UNIT 3 COMPACTNESS

Structure **Page No.**

3.1 INTRODUCTION

In the last unit, we discussed an important class of metric spaces called complete metric spaces. In this unit, we shall discuss another important class known as the class of compact metric spaces. A compact set in a metric space is a set that resembles a closed and bounded subset of **R**, it is "small" in a certain sense and "contains" all its adherent points.

One of the main reasons for studying compact sets is that they are in some ways very similar to **finite sets**. In other words, there are many results which are easy to show for finite sets, the formulations as well as the proofs of which carry over with minimal changes to compact sets. It is often said that "compactness is the next best thing to finiteness".

In this unit, we shall discuss the notion of compactness in a metric space. In **Section 3.2**, we shall define compact sets and consider examples of these sets in different metric spaces. In **Section 3.3**, we discuss certain theorems which characterise compact sets and give a complete description of compact sets in a metric space. We first give a characterisation in terms of convergence of sequences and then in terms of completeness. In this connection, we introduce the concept of "totally bounded sets" which is a stronger version of bounded sets. We show that a set is compact if and only if it is complete and totally bounded. We also discuss the analogue of the famous "Heine Borel theorem" in R which characterises compact sets in terms of closed and bounded sets. **Section 3.4** deals with special properties of compact sets. Here we discuss relationship between continuity and compactness.

Objectives

After studying this unit, you should be able to

- use the definition of compact sets to check whether a given set in a metric space is compact or not;
- explain the connection between compactness and sequential convergence;
- explain the relationship between compact sets and totally bounded sets; and that between compact sets and sets having finite intersection property;
- state and prove Heine-Borel theorem for $\mathbf{R}^n$;
- explain the relationship between continuity and compactness.

3.2 BASIC PROPERTIES OF COMPACT SETS

In this section, we define compact sets and discuss various properties.

We shall begin with recalling the definition of compact sets in **R** (Refer **MTE-09, Block 1**).

Definition 1: A compact set in **R** is a set E satisfying the property that if $\mathcal{U}$ is a collection of open sets in **R** whose union contains E, then there is a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ whose union contains E.

Recall that such a collection is called an **open cover** and $\mathcal{V}$ is called a **finite subcover of** $\mathcal{U}$ for E. In terms of this, a set E in **R** is compact if every open cover of the set E has a finite subcover for E.

Because of this criteria, compact sets are also viewed as a generalisation of finite sets. Here we try to extend this idea to metric spaces. For that we first introduce the concept of an indexed family of subsets of a set. A thorough knowledge of this will be needed to understand open covers in a metric space.

Definition 2: Let I be a non-empty set. A family (or collection) of subsets of a set X **indexed by the index set I** is a map of I into the power set $\mathcal{P}(X)$ of X, that is, the set of all subsets of X.

Usually we denote the image of $i \in I$ by A_i or some such notation. Further, the indexed family is denoted by $\{A_i : i \in I\}$. Sometimes we use $\{A_i\}_{i \in I}$ or simply $\{A_i\}$ for this indexed family. This resembles the notation for a sequence in a set. We note that A_i can be A_j for $i \neq j$.

Let us look at some examples.

Example 1: Let $X = \mathbf{R}^2$ and $I = [0, \infty)$. For any $r \in I$, we let $C_r := \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = r^2\}$. Then, for $r > 0$, C_r is the circle of radius r with centre at the origin and C_0 is simply the point circle $\{0\}$. Here the family is $\{C_r : r \in [0, \infty)\}$.

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Example 2: Let $X = \mathbf{R}$ and $I = \mathbf{Q}^+$, the set of positive rational numbers. For any $r \in \mathbf{Q}^+$, let J_r denote the interval $(-r, r)$. Here the indexed family is $\{J_r : r \in \mathbf{Q}^+\}$.

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Definition 3: Let $\{A_i : i \in I\}$ be a family of subsets of X indexed by a non-empty set I. Let $\Lambda \subset I$ be a non-empty subset. Then the family $\{A_i : i \in \Lambda\}$ is called a **subfamily** of the given family $\{A_i : i \in I\}$.

For example, $\{J_n : n \in \mathbf{N}\}$ is a subfamily of $\{J_r : r \in \mathbf{Q}^+\}$ given in Example 2.

Definition 4: Let $[A_i : i \in I]$ be a family of subsets of X indexed by I. The subset $\bigcup_{i\in I} A_i$ of X, defined by,

$$\bigcup_{i\in I} A_i := \{x \in X : x \in A_i \text{ for some } i \in I\}$$

is called the **union of the indexed family**.

For example in Example 1, if we take $\Lambda = [1, 2]$, then the union of the subfamily $\{C_r : r \in [1, 2]\}$ is the annular region $\{(x, y) \in \mathbf{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. If we take $\Lambda = [0, 1]$, then the union is the closed disk or the closed ball $B(0, 1)$ in $\mathbf{R}^2$.

