UNIT 1 SIMILARITY

Structure

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1.1 INTRODUCTION

In your undergraduate course on linear algebra you would have studied the basics of vector spaces: linear independence, bases, linear transformations, eigenvalues and eigenvectors, characteristic and minimal polynomials. In this unit, we start by recalling how a linear operator T on an n-dimensional vector space V can be seen as an $n \times n$ matrix w.r.t. a fixed **ordered basis**. This gives a one-one onto correspondence between the set of linear transformations and the set of $n \times n$ matrices. This is actually not a canonical map, that is, the matrix of T changes if the ordered basis changes.

Interestingly, the matrices of linear transformations w.r.t. the different ordered bases will be similar, as you will see in Sec. 1.3. Similarity is important for us to study, chiefly because two similar matrices will have the same eigenvalues. Translating this to linear operators, it means that a change of basis will not alter the set of eigenvalues of the linear operator T.

Similarity leads us very naturally to the process of diagonalising a matrix or a linear transformation, which we shall discuss in Sec. 1.4. Apart from similar operators, we will also discuss nilpotent operators here.

We emphasise here, as we did in the course introduction, that you must try every exercise as you get to it. This will help you check whether you have understood the concepts and results discussed upto that point. Further, after studying the unit, you need to re-check whether you have achieved the following unit objectives.

Objectives

After going through this unit, you should be able to

- explain, and give examples of, similar matrices;
- prove, and apply, the result that similarity preserves trace, determinant, eigenvalues, and hence the minimal polynomial;
- define the algebraic and geometric multiplicity of eigenvalues;
- obtain, and apply, a characterisation of diagonalisable operators.

1.2 MATRIX OF A LINEAR TRANSFORMATION

The purpose of this section is to review some results from linear algebra that are needed in this course, and to establish certain notations, which we will use throughout this text.

Let V be a vector space of dimension n over a field F, and let $B = \{v_1, \ldots, v_n\}$ be a fixed ordered basis of V. For each vector v in V, there are unique $\alpha_1, \ldots, \alpha_n$ in F such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. We write these scalars in the form of a column matrix as $[v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in F^n$.

Since, for a fixed B, these scalars are uniquely determined by $v \in V$, this defines a mapping $[\ ]_B : V \to F^n$. Verify that the mapping $[\ ]_B$ is actually an isomorphism of vector spaces.

Let us look at some examples.

Example 1: Let V be the vector space of all polynomials of degree at most 2, with coefficients from the field of rational numbers, **Q**. Let $p(t) \in V$ be given by $p(t) = t^2 - 4t + 3$.

i) Find $[p(t)]_B$, where $B = \{1, t, t^2\}$.

ii) Find $[p(t)]_{B'}$, where $B' = \{t^2, t, 1\}$.

Solution i) Since $p(t) = 3 - 4t + t^2$, $[p(t)]_B = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$.

ii) Since $p(t) = t^2 - 4t + 3$, $[p(t)]_{B'} = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$.

Note: From Example 1, you can see that the change in the order of the elements of the basis changes the matrix representation of p(t). This is why we usually fix the ordering of a basis for a given situation.

Example 2: Find $[x]_B$, where $x = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbf{Q}^3$, and $B = \{e_1, e_2, e_3\}$, where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution: Since $x = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, $x = \alpha e_1 + \beta e_2 + \gamma e_3$, and so $[x]_B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$.

Since $[x]_B = x \ \forall x \in \mathbf{Q}^3$, $\{e_1, e_2, e_3\}$ is called the **standard basis of $\mathbf{Q}^3$**.

Remark: If F is a field and e_i is a column matrix of size n whose ith entry is 1 and other entries are 0, then the ordered basis $\{e_1, \ldots, e_n\}$ is called the **standard basis of F^n**. The reason for this is the same as given in Example 2.

Example 3: Let $V = \mathbf{Q}^3$ and $y = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$, with respect to the standard basis of $\mathbf{Q}^3$. Find $[y]_{B'}$, where $B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: Write $y = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha+\beta+\gamma \\ \alpha+\beta \\ \alpha \end{bmatrix}$

This gives $\alpha = 3, \beta = 1, \gamma = 2$, and $[y]_{B'} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Now, consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2 : T(x, y, z) = (x + y, z)$. Let the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^2$ be B and B', respectively.

