n} A_n \right) = m(A_1) - \lim_{n \to \infty} m(A_n).$$

The result follows.

Try some exercises.

---

E4) Prove Proposition 7.

E5) Prove Proposition 8.
8.4 LEBESGUE MEASURABLE FUNCTIONS

In the last section, we defined Lebesgue measure for sets of infinite as well as finite measures. In this section, we shall discuss the notion of measurability of functions defined on measurable sets. In order to handle such functions it is convenient to allow functions which take infinite values: we take their range to be (part of) the 'extended real line' \( \mathbb{R}^\infty \), obtained by adding the 'points at infinity' \(-\infty\) and \(+\infty\) to \(\mathbb{R}\). Arithmetic in this set needs a little care, we assume that \(a + \infty = \infty\) for all real \(a\), \(a \times \infty = \infty\) for \(a > 0\), \(A \times \infty = -\infty\) for \(a < 0\), \(\infty \times \infty = \infty\) and \(0 \times \infty = 0\), with similar definitions for \(-\infty\). These are all 'obvious' intuitively (except possibly \(0 \times \infty\)), and (as for measures) we avoid ever forming 'sums' of the form \(\infty + (-\infty)\). With these assumptions we shall begin our discussion on measurable functions. Here we shall be considering functions whose domain is \(\mathbb{R}\). Now we have the freedom of defining \(f\) only 'up to null sets': once we have shown two functions \(f\) and \(g\) to be equal on \(\mathbb{R}\backslash E\) where \(E\) is some null set, then \(f = g\) for all practical purposes. To formalise this, we say that \(f\) has a property (P) almost everywhere (a.e.) if \(f\) has this property at all points of its domain except possibly on some null set.

For example, the function

\[
 f(x) = \begin{cases} 
 1 & \text{for } x \neq 0 \\
 0 & \text{for } x = 0 
\end{cases}
\]

is almost everywhere continuous, since it is continuous on \(\mathbb{R}\backslash \{0\}\) and the exceptional set \(\{0\}\) is null.

The next definition will introduce the class of Lebesgue-measurable functions. The condition imposed on \(f: \mathbb{R} \to \mathbb{R}\) will be necessary (though not sufficient) to give meaning to the concept (Lebesgue) integral \(\int f\,dm\) which we will discuss in Unit 9. Let us first give some motivation.

Integration is always concerned with the process of approximation. In the Riemann integral we partitioned the interval \(I = [a, b]\), over which we integrate, into small intervals \(I_n\). The simplest method of doing this is to divide the interval into \(N\) equal parts. Then we construct approximating sums by multiplying the lengths of the small intervals by certain numbers \(c_n\) (related to the values of the function in question; for example \(c_n = \inf I_n f\), \(c_n = \sup I_n f\), or \(c_n = f(x)\) for some \(x \in I_n\) :)

\[
 \sum_{n=1}^{N} c_n I_n. 
\]

For large \(N\) this sum is close to the Riemann integral \(\int_a^b f(x)\,dx\) (given some regularity of \(f\)).

The approach to the Lebesgue integral is similar but there is a crucial difference. Instead of splitting the integration domain into small parts directly, we induce a partition on the domain, not necessarily as intervals, by partitioning the range as disjoint intervals and then using the given function.

Again, a simple way is to introduce short intervals \(J_n\) of equal length. To build the approximating sums we first take the inverse images of \(J_n\) by \(f\), i.e. \(f^{-1}(J_n)\).
These may be complicated sets, not necessarily intervals. Here the theory of measure developed previously comes into its own. We are able to measure sets provided they are measurable, i.e., they are in $\mathcal{M}$. Given that, we compute

$$\sum_{n=1}^{N} c_n m(f^{-1}(J_n)),$$

where $c_n \in J_n$ or $c_n = \inf J_n$, for example.

The following definition guarantees that this procedure makes sense. (Though some extra care may be needed to arrive at a finite number as $N \to \infty$.)

**Definition 8:** Suppose that $E$ is a measurable set. We say that a function $f : E \to \mathbb{R}$ is (Lebesgue-)measurable if for any interval $I \subseteq \mathbb{R}$

$$f^{-1}(I) = \{ x \in \mathbb{R} : f(x) \in I \} \in \mathcal{M}.$$

In what follows, the term measurable (without qualification) will refer to Lebesgue-measurable functions.

It is also important to note that, since all open sets in $\mathbb{R}$ are Lebesgue measurable it follows that every **continuous real-valued function is a measurable function**. This assures that all continuous functions (a large class of functions that we normally encounter) are measurable functions. Hence the Lebesgue Integration theory applies to them. In the next unit, we shall look into Riemann integral more closely and then introduce Lebesgue integration.