Definition 5: Let X be a metric space and $E \subset X$. A family of subsets of X, $\mathcal{U} = \{U_i : i \in I\}$ is called an open cover for E if each U_i is an open set in X and $E \subset \bigcup_{i\in I} U_i$. If there exists a subfamily $\mathcal{V}$ of $\mathcal{U}$ which also covers E, then $\mathcal{V}$ is called a **subcover of $\mathcal{U}$** for E. In this case, we also say that $\mathcal{U}$ admits a subcover $\mathcal{V}$ for E. If $\mathcal{U}$ has finite number of sets, $\mathcal{U}$ is called a **finite cover**.

Let us see some examples.

Example 3: Let (X, d) be a metric space. For any $r_0 > 0$, $\{B(x, r_0) : x \in X\}$ is an open cover for X. If we fix an $x_0 \in X$, then $\{B(x_0, r) : r \in [0, \infty)\}$ is also an open cover for X.

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Example 4: Consider the interval $]0, 1[\subset \mathbf{R}$. The family $\mathcal{U} = \{]1/n, 1[: n \in \mathbf{N}, n \geq 2\}$ is an open cover for $]0, 1[$.

* * *

Example 5: The family $\{]-n, n[: n \in \mathbf{N}\}$ is an open cover for $\mathbf{R}$. Further, $\{(-2n, 2n) : n \in \mathbf{N}\}$ is a subcover.

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We are now ready to define compact sets.

Definition 6: A subset $E \subset X$ of a metric space is said to be **compact** if corresponding to every open cover of E there is a finite subcover for E. A metric space (X, d) is said to be a **compact metric space** if every open cover of X admits a finite subcover for X.

As a consequence of the definition we get that the empty set is compact in any metric space. In fact we have the following proposition.

Proposition 1: A finite subset of a metric space is compact.

Proof: Let (X, d) be a metric space and E be a non-empty finite subset of X. We denote the elements of E by $x_1, \ldots, x_N$. Let $\mathcal{U} = \{A_i\}_{i\in I}$ be an open cover of E. That is, $E \subset \bigcup_{i\in I} A_i$. This implies that each element $x_j, j = 1, \ldots, N$, belongs to $\bigcup_{i\in I} A_i$. That means for each j, $x_j \in A_i$ for some $i \in I$. Let us denote these A_i's by $A_{i_k}, k = 1, \ldots, m$. Let $\mathcal{U}_0 = \{A_{i_k} : 1 \leq k \leq m\}$. Then $\mathcal{U}_0$ is a finite subcover of for E. This shows that every open cover of E admits a finite subcover for E. Hence E is compact. □

Can you find a metric space in which the compact sets are finite sets only? We leave it as an exercise for you to check (see E3).

Let us see some examples.

Example 6: Let E denote the subset of **R** defined by

$$E = \{0\} \cup \{1/n : n \in \mathbf{N}\}$$

Then E is a compact subset of **R**.

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You may recall from your undergraduate Real Analysis course that **R is not compact** (Refer **MTE-09, Block 1**). So can we expect $\mathbf{R}^2$ also to be non-compact? Let us see.

Example 7: $\mathbf{R}^2$ with the usual metric is not a compact space. In fact $\mathcal{U} = \{B(0, n) : n \in \mathbf{N}\}$ is an open cover for $\mathbf{R}^2$ which has no finite subcover.

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Example 8: The unit ball B(0, 1) in $(C[0, 1], d_\infty)$ is not compact.

Let us recall that the unit ball $B(0, 1)$ in C[0, 1] is the set given by $B(0, 1) = \{f \in C[0, 1] : d(0, f) < 1\}$ where the metric d_∞ is given by

$$d_\infty(f, g) = \sup_{x \in [0,1]} \{|f(x) - g(x)|\}$$

Let us assume that $B(0, 1)$ is compact. Consider the sequence $\{f_n\}$ of functions in C[0, 1] defined by

$$f_n(t) = \begin{cases} 0, & 0 \le t \le \dfrac{1}{n+2} \\ (n+1)(n+2)\left(t - \dfrac{1}{n+2}\right), & \dfrac{1}{n+2} \le t \le \dfrac{1}{n+1} \\ n(n+1)\left(\dfrac{1}{n} - t\right), & \dfrac{1}{n+1} \le t \le \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \le t \le 1 \end{cases}$$

Then $d(f_m, f_n) = 1$ if $m \neq n$. Consider the open cover $\left\{B\left(f, \frac{1}{2}\right) : f \in C[0, 1]\right\}$. This does not admit a finite subcover for $B(0, 1)$. For if $\left\{B\left(f_j, \frac{1}{2}\right) : 1 \le j \le p\right\}$ is a finite subcover, then there exists j_0, $1 \le j_0 \le p$ such that $f_m, f_n \in B\left(f_{j_0}, \frac{1}{2}\right)$ for $m \neq n$. This implies that

$$\begin{aligned} d(f_m, f_n) &\le d(f_m, f_{j_0} + d(f_n, f_{j_0}) \\ &< 1, \end{aligned}$$

which is a contradiction. This shows that $\left\{B(f, \frac{1}{2}) : f \in C[0, 1]\right\}$ cannot have a finite subcover. Hence $B(0,1)$ is not a compact subset of $C[0,1]$.