Then $[T(1,0,0)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[T(0,1,0)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $[T(0,0,1)]_{B'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We call the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ the matrix of T with respect to the bases B and B'. This is the matrix whose columns are $[T(e_i)]_{B'}$, $i = 1,2,3$. More generally, we have the following definition.

Definition: Let V and W be vector spaces over F of dimensions n and m, respectively. Let $T : V \to W$ be a linear transformation. Let $B = \{v_1, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_m\}$ be ordered bases of V and W. Then, for each $v_j \in V, T(v_j) \in W$, and so $[T(v_j)]_{B'} \in F^m$. By ${}_{B'}[T]_B$, we denote the $m \times n$ matrix whose jth column is $[T(v_j)]_{B'}$ e.g. $\forall\ j = 1,2,\ldots,n$. The matrix ${}_{B'}[T]_B$ is called **the matrix of the linear transformation T** with respect to the bases B and B'.

If $T : V \to V$, that is, $T \in L(V)$, and $B = B'$, we write $[T]_B$ for ${}_{B'}[T]_B$, which is called **the matrix of the linear operator T** with respect to B.

For example, if I denotes the identity linear operator on the n-dimensional vector space V, then for any basis B of V, $[I]_B = I_n$, the $n \times n$ identity matrix. So, the matrix of the identity operator on V with respect to any basis of V is the identity matrix.

Here are some exercises for you now.

E1) Find the matrices of the following linear transformations with respect to the standard bases of the spaces concerned.

(i) $T_1 : \mathbf{R}^n \to \mathbf{R} : T_1(x_1, \cdots, x_n) = \frac{1}{n} x_1$

(ii) $T_2 : \mathbf{R}^4 \to \mathbf{R}^2 : T_2(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2)$

Let us now go further into what happens with a change of basis. Let B and B' be bases of an n-dimensional vector space V over F. Let $B = \{v_1, \ldots, v_n\}$. Then, for any $v \in V$,

$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_1, \ldots, \alpha_n$ in F.

So, $Tv = \alpha_1 Tv_1 + \cdots + \alpha_n Tv_n$.

Since the mapping $[\ ]_{B'}$ is linear, we have

$$[Tv]_{B'} = \alpha_1 [Tv_1]_{B'} + \cdots + \alpha_n [Tv_n]_{B'}$$

$$\text{So, } [Tv]_{B'} = {}_{B'}[T]_B \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = {}_{B'}[T]_B [v]_B.$$

Hence, for $T \in L(V)$, $v \in V$ and bases B and B' of V,

$$[Tv]_{B'} = {}_{B'}[T]_B [v]_B \qquad \ldots(1)$$

Next, if S, $T \in L(V)$ and B, B', B'' are ordered bases of V, then for $v \in V$, using (1), we have

$$[(S \circ T)(v)]_{B''} = {}_{B''}[S \circ T]_B [v]_B.$$

Also $(S \circ T)(v) = S(Tv)$, and so again by (1),

$$[(S \circ T)(v)]_{B''} = {}_{B''}[S]_{B'} [T(v)]_{B'} = {}_{B''}[S]_{B'}\, {}_{B'}[T]_B [v]_B$$

Thus

$${}_{B''}[S \circ T]_B [v]_B = {}_{B''}[S]_{B'}\, {}_{B'}[T]_B [v]_B.$$

Since the identity above holds for every $v \in V$, and $[\ ]_B$ is one-one, we have the following identity:

$${}_{B''}[S \circ T]_B = {}_{B''}[S]_{B'}\, {}_{B'}[T]_B \qquad \ldots(2)$$

The next theorem states how a change of basis changes the matrix of the linear transformation.

Theorem 1: Let V be an n-dimensional vector space over a field F and let $T \in L(V)$. If B and B' are ordered bases of V, then there is an invertible matrix P such that $[T]_{B'} = P^{-1} [T]_B P$.