With this we come to an end of this unit.

Let us summarise the points covered in this unit.

**8.5 SUMMARY**

In this unit, we have covered the following points.

1. We have defined $\sigma$-algebra of subsets of a set $X$.
2. We have defined our measure and discussed how to compute the outer measure of set.
3. We have discussed how to check whether a set is measurable or not.
4. We have given an example of a set which is not measurable.
5. We have defined measurable functions.

**8.6 HINTS/SOLUTIONS**

E1) $X \in \mathcal{C}_1$

$$\{1\}' = \{2, 3\} \notin \mathcal{C}_1$$

$\therefore \mathcal{C}_1$ is not an algebra.

$X \in \mathcal{C}_2, \{1, 2\}' = \{3\} \in \mathcal{C}_2$
Measure and Integral

\{3\}^c = \{1, 2\} \in \mathcal{C}_2

\{1, 2\} \cup \{3\} = X \in \mathcal{C}_2

\therefore \mathcal{C}_2 \text{ is an algebra}

and \(\mathcal{C}_1 \cup \mathcal{C}_2 = \{(1), \{1, 2, 3\}, X, \phi, \{1, 2\}, \{3\}\}\)

\(X \in \mathcal{C}_1 \cup \mathcal{C}_2\), \(\{1\}^c = [2, 3] \notin \mathcal{C}_1 \cup \mathcal{C}_2\)

\therefore \mathcal{C}_1 \cup \mathcal{C}_2 \text{ is not an algebra.}

E2) i) \(X \in A \implies X \cap X = \phi\) which is countable.

ii) Let \(E \in A \implies E\) is countable or \(X \setminus E\) is countable

or \(X \setminus (X \setminus E)\) is countable or \(X \setminus E\) is countable

\(\implies X \setminus E \in A\)

iii) Let \(E_n \in A, n = 1, 2, \ldots\)

\(E_n\)'s are countable or \(X \setminus E_n\) is countable

\(X \setminus (\bigcup E_n) = \cap (X \setminus E_n)\) is countable and we know intersecting countable set is countable.

E3) **Hint:** We have \(m^*(\mathbb{R}) \leq m^*(\mathbb{R} \setminus \mathbb{Q}) + m^*(\mathbb{Q})\) and \(m^*(\mathbb{Q}) = 0\)

E4) i) Try it by yourself

ii) **Hint:** \(B = (B \sim A) \cup A\) is disjoint union of

\(m^0B = m^0(B \sim A) + m^0A\)

iii) The Lebesgue outer measure \(m\) obviously has the property that

\(m(E + x) = m(E)\) for every \(E \subseteq \mathbb{R}\) and \(x \in \mathbb{R}\). Next, suppose \(E \in \mathcal{L}\).

Then for every \(A \subseteq X\) and \(x \in \mathbb{R}\),

\[m(A) = m(A - x) = m((A - x) \cap E) + m((A - x) \cap E^c).\]

Since

\((A - x) \cap E = (A \cap (E + x)) - x\) and

\((A - x) \cap E^c = (A \cap (E + x)^c) - x,\)

we have

\[m(A) = m(A \cap (E + x)) + m(A \cap (E + x)^c).\]

Thus, \(E + x \in \mathcal{L}\) and \(m(E + x) = m(E)\) i.e., \(m(E + x) = m(E)\).

Hence the result.

E5) As \(A\) be measurable set and \(B\) be such that \(m^*(A \Delta B) = 0\)

then \(B\) is measurable

\(A \Delta B = (A \sim B) \cup (B \sim A)\) and \(A \sim B \subseteq A \Delta B, B \sim A \subseteq A \Delta B\)

\(\implies m^*(A \sim B) \leq m^*(A \Delta B) = 0\)

\(\implies m^*(A \sim B) = 0\)

and \(m^*(B \sim A) = 0\)

\(\implies A \sim B\) and \(B \sim A\) are measurable sets

also \(A\) is measurable

\(\implies A \sim (A \sim B)\) is measurable

\[A \sim A \sim B = A \sim (A \cap B) = A \cap (A \cap B) = A \cap (A \cap B) = (A \cap A) \cup (A \cap B) = A \cap B\]

\(\implies A \cap B\) is measurable.
Now $B = (B \sim A) \cup (A \cap B)$ measurable

$\Rightarrow$ $B$ is measurable

To show $m^*A = m^*B$

$m^*(B \sim A) = m(B \sim A)$

$0 = m^*(B \sim A) = mB - mA$

$\Rightarrow mB = mA$

or $m^*B = m^*A$