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Try these exercises now.

E1) For each pair (m, n) of integers, let $B_{m,n}$ be the open ball in $\mathbf{R}^2$ with centre (m, n) and radius $\frac{1}{2}$. Let $\mathcal{U} = \{B_{m,n}, , m, n \in \mathbf{Z}\}$. Show that $\mathcal{U}$ is not an open cover of $\mathbf{R}^2$.

E2) For each pair (m, n) of integers, let $C_{(m,n)}$ be the open ball in $\mathbf{R}^2$ with centre (m,n) and radius 1. Show that $\mathcal{U}_0 = \{C_{(m,n)} : m, n \in \mathbf{Z}\}$ is an open cover of $\mathbf{R}^2$.

E3) Find all compact sets in a discrete metric space.

Now we shall prove some simple results on compactness.

Theorem 1: Every compact set in a metric space is closed and bounded.

Proof: Let X be a metric space and E be a compact subset. To show that E is closed, it is enough to show that E^c is open. Let $x_0 \in E^c$. Now we apply Hausdorff property (Refer Proposition No. 3 in unit 1) to each element $y \in E$. Then we get that for each $y \in E$, there exists open sets U_y and V_y of the points x_0 and y respectively such that $U_y \cap V_y = \phi$. Then the collection

$$\mathcal{V} = \{V_y : y \in E\}$$

is an open cover for E. Since E is compact, $\mathcal{V}$ admits a finite subcover for E. Then there exists sets $V_{y_1}, \ldots, V_{y_n}$ such that

$$E \subset \bigcup_{i=1}^{n} V_{y_i}$$

Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$

where U_{y_i}'s are the neighbourhoods of x_0 corresponding to $V_{y_1}, \ldots, V_{y_n}$.

Since U is a intersection of finite open sets, U is open and $x_0 \in U$. Also $U \cap V = \phi$ (by the choice of U_{y_i}'s and V_{y_i}'s) which implies that $U \cap E = \phi$. Thus U is a neighbourhood of x_0 which is fully contained in E^c. Hence E^c is open.

Now we have to show that E is bounded. Fix any $x_0 \in E?$. Consider the open cover $\mathcal{U} = \{B(x_0, n) : n \in \mathbf{N}\}$ of E. This admits a finite subcover for E, say, $\{B(x_0, n_j) : 1 \leq j \leq p\}$. Since $B(x_0, m) \subset B(x_0, n)$ for $n \geq m$, it follows that $E \subset B(x_0, M)$ for $M = \max\{n_j : 1 \leq j \leq p\}$. Thus, E is bounded. □

Remark 1: But the converse of the above theorem is not true. For example, let X be an infinite set with discrete metric space. Then every subset of X is closed and bounded, and we have seen in E3 that only the finite subsets of X are compact. This shows that a closed and bounded set need not be compact in a general metric space.

Now in the case of the real-line, we have the following result.

Theorem 2: A subset of R is compact if and only if it is closed and bounded.

The theorem above is known as **Heine-Borel Theorem**. You must have already studied this theorem in your undergraduate Analysis course (Refer IGNOU course **MTE-09 Block 1**).

The next theorem shows that Heine Borel Theorem is true for all Euclidean spaces.

Theorem 3: **(Heine-Borel Theorem for R^n):** A subset of R^n is compact if and only if it is closed and bounded.

We will not give a proof of this theorem now. You will see that this follows from certain characterizations of compact sets which shall consider in the next section. Before we go to the next section we shall discuss some more basic properties of compact sets.

Theorem 4: Closed subsets of compact sets in a metric space are compact.

Proof : Let X be a metric space and K is a compact set in X. Suppose F is a closed subset of K. Since K is compact it is closed in X. Since F is closed in K, it is closed in X. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have, thus, shown that a finite subcollection of $\{V_\alpha\}$ covers F. Therefore, F is compact. Hence the result. □

Theorem 5: If A and B are compact sets in a metric space X, then $A \cup B$ and $A \cap B$ are compact sets in X.

Proof : We shall first consider $A \cup B$. Let $\mathcal{U}$ be an open cover for $A \cup B$. Then $\mathcal{U}$ is an open cover for A as well as for B. Since A is compact, $\mathcal{U}$ admits a finite cover, say $\mathcal{U}_1$, for A. Since B is compact, $\mathcal{U}$ admits a finite subcover, say $\mathcal{U}_2$, for B. Then the collection obtained by adjoining the sets in $\mathcal{U}_2$ to $\mathcal{U}_1$, becomes a finite subcover for $A \cup B$. This shows that $A \cup B$ is compact.

Since A and B are compact sets, by Theorem 2, they are closed. So $A \cap B$ is closed, and it is a subset of A. Since any closed subset of a compact set is compact, it follows that $A \cap B$ is compact. Hence the result. □

Remark 2: In fact we can show that a finite union of compact sets are compact. Here we ask the question whether arbitrary union of compact sets are compact. The answer is "No", as you might be already knowing (think of an example, see E5).