Proof. The proof is actually a repeated use of the identity (2).

$$[T]_{B'} = [I \circ T]_{B'} = {}_{B'}[I]_B\, {}_B[T]_{B'} = {}_{B'}[I]_B\, {}_B[T \circ I]_{B'} = {}_{B'}[I]_B [T]_B\, {}_B[I]_{B'}$$

Also, $I_n = [I]_B = {}_B[I]_{B'\,B'}[I]_B$, and $I_n = [I]_{B'} = {}_{B'}[I]_{B\,B}[I]_{B'}$.

Therefore, if $P = {}_B[I]_{B'}$, then $P^{-1} = {}_{B'}[I]_B$. This proves the proposition.

Theorem 1 can be restated as follows.

Theorem 1: Let B and B' be ordered bases of V and let $T \in L(V)$. Then

$$[T]_{B'} = {}_{B'}[I]_{B}\,[T]_{B\,B}\,[I]_{B'}. \qquad \ldots(3)$$

(3) is called the **change of basis formula**.

We illustrate this with an example.

Example 4: Consider the vector space $\mathbf{P}_3(\mathbf{R})$ of polynomials with real coefficients and having degree at most 3. Let $B = \{1, t, t^2, t^3\}$ and
$B' = \{q_1(t) = 1 - t,\ q_2(t) = 1 + t,\ q_3(t) = t^2 - t^3,\ q_4(t) = t^2 + t^3\}$ be ordered bases of $\mathbf{P}_3(\mathbf{R})$.

Let D be the differential operator on $\mathbf{P}_3(\mathbf{R})$. Find

i) $[D]_B$,

ii) $[D]_{B'}$,

iii) an invertible matrix P such that $[D]_{B'} = P^{-1}[D]_B P$.

Solution: i) Since $D(1) = 0,\ D(t) = 1,\ D(t^2) = 2t,\ D(t^3) = 3t^2$,

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

ii) You can see that

$$Dq_1(t) = -1 = -\frac{1}{2}q_1(t) - \frac{1}{2}q_2(t),$$

$$Dq_2(t) = 1 = \frac{1}{2}q_1(t) + \frac{1}{2}q_2(t),$$

$$Dq_3(t) = 2t - 3t^2 = -q_1(t) + q_2(t) - \frac{3}{2}q_3(t) - \frac{3}{2}q_4(t),$$

$$Dq_4(t) = 2t + 3t^2 = -q_1(t) + q_2(t) + \frac{3}{2}q_3(t) + \frac{3}{2}q_4(t).$$

Thus,

$$[D]_{B'} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 1 \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

iii) Next, we find P such that $[D]_{B'} = P^{-1}[D]_B P$. As we have seen in (3), P is actually ${}_B[I]_{B'}$. Thus, P is the 4×4 matrix with columns $[q_1(t)]_B, [q_2(t)]_B, [q_3(t)]_B$ and $[q_4(t)]_B$. This gives

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

You should similarly find ${}_{B'}[I]_B$, and verify that it is actually P^{-1}, and also that $[D]_{B'} = P^{-1}[D]_B P$.

You can try the following exercises now.

E2) Let $T: \mathbf{R}^3 \to \mathbf{R}^2 : T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x-y+2z \end{bmatrix}$. Find the matrix of T with respect to the bases $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

E3) Let $T: \mathbf{R}^3 \to \mathbf{R}^3 : T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x-y+z \\ x+y-z \end{bmatrix}$. Find $[T]_B$, where $B = \left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$, and $[T]_{B'}$, where $B' = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Also find an invertible matrix P such that $[T]_{B'} = P^{-1}[T]_B P$.

The change of basis formula leads us to the concept of similarity, which we shall now discuss.

1.3 SIMILAR MATRICES

In Theorem 1, you saw that if B and B′ are two bases of a vector space V, and if $T \in L(V)$, then $[T]_{B'} = P^{-1}[T]_B P$ for some non-singular matrix P. In fact, this relationship between $[T]_B$ and $[T]_{B'}$ shows that they are similar to each other, as the following definition tells us.