Theorem 6: If S is an infinite subset of a compact metric space X, then S has a limit point in X.

Proof : We prove this by a contradiction argument. Let, if possible, S have no limit point in X. Then given any point x in X, there exists U_x open in X such that $x \in U_x$ and U_x contains no points of S except possibly x itself. The family $\{U_x : x \in X\}$ is an open cover for X and, since X is compact, has a finite subcover, say

$$\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}.$$

So,

$$S = S \cap X \subseteq S \cap \left(\bigcup_{i=1}^{n} U_{x_i}\right) \subseteq \bigcup_{i=1}^{n} (S \cap U_{x_i}) \qquad (1)$$

Each of the sets U_{x_i} contains at most one point of S. So the R.H.S. of (1) has at most n points which in turn implies that S is a finite set. This is a condraction. Therefore, S has a limit point in X. □

You can try some exercises now.

E4) If F is closed and K is compact, then show that $F \cap K$ is compact.

E5) Give an example to show that arbitrary union of compact sets need not be compact.

In the next section, we shall discuss some characterisation of compact sets.

3.3 CHARACTERISATION OF COMPACT SETS

In this section, we shall discuss three characterisations of compact sets in a metric space. We shall first give a characterisation using convergence of sequences.

Theorem 7: Let X be a metric space. Then the following are equivalent

i) X is compact

ii) Every sequence in X has a convergent subsequence.

Proof : We shall first show that **(i)** $\Rightarrow$ **(ii)**.

Suppose X is compact. We have to show that every sequence in X has a convergent subsequence. On the contrary, assume that it is not so. Then there exists a sequence $\{x_n\}$ such that it has no subsequence which converges in X. This implies that for each $x \in X$, there is some $r_x > 0$ and a positive integer n_x such that $x_n \notin B(x, r_x)$ for all $n \geq n_x$. To see this, let us assume that it is not true. Then $\exists x \in X$ such that $\forall r > 0$ and $j \in N$, $\exists m_{j,r} \geq j$ with $x_{m_{j,r}} \in B(x, r)$. Let us take $r_1 = 1$ and $j_1 = 1$. Then $\exists n_1 \geq j$, with $x_{n_1} \in B(x, 1)$. Note $n_1 \geq 1$. Now we take $r_2 = \frac{1}{2}, j_2 = n_1 + 1$. Note that $j_2 > n_1$. Then $\exists n_2 \geq j_2$ with $x_{n_2} \in B\left(x, \frac{1}{2}\right)$. Note that $n_2 > n_1$.

Then we take $r_3 = \frac{1}{3}$, $j_3 = n_2 + 1$. Note that $j_3 > n_2$. Then $\exists n_3 \geq j_3$ with $x_{n_3} \in B\left(x, \frac{1}{3}\right)$. Note that $n_3 > n_2$.

Proceeding like this we get an increasing sequence $\{n_i\}_{i=1}^{\infty}$ for which $x_{n_i} \in B\left(x, \frac{1}{i}\right)$ $\forall i \in N$. $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence of the sequence $\{x_n\}$ and $x_{n_i} \to x$ as $i \to \infty$. This is not possible by our assumption. Therefore, we got that for each $x \in X$, there is some $r_x > 0$ and a positive integer n_x such that $x_n \notin B(x, r_x)$ for $n \geq n_x$.

Then $\{B(x, r_x) : x \in X\}$ is an open cover for X. Since X is compact, there exist $y_1, y_2, \ldots, y_m \in X$ such that

$$X \subset \bigcup_{i=1}^{m} B(y_i, r_{x_i})$$

Let $n_0 = \max[n_{x_1}, \ldots, n_{x_m}]$. Then $x_{n_0} \notin \bigcup_{j=1}^{m} B(y_j, r_{x_j})$ so that $x_{n_0} \notin X$, which is impossible. Therefore, our assumption is wrong. Hence the claim.

Next, we shall show that **(ii)** $\Rightarrow$ **(i)**.

Suppose that every sequence in X has a convergent subsequence we have to show that X is compact. We shall prove this in three steps.

Step 1: We shall first claim that given any open cover $\mathcal{U}$ of X, there exists a number $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of $\mathcal{U}$ containing B.

We shall refer to such a number δ as a **Lebesgue number for** $\mathcal{U}$.

Let, if possible, there be no such δ for some open cover $\mathcal{U}$. That means for any $\delta > 0$, there exists a subset, whose diameter is less than δ, and this subset does not lie inside any element of $\mathcal{U}$. In particular, for each $n \in N$, we can choose a set C_n having diameter less than $\frac{1}{n}$ which is not contained in any element of $\mathcal{U}$. Now for each n, choose an element $x_n \in C_n$. Then $\{x_n\}$ so obtained has a convergent subsequence $\{x_{n_i}\}$, converging to x, say, in view of the fact that (ii) holds. Now, x belongs to some element A of $\mathcal{U}$, and because A is open, there is an $r > 0$ such that $B(x, r) \subset A$. Choose i large enough that satisfies

$$d(x_{n_i}, x) < r/2 \text{ and } 1/n_i < r/2.$$

Because the diameter of C_{n_i} is less than $\frac{1}{n_i}$, we have,

$$C_{n_i} \subset B\left(x_{n_i}, \frac{1}{n_i}\right)$$

which in turn is contained in $B\left(x_{n_i}, \frac{r}{2}\right)$. It follows then, that

$$C_{n_i} \subset B(x, r)$$

Therefore, $C_{n_i} \subset A$, this is not so. Hence the claim.

This establishes the claim made in the beginning of Step 1.

Step 2: We now claim that for every $\epsilon > 0$, there exists a finite covering of X by ϵ- balls.

Let, if possible, there be an $\epsilon > 0$ for which such a covering does not exist. Now, we construct a sequence $\{x_n\}$ in X. Choose an element $x_1 \in X$. Since $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ ball), $\exists$ an $x_2 \in X$ such that $x_2 \notin B(x_1, \epsilon)$. So $d(x_2, x_1) \geq \epsilon$. Then $B(x_1, \epsilon) \cup B(x_2, \epsilon)$ is not all of X. So there exists $x_3 \in X$ with $x_3 \notin B(x_1, \epsilon) \cup B(x_2, \epsilon)$. Therefore $d(x_2, x_1), d(x_3, x_2) \geq \epsilon$. We already have $d(x_1, x_2) \geq \epsilon$.

Proceeding similarly we get that given $x_1, x_2, \ldots, x_n \in X$ with $d(x_j, x_k) \geq \epsilon$ for $1 \leq j \neq k \leq n$, there exists $x_{n+1} \in X$ such that

$$x_{n+1} \notin \bigcup_{i=1}^{n} B(x_i, \epsilon)$$

Therefore, $d(x_j, x_{n-1}) \geq \epsilon$ for $1 \leq j \leq n$, thus by induction we get a sequence (x_n) for which $d(x_j, x_k) \geq \epsilon$ for $j \neq k$. Thus $\{x_n\}$ can have no convergent subsequence. This is a contradiction because (ii) holds. Hence the claim.

Step 3: Now we claim that X is compact.

Let $\mathcal{U}$ be an open covering of X. Then the covering $\mathcal{U}$ has a Lebesgue number δ as specified in Step 1. Corresponding to this δ, using Step 2 $\epsilon = \frac{\delta}{3}$, there exists using with $\epsilon = \frac{\delta}{3}$ a finite covering $\mathcal{F}$ of X by balls of radius $\frac{\delta}{3}$. Then each of these balls has diameter at most $\frac{2\delta}{3}$. So, we can choose, for each of these balls, an element of $\mathcal{V}$ containing it using the fact that $\frac{2\delta}{3} < \delta$ a Lebesgue number for $\mathcal{V}$, as in Step 1. If $\mathcal{U}'$ denote this collection, then $\mathcal{U}'$ becomes a finite subcover of $\mathcal{U}$ for X. Thus, we get that every open covering of X has a finite subcover. Hence X is compact. □

Remark 3: As already mentioned in Step 1 of the proof of Theorem 7, given any open cover $\mathcal{U}$ of X, there exists a number $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of $\mathcal{U}$ containing B. Clearly for $0 < \delta' < \delta$, δ' satisfies the same requirement again. Any such number is called a Lebesgue number for $\mathcal{U}$ and supremum of all such numbers can also be the Lebesgue number for $\mathcal{U}$. The above theorem asserts that if X is compact, then corresponding to each open cover for X there exists a lebesgue number. This is an important result and will be used in proving the other theorems.

Remark 4: In literature any metric space satisfying (ii) of Theorem 7 is called sequentially compact. Accordingly, Theorem 7 can be reworded as: A metric space S is compact iff it is sequentially compact. This is an important and useful characterization of compactness. You will see that it is more easy to deal with sequences than "open covers". Further Theorem 7 combined with the fact that a Cauchy sequence is convergent if and only if it has a convergent subsequence (Refer E5 in Unit 2) gives that a compact metric space is complete. But you can easily see that the converse does not hold, as **R** is an example of a complete space which is not compact. Here the question arises under what extra conditions a complete metric space becomes a compact metric space. Such a condition is the one called total boundedness which is given in the following definition.

Definition 7: A subset S of a metric space is **totally bounded** if given any $\epsilon > 0$ there exists a natural number m and finite number of points $x_1, x_2, \ldots, x_m$ in X such that $A \subset \cup_{i=1}^{m} B(x_i, \epsilon)$. A metric space X is said to be totally bounded if it is a totally bounded set when considered as a subset of itself.

The definition says that a totally bounded space is a space that can be covered by finitely many open balls of any fixed radius.

From the definition it is clear that every totally bounded set is bounded. But the converse is not true. For an example, consider $\mathbf{R}$ with the metric $d(x, y) = \min\{|x - y|, 1\}$. Then $\mathbf{R}$ is bounded but not totally bounded as can be seen by taking $0 < \epsilon \leq 1$.

We also note that a **set in an Euclidean space is totally bounded if and only if it is bounded**. That means the concepts of bounded sets and totally bounded sets are equivalent in Euclidean spaces.