Definition: Let A and B be $n \times n$ matrices. We say that B is **similar** to A if there is an $n \times n$ invertible matrix P such that $B = P^{-1}AP$.

Now, if B is similar to A, is A similar to B? Note that, if $B = P^{-1}AP$, then $A = Q^{-1}BQ$, where $Q = P^{-1}$. So A is also similar to B. In other words, B is similar to A if and only if A is similar to B. Therefore, we also say that A and B are similar matrices.

A few short exercises here.

E4) Check whether "is similar to" is an equivalence relation on $\mathbf{M}_n(\mathbf{R})$.

E5) Show that the set $S = \{P^{-1}AP \mid P \text{ is invertible}\}$ is the set of all those matrices which are similar to $A \in \mathbf{M}_n(\mathbf{R})$.

E6) Find all the matrices similar to the identity matrix I_n, and all the matrices similar to $\mathbf{O} \in \mathbf{M}_n(\mathbf{R})$.

Now some examples for showing how we can check if two given matrices are similar or not.

Example 5: Check whether the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ are similar.

Solution: Looking at the elements of both the matrices, and their positions, we see that if $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $P^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

So, the given matrices are similar.

Example 6: Show that the matrices $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are not similar.

Solution: If these matrices were similar, then for some invertible matrix

$$P = [p_{ij}],\ P^{-1}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ i.e., } \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}P = P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, the first column of $P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is $P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$. Also, the first column of

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} P$ is $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$. Thus, on equating these columns, we get

$$\begin{bmatrix} 2p_{11}+p_{21} \\ 2p_{21} \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}.$$

So, we have $p_{11} = 0 = p_{21}$. Therefore, there is no such invertible matrix P.

From the examples above, you can see that showing that two matrices are not similar could be tedious. There are some properties that similar matrices share, that can help us in cutting short this tedious process. In the following theorem, we discuss some of them. But first, some remarks.

Remarks: 1) Recall that if $X,\ Y \in \mathbf{M_n}(F)$, then $\det(XY) = \det(X)\det(Y)$. So if X is invertible, $\det(X^{-1}) = 1/\det(X)$.

2) The **trace of the matrix X**, written as **tr X**, is the sum of the diagonal entries of X, that is, $\operatorname{tr} X = \sum_{i=1}^{n} x_{ii}$, where $X = [x_{ij}]$. You can verify that $\operatorname{tr}(XY) = \operatorname{tr}(YX)$.

Now, let us state the theorem.

Theorem 2: Let $A, B \in \mathbf{M_n}(F)$ be similar matrices. Then

i) $\det A = \det B$,

ii) $\operatorname{tr} A = \operatorname{tr} B$,

iii) A and B have the same characteristic polynomials.

Proof: i) Let P be an invertible matrix so that $B = P^{-1}AP$. Then
$\det B = \det(P^{-1}AP) = \det P^{-1} \det A \det P = (1/\det P)\det P \det A = \det A$.

ii) $\operatorname{tr} B = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(P^{-1}(AP)) = \operatorname{tr}((AP)P^{-1}) = \operatorname{tr}(A(PP^{-1})) = \operatorname{tr} A$.

iii) If $C_A(t)$ and $C_B(t)$ are the characteristic polynomials of A and B, then

$$C_B(t) = \det(tI_n - B) = \det(tI_n - P^{-1}AP) = \det(P^{-1}(tI_n - A)P)$$

$$= \det(tI_n - A).$$

$$= C_A(t).$$

Using Theorem 2, we are now in a position to make the following definitions.

Definition: Let V be an n-dimensional vector space over a field F, and let $T \in L(V)$. Let B be an ordered basis of V. We define **the determinant and the trace of T** as $\mathbf{\det T = \det[T]_B}$ and $\mathbf{\operatorname{tr} T = \operatorname{tr}[T]_B}$.

Now, how does Theorem 2 help us in ensuring that these terms are well-defined? We need to know that the definition does not depend on the basis we choose for B. So, if

we take two different bases B and B′ of V, is det $[T]_B = [T]_{B'}$ and $tr[T]_B = tr[T]_{B'}$?
Since the change of basis is a similarity transformation and since similar matrices have the same determinant and the same trace, it follows that these definitions of the determinant and the trace of a linear transformation are independent of the choice of basis, and so are well-defined.