Our next theorem connects the notion of compactness, completeness and total boundedness

Theorem 8: Let (X, d) be a metric space. Then the following are equivalent

i) X is compact

ii) X is complete and totally bounded.

Proof : We first show that (i) implies (ii).

Suppose that X is compact. Then by Remark 3, we get that X is complete. Now we have to show that X is totally bounded. Let $\epsilon > 0$ be given. Then we consider covering $\mathcal{U}$ of X by ϵ-balls around each point of X i.e., $\mathcal{U} = \{B(x, \epsilon) : x \in X\}$. Since X is compact the covering $\mathcal{U}$ has a finite subcover. This shows that given $\epsilon > 0$ there exists a natural number m and a finite number of points $x_1, x_2, \ldots, x_m$ in X such that $X \subset \cup_{i=1}^{m} B(x_i, \epsilon)$. Thus X is totally bounded. Hence (ii) holds.

Next, we shall show that (ii) $\Rightarrow$ (i).

Suppose that X is complete and totally bounded. To show that X is compact, by Theorem 7, it suffices to show that each infinite sequence $\{x_n\}$ has a convergent subsequence.

Let $\{x_n\}$ be a sequence in X. We shall construct a subsequence of $\{x_n\}$ that is a Cauchy sequence, so that in view of completeness of X, it necessarily converges. First cover X by finitely many balls of radius 1. At least one of these balls, say B_1, contains x_n for infinitely many values of n. Let J_1 be the subset of $\mathbf{N}$ consisting of those indices n for which $x_n \in B_1$.

Next, cover X by finitely many balls of radius $\frac{1}{2}$. Because J_1 is infinite, at least one of these balls, say B_2, must contain x_n for infinitely many values of n in J_1. Choose J_2 to be the set of those indices n for which $n \in J_1$ and $x_n \in B_2$. Proceeding like this, we get a sequence of $\{B_k\}$ of open balls and sequence $\{J_k\}$ of infinite sets of natural numbers such that for each $k \in \mathbf{N}$, B_k is a ball of radius $\frac{1}{k}$ and

$$J_{k+1} = \{n \in J_k : x_n \in B_{k+1}\}$$

We shall now construct a strictly increasing sequence $\{n_k\}$ with $n_k \in J_k \forall\, k$. For this, we start with any $n \in J$. Suppose we have chosen n_j, $1 \leq j \leq k$ satisfying $n_j \in J_j, 1 \leq j \leq k$ and $n_j < n_{j+1}$ for $1 \leq j \leq k-1$. Choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$; this we can do because J_{k+1} is an infinite set. Now for $i, j \geq k$, the indices n_i and n_j both belong to J_k (because $J_1 \supset J_2 \supset \ldots$ is a nested sequence of sets). Therefore, for all $i, j \geq k$, the points x_{n_i} and x_{n_j} are contained in a ball B_k of radius 1/k. It follows that the sequence (x_{n_i}) is a subsequence of $\{x_n\}$ which is a Cauchy in X, as desired. Since X is complete, the sequence $\{x_{n_i}\}$ is convergent.

This shows that the seuqnce $\{x_n\}$ is convergent. Thus, X is sequentially compact. Hence, X is compact. □

Remark 5: The above theorem and the fact that totally bounded sets are bounded in $\mathbf{R}^n$ asserts that a set in $\mathbf{R}^n$ is compact iff it is complete and bounded. Thus, we have shown that Heine-Borel theorem holds for $\mathbf{R}^n$

Next, we shall discuss another characterization for compact sets. Before that we make a definition.

Definition 8: A family $\mathcal{F}$ of subsets of a set X is said to have the finite intersection property (f.i.p.), if the intersection of any finite number of sets in $\mathcal{F}$ is non-empty, that is, for every finite collection $F_1, F_2, \ldots, F_n \in \mathcal{F}$,

$$\bigcap_{i=1}^{n} F_i \neq \phi$$

Here are some examples.

Example 9:

i) The family of closed intervals given by

$$\mathcal{F} = \{[-1/n, 1/n] : n \in \mathbf{N}\}$$

has the finite intersection property.

ii) The family of open intervals

$$\mathcal{F} = \{]0, 1/n[: n \in \mathbf{N}\}$$

has finite intersection property, since for any finite natural numbers $n_1, n_2, \ldots, n_p$

$$\bigcap_{i=1}^{p}]0, n_i[=]0, b[\neq \phi, \ i = 1$$

where $b = \min\left\{\frac{1}{n_1}, \ldots \frac{1}{n_p}\right\} > 0$. Note that $\mathcal{F}$ itself has empty intersection.

* * *

Theorem 9: A metric space X is compact iff every family $\mathcal{F}$ of closed subsets of X with finite intersection property, has itself non-empty intersection, that is

$$\bigcap_{F \in \mathcal{F}} F \neq \phi \qquad (2)$$

Proof : Let X be a compact metric space and $\mathcal{F}$ be a family of closed subsets of X with finite intersection property. We have to show that (2) holds.