Here are some exercises now.

E7) Check whether $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ are similar.

E8) Give an example of two matrices which have the same determinant and trace, but are not similar.

E9) Find the determinant and the trace of the linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$, defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x-y+z \\ x+y-z \end{bmatrix}.$$

E10) Let V be a finite-dimensional vector space over a field F. Let S and T be linear operators on V. Show that if $\alpha \in F$ and B is an ordered basis of V, then $[\alpha S + T]_B = \alpha[S]_B + [T]_B$.

E11) Show that the matrices $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are not similar.

Now that you have studied similar matrices, let us see when a matrix is similar to a diagonal matrix.

1.4 DIAGONALISABILITY

In your undergraduate studies you must have already studied a bit about diagonalisable matrices. You can also refer to Block 3, MTE-02, which is the IGNOU course "Linear Algebra".

Let V be an n-dimensional vector space over F, and let $T \in L(V)$. Then, as you know, V cannot have more than n linearly independent eigenvectors. If T has n linearly independent eigenvectors, $v_1, v_2, \cdots, v_n$, then $B = \{v_1, \ldots, v_n\}$ is a basis of V. Suppose $Tv_i = \lambda_i v_i$ for $i = 1, \ldots, n, \lambda_i \in F$, where not all λ_i's need be distinct. Then

$$[T]_B = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}, \text{ a diagonal matrix.}$$

This leads us to the following definitions.

Definitions: A linear operator T on a vector space V is called **diagonalisable** if V has a basis consisting of eigenvectors of T. An $n \times n$ matrix is called **diagonalisable** if it is similar to a diagonal matrix.

Now, how do we find out if a linear operator has enough linearly independent eigenvectors to be diagonalisable? The following theorem helps us in this.

Theorem 3: Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof: Let $\lambda_1,\ldots,\lambda_k$ be distinct eigenvalues of T, and let $u_1,\ldots,u_k$ be corresponding eigenvectors, i.e., $Tu_i = \lambda_i u_i$, $i = 1,\ldots,k$.

For each i, define $S_i = \dfrac{(T-\lambda_1 I)\cdots(T-\lambda_{i-1}I)(T-\lambda_{i+1}I)\cdots(T-\lambda_k I)}{(\lambda_i-\lambda_1)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_k)}$.

Then S_i is a linear operator on V, $S_i u_j = 0$ for $i \neq j$ and $S_i u_i = u_i$.

Now suppose that $\alpha_1 u_1 + \cdots + \alpha_k u_k = \mathbf{0}$, $\alpha_i \in F \forall i$. Then $0 = S_i(\alpha_1 u_1 + \cdots + \alpha_k u_k) = \alpha_i u_i$, and so $\alpha_i = 0$.
Hence $u_1,\ldots,u_k$ are linearly independent.

It is immediate from the theorem above that **if T has n (= dimV) distinct eigenvalues, then T is diagonalisable**. Let us use this fact in an example.

Example 7: Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalisable.

Solution: The characteristic polynomial of A is $(x-2)(x-1)(x+1)$. Therefore, A has 3 distinct eigenvalues, and hence is diagonalisable.

Example 8: Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalisable.

Solution: If A is diagonalisable, then A must be similar to I_2 (Why?). If there is an invertible matrix P such that $P^{-1}AP = I_2$, then $A = PI_2P^{-1} = I_2$, a contradiction. So, A is not diagonalisable.

Note: You know that if a matrix has all its eigenvalues distinct, then it is diagonalisable. However, even if it doesn't have all its eigenvalues distinct, it can still be diagonalisable. The only condition is to find **enough** linearly independent eigenvectors. Let us consider an example of this situation.

Example 9: Show that $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalisable.

Solution: A has eigenvalues 1 and 2. Let us find bases for the eigenspaces of A, W_1 and W_2.

Now, $Ae_1 = 2e_1$, $Ae_2 = 2e_2$, where $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. So, W_2 has dimension 2, with a basis $\{e_1, e_2\}$.