Suppose, if possible

$$\bigcap_{F \in \mathcal{F}} F = \phi$$

Then we can write $X = X - \bigcap_{F\in\mathcal{F}} F$. For our convenience, we shall write $\bigcap$ for $\bigcap_{F\in\mathcal{F}}$. By DeMorgan law

$$X - \bigcap_{F\in\mathcal{F}} F = \bigcup_{F\in\mathcal{F}} (X - F)$$

Therefore, $X = \bigcup_{F\in\mathcal{F}} (X - F)$. Since each F is closed, $X - F$ is open and so $\{(X - F)\}_{F\in\mathcal{F}}$ forms an open covering for X. Since X is compact, there exists a finite number of sets $F_1, F_2, \ldots, F_n$ such that $X = \bigcup_{i=1}^{n} (X - F_i)$ which implies that $X = X - \bigcap_{i=1}^{n} F_i$. Hence $\bigcap_{i}^{n} F_i = \phi$. This contradicts the fact that $\mathcal{F}$ has the finite intersection property. Therefore, our assumption is wrong. Hence (2) holds.

Conversely, let every collection of closed subsets of X with finite intersection property have non-empty intersection. We shall show that X is compact. Let $\{G_i\}$ be an open covering of X so that $X = \cup G_i$ which gives $X - \cup G_i = \phi$. This implies by DeMorgan law, that $\cap(X - G_i) = \phi$. Thus, $\{(X - G_i)\}$ is a collection of closed sets with empty intersection and so by hypothesis this collection does not have finite intersection property. Hence, there exists a finite number of sets

$$X - G_{i_1}, X - G_{i_2}, \ldots, X - G_{i_n}$$

such that

$$\bigcap_{j=1}^{n} (X - G_{i_j}) = \phi$$

which implies that

$$X - \bigcup_{i=1}^{n} G_i = \phi$$

So that we have

$$X = \bigcup_{i=1}^{n} G_i$$

that is, the open covering $\{G_i\}$ has a finite subcovering and hence X is compact. □

You can try some exercises now.

E6) Which of the following sets are compact?

i) $\{(x_1, y_1) \in \mathbf{R}^2 : x_1^2 + y_1^2 < 1\}$

ii) $\{(x_1, y_1) \in \mathbf{R}^2 : x_1^2 + y_1^2 \leq 1\}$

iii) $\{(x_1, y_1) \in \mathbf{R}^2 : x_1^2 + y_1^2 = 1\}$

iv) $\{(x_1, y_1 \in \mathbf{R}^2 : x_1^2 + y^1 > 1\}$

v) $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 - z^2 = 1\}$.

E7) Which of the following are totally bounded?

i) (0, 1]

ii) $\mathbf{R}^n$ (with the usual metric)

iii) $\mathbf{Z}$ (with the discrete metric)

E8) Show that every finite subset of a metric space is totally bounded.

In the next section, we shall discuss connection between continuity and compactness.

3.4 CONTINUITY AND COMPACTNESS

You can try some exercises now.

Here we shall look at the relationship between continuity and compactness. We shall start with a theorem.

Theorem 10: The continuous image of a compact metric space is compact.

Proof : Let $f : X \to Y$ be continuous and X be compact. We have to show that Y is compact. Let $\mathcal{U} = \{A_i\}_{i \in I}$ be an open cover for Y. Then the collection

$$\mathcal{F} = \{f^{-1}(A_i) : A_i \in \mathcal{U}\}$$

is an open cover for X. Since X is compact, there exists finitely many $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ such that the collection

$$\{f^{-1}(A_{i_j}\ j = 1, \ldots, n\}$$

covers X. Then the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ cover Y. Hence the result. □

As a corollary of the above theorem we have the following result.

Corollary: A continuous real-valued function defined on a compact metric space attains its bounds.

Proof : Let X be a compact space and $f : X \to \mathbf{R}$ be continuous. Then by the theorem above, f(X) is a compact subset of $\mathbf{R}$. Therefore, by Heine-Borel theorem it is closed and bounded. Then there exists a_1, a_2 in f(X) such that $\text{lub}(f(X)) = a_1$ and $\text{glb}\,(f(X)) = a_2$. This means that there exist x_1, x_2 in X such that $f(x_1) = a_1$ and $f(x_2) = a_2$. Hence the result. □

We shall start with a theorem.

Theorem 11: Any continuous function f from a compact (metric) space X to another metric space Y is bounded, that is, f(X) is a bounded subset of Y.

Proof : Fix any $y \in Y$. For each $n \in \mathbf{N}$, the ball $B(y, n)$ is an open set in Y and, therefore,

$$U_n := f^{-1}(B(y, n))$$

is open in X. Note also that for each n $U_n \subset U_{n+1}$. The collection $\{U_n : n \in \mathbf{N}\}$ is an open cover of the compact space X. Let $\{U_{n_i} : 1 \leq i \leq m\}$ be a finite subcover for X and let $N := \max\{n_i\}$. Then $X = U_N$, thus, $f(X) \subset B(y, N)$. Hence f(X) is bounded. □

In Unit 1, we saw that a **continuous function need not be uniformly continuous**. Our next theorem shows that for functions whose domain is compact continuity and uniform continuity are equivalent concepts.