Next, W_1 will have at least one eigenvector, say x. According to Theorem 3, $\{x, e_1, e_2\}$ is a basis of $\mathbf{R}^3$. Hence A is diagonalisable.

Here are some exercises now.

E12) Check whether the following matrices are diagonalisable.

i) $\begin{bmatrix}0 & 1 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}$, ii) $\begin{bmatrix}2 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$

Our aim now is to establish necessary and sufficient conditions for a linear operator to be diagonalisable. For this we require the following definitions.

Definitions: Let λ be an eigenvalue of a linear operator T on a vector space V. The dimension of $\text{Ker}(T - \lambda I)$ is the **geometric multiplicity** of the eigenvalue λ. The multiplicity of λ, as a root of the characteristic polynomial of T, is the **algebraic multiplicity** of λ.

For instance, in Example 8, the geometric multiplicity of the eigenvalue 1 is 1 (Why?) and the algebraic multiplicity of 1 is 2. Again, in Example 9, the algebraic and geometric multiplicities of the eigenvalue 2 are both 2.

Why don't you try to find some multiplicities yourself?

E13) Obtain the algebraic and geometric multiplicities of the operators given by the matrices in E12. Do you see any relationship between the two kinds of multiplicities?

While doing E13, you may have found the following relationship between the two kinds of multiplicities.

Theorem 4: The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.

Proof: Let λ be an eigenvalue of an operator T on a vector space V of dimension n. Let the geometric multiplicity of λ be m. Let $\{u_1, \cdots, u_m\}$ be a basis of $\ker(T - \lambda I)$. Then $Tu_i = \lambda u_i$ for $i = 1, \ldots, m$. So, T has m linearly independent eigenvectors $u_1, \ldots, u_m$. Extend this basis of $\ker(T - \lambda I)$ to a basis of V, say, $B = \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$.

Then $[T]_B = \begin{bmatrix}\lambda I_m & C\\ \mathbf{0} & A\end{bmatrix}$.

Therefore, $c_T(t) = (t - \lambda)^m \det(tI_{n-m} - A)$.

So the algebraic multiplicity of λ is at least m.

Finally, we give a necessary and sufficient condition for T to be a diagonalisable operator.

Theorem 5: Let T be a linear operator on an n-dimensional vector space over a field F. Assume that the characteristic polynomial of T has all its roots in F. Then T is diagonalisable if and only if for each eigenvalue $\lambda \in F$, its algebraic multiplicity **is equal to** its geometric multiplicity.

Proof: Assume first that the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T with geometric multiplicity $n_1, \ldots, n_k$, respectively. Since the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, the characteristic polynomial of T is $(t-\lambda_1)^{n_1} \cdots (t-\lambda_k)^{n_k}$, and $n_1 + \cdots + n_k = n$. Let $B_i = \{u_{i1}, \ldots, u_{in_i}\}$ be the set of linearly independent eigenvectors corresponding to the eigenvalue λ_i. Then, to show that T is diagonalisable, we need to verify that $B = \bigcup_{i=1}^{k} B_i$ is a basis of V.

Since B has n elements, all we need to check is that B is a linearly independent set. For this purpose, let S_i be a linear operator, as in Theorem 3. Then $S_i u_{rs} = 0$ if $i \neq r$, and $S_i u_{is} = u_{is}$. To check the linear independence of B, assume that

$$\sum_{p=1}^{k} \sum_{q=1}^{n_p} \alpha_{pq} u_{pq} = \mathbf{0}.$$

Then, operating S_i on both sides, you can see that $\sum_{q=1}^{n_i} \alpha_{iq} u_{iq} = 0$.

Since the elements of B_i are linearly independent, this proves the result.