Theorem 12: Any continuous function from a compact metric space to any other metric space is uniformly continuous.

Proof: Let $f : (X, d) \to (Y, d)$ be continuous. Assume that X is compact. We need to prove that f is uniformly continuous on X.

For a given $\epsilon > 0$, for each x, there exists $\delta_x > 0$ by continuity of f at x. Since X is compact the open cover $\{B(x, \delta_x) : x \in X\}$ admits a finite subcover, say $\{B(x_i, \delta_i) : 1 \le i \le n\}$ where $\delta_i = \delta_{x_i}$. One may be tempted to believe that if we set $\delta := \min\{\delta_i : 1 \le i \le n\}$, it might work. See where the problem is. Once you arrive at a complete proof, you may refer to the proof below.

We now modify the argument and complete the proof. Given $\epsilon > 0$, by the continuity of f at x, there exists an $r_x > 0$ such that

$$d(x, y) < r_x \Rightarrow d(f(x), f(y)) < \epsilon/2.$$

Instead of the open cover $\{B(x, r_x) : x \in X\}$, we consider the open cover $\{B(x, r_x/2) : x \in X\}$ and apply compactness. Let $\{B(x_i, r_i/2) : 1 \le i \le n\}$ be a finite subcover. (Here $r_i = r_{x_i}, 1 \le i \le n$.) Let $\delta := \min\{r_i/2 : 1 \le i \le n\}$. Let $x, y \in X$ be such that $d(x, y) < \delta$. Now $x \in B(x_i, r_i/2)$ for some i. Since $d(x, y) < \delta$, we see that

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + r_i/2 \le r_i.$$

Thus, $y \in B(x_i, r_i)$. It follows that

$$d(f(x), f(y)) \le d(f(x), f(x_i)) + d(f(x_i) + f(y)) < \epsilon/2 + \epsilon/2,$$

by our choice of r_i. Thus, f is uniformly continuous. □

Try these exercises now.

E9) Give another proof of the Theorem 10 using Theorem 7.

With this we come to an end of this unit. Let us summarise the points we have discussed in this unit.

3.5 SUMMARY

In this unit, we have covered the following points.

1. We have introduced the notions of open cover, subcover and finite cover in a metric space.
2. We have defined compact sets in a metric and discussed the following properties:
   i) Every compact set in a metric space is closed and bounded.
   ii) Closed subsets of compact sets are compact.

iii) If A and B are compact sets in a metric space X, then $A \cup B$ and $A \cap B$ are compact.

iv) An infinite subset of a compact metric space has a limit point.

3. We have shown that Heine-Borel theorem holds for $\mathbf{R}^n$.

4. We have obtained the following three characterisations of compact sets.

   i) X is compact if and only if X is sequentially compact.

   ii) X is compact if and only if X is complete and totally bounded.

   iii) X is compact iff every family $\mathcal{F}$ of closed subsets of X with finite intersection property, has itself non-empty intersection, that is

$$\bigcap_{F \in \mathcal{F}} F \neq \phi$$

5. We have explained the relationship between continuity and compactness.

6. We have shown that any continuous function from a compact metric space to any other metric space is uniformly continuous.

3.6 HINTS/SOLUTIONS

E1) **Hint:** There is no member of the collection which contains the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

E2) **Hint:** Note that each point of $\mathbf{R}^2$ is at a distance less than 1 from at least one point of $\mathbf{R}^2$ both of whose coordinates are integers.

E3) Since singleton sets are open sets in a metric space, the finite sets are the only compact sets in a metric space.

E4) Theorems 1 shows that $F \cap K$ is closed; since $F \cap K \subset K$ and K is compact, we get that $F \cap K$ is compact.

E5) **Hint:** For any $n \in \mathbf{N}$, let $A_n = [-n, n]$. Then each A_n is compact and $\bigcup_{n \in \mathbf{N}} A_n = \mathbf{R}$ is not compact.

E6) i) This set is not closed in $\mathbf{R}^2$ and so is not compact.

   ii) This set is closed and bounded and hence compact.

   iii) It is compact by Heine Borel theorem.

   iv) It is not bounded and so is not compact.

E7) i) Since (0, 1] is a bounded set in $\mathbf{R}$, it is totally bounded.

   ii) $\mathbf{R}^n$ is not bounded since it is not compact.

   iii) $\mathbf{Z}$ is not compact and, therefore, not totally bounded.

E8) **Hint:** It follows from the definition.

E9) Suppose that $\{y_n\}$ is a sequence in f(X). Then $\exists$ a sequence $\{x_n\}$ such that $f(x_n) = y_n$. Since X is compact, there exists a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ which converges to some point $x \in X$. Since f is continuous we have

$$f(x_{n_i}) \to f(x)$$

Let $y_{n_i} = f(x_{n_i})$. Then $\{y_{n_i}\}$ is a subsequence of $\{y_n\}$ which converges to $f(x)$. Hence by Theorem 7 $f(X)$ is compact.