Conversely, assume that T is diagonalisable. Then V has a basis consisting of eigenvectors of T. Let $B = \bigcup_{i=1}^{k} B_i$, where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T and $B_i = \{u_{i1}, \ldots, u_{in_i}\}$ is the set of linearly independent eigenvectors of T corresponding to λ_i. Then $Tu_{ij} = \lambda_i u_{ij}$ for $j = 1, \ldots, n_i$, $i = 1, \ldots, k$. Thus,

$$[T]_B = \begin{bmatrix} \lambda_1 I_{n_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_k I_{n_k} \end{bmatrix}.$$

Therefore, the characteristic polynomial of T is $(t-\lambda_1)^{n_1} \cdots (t-\lambda_k)^{n_k}$. Hence the algebraic multiplicity of each λ_i is n_i. Since $u_{i1}, \ldots, u_{in_i}$ are linearly independent eigenvectors corresponding to the eigenvalue λ_i, it follows that the geometric multiplicity of λ_i is at least n_i. Since the geometric multiplicity cannot exceed the algebraic multiplicity, it follows that both are equal for each eigenvalue λ_i.

Now an example to show how Theorem 5 can be useful.

Example 10: Show that the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not diagonalisable.

Solution: The algebraic multiplicity of 1 is 2, and the geometric multiplicity of 1 is one. So, by Theorem 5, the matrix cannot be diagonalisable.

Try some exercises now.

E14) Let $T: \mathbf{C}^3 \to \mathbf{C}^3$ be defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ y-z \\ z-x \end{bmatrix}$.

Find the eigenvalues and eigenvectors of T. Is T a diagonalisable linear operator? Why?

E15) Give an example of two matrices which have the same characteristic polynomial, but one is diagonalisable and the other is not.

In Example 4, Section 1.2, we discussed the differential operator D. You know that $D^4 = \mathbf{0}$ and $D^3 \neq \mathbf{0}$. In fact, this kind of property is found in many operators of the kind we define below.

Definition: A non-zero linear operator T, on a vector space V, is called **nilpotent** if for some positive integer r, $T^r v = 0$ for all $v \in V$. The positive integer k such that $T^k = \mathbf{0}$ and $T^{k-1} \neq \mathbf{0}$ is called the **nilpotency index** of T.

Note: From the definition above, it follows that the minimal polynomial of a nilpotent operator T is t^k, where k is the nilpotency index of T.

Corresponding to the definition for operators, we have the following definition for matrices.

Definition: A non-zero matrix A is **nilpotent** if $A^k = \mathbf{0}$ for some $k > 0$. The least positive integer k such that $A^{k-1} \neq \mathbf{0}$ and $A^k = \mathbf{0}$ is called the **nilpotency index of** A.

Let us consider some examples.

Example 11: Find the nilpotency index of the following matrices:

i) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ii) $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, iii) $C = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: i) You should verify that $A^2 = \mathbf{0}$. Since $A \neq \mathbf{0}$, the nilpotency index of A is 2.

ii) $B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $B^3 = \mathbf{0}$. Thus the nilpotency index of B is 3.

iii) You can verify that the nilpotency index of C is 4.

Remarks: A nilpotent operator is never diagonalisable. Can you see why? Indeed if T is nilpotent and diagonalisable, then for any ordered basis B of V, the matrix $[T]_B$ is similar to the zero matrix (why?). But we have seen that the only matrix similar to the zero matrix is the zero matrix itself.

Here's an exercise now.

E16) Let $A = \begin{bmatrix} -2 & 0 & 3 \\ -2 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$. Check whether or not $A - I_3$ is a nilpotent matrix.

We have now come to the end of this unit. Let us look at a brief overview of what we have covered in it.

1.5 SUMMARY

In this unit, we have covered the following points.

1. Change of basis is a similarity transformation.
2. Similar matrices have the same eigenvalues.
3. T is diagonalisable if the algebraic multiplicity of every eigenvalue λ of T equals its geometric multiplicity.
4. A nilpotent matrix (operator) is not diagonalisable.

1.6 SOLUTIONS/ANSWERS

E1) i) $\left[\frac{1}{n}, 0, \ldots, 0\right]$, ii) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

E2) $T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\1\end{bmatrix}-2\begin{bmatrix}0\\1\end{bmatrix}$$

$\therefore$, the required matrix is $\begin{bmatrix}2 & 2 & 2\\-1 & 1 & -2\end{bmatrix}$.

E3) $T\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}2\\0\\0\end{bmatrix}=a\begin{bmatrix}0\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\0\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\0\end{bmatrix}$, say.

Then, we get the equations

$b+c=2,\ a+c=0,\ a+b=0$.

On solving these equations, we get $b=c=1,\ a=-1$.

So, $T\begin{bmatrix}0\\1\\1\end{bmatrix}=-\begin{bmatrix}0\\1\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Similarly, you can check that

$$T\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}2\\2\\0\end{bmatrix}=2\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}=2\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

So, $[T]_B=\begin{bmatrix}-1 & 0 & 0\\1 & 0 & 2\\1 & 2 & 0\end{bmatrix}$.

You should verify that

$$[T]_{B'}=\begin{bmatrix}3 & 2 & 1\\-2 & -2 & 0\\0 & 0 & -2\end{bmatrix}.$$

Now, $P={}_B[I]_{B'}$. To find this, we write the columns of B' as a linear combination of the columns of B, as follows.

$$\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\text{So, } P=\begin{bmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

Check that $[T]_{B'}=P^{-1}[T]_B P$.

E4) Let us say $A \sim B$ iff A is similar to B for $A, B \in \mathbf{M}_n(\mathbf{R})$.

Now, $A \sim A$.

Also, $\sim$ is a symmetric and transitive relation.

Hence $\sim$ is an equivalence relation on $\mathbf{M}_n(\mathbf{R})$.

E5) By definition, any element of S is similar to A.

Conversely, if $B \sim A$, then $\exists$ an invertible matrix P such that $B=P^{-1}AP$, i.e., $B \in S$.

E6) From E5 you can see that the only matrix similar to I_n is I_n. And the only matrix similar to $\mathbf{0}$ is $\mathbf{0}$.

E7) If they were similar, they would have had the same determinant, which is not so.

E8) e.g., $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ (refer to E6).

E9) For this we find $A=[T]_B$, where B is the standard basis of $\mathbf{R}^3$. Then $\det A$ and $\operatorname{tr} A$ are what we require.

$$A=\begin{bmatrix}1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}$$

$\operatorname{Tr} A=-1$, $\det A=4$.

E10) Let $B=\{u_1, \ldots, u_n\}$. Then the jth column of $[\alpha S+T]_B$ will be $\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$, where

$$\sum_{i=1}^{n} a_i u_i = (\alpha S+T)(u_j) = \alpha S(u_j)+T(u_j) = \alpha\sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} c_i u_i,$$

where $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ are the jth columns of $[S]_B$ and $[T]_B$, respectively.

Hence the result.

E11) The reason is as in E7.

E12) i) You can check that its eigenvalues are $-1, -1$ and 2. So, to see if it is diagonalisable, we need to see if it has 2 linearly independent eigenvectors corresponding to -1.

Now, $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\Rightarrow y = -x,\ 2x + y = -y,\ -z = -z$.

So, the elements of W_{-1} are $\begin{bmatrix} x \\ -x \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So, there are two linearly independent eigenvectors, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, corresponding to $\lambda = -1$.

Therefore, the given matrix is diagonalisable.

ii) The eigenvalues are $2, 1, 1$.

Check that any element of W_1 is $\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

So, there is only one linearly independent eigenvector corresponding to $\lambda = 1$. Therefore, the matrix is not diagonalisable.

E13) i) In E12 (i), you have shown that the geometric multiplicity of $\lambda = -1$ is 2, the same as its algebraic multiplicity.

Again, both the multiplicities of $\lambda = 2$ are 1.

ii) The geometric multiplicity of $\lambda = 1$ is 1, and its algebraic multiplicity is 2.

The geometric and algebraic multiplicities of $\lambda = 2$ are 1.

Regarding the relationship, see Theorem 4.

E14) $[T]_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

Since the eigenvalues of this are all distinct, i.e., 0, $\frac{-3 \pm \sqrt{21}}{2}$, T is diagonalisable. (You can also apply Theorem 5 to show this.)

$$W_0 = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbf{C} \right\}$$

$$W_{\frac{-3+\sqrt{21}}{2}} = \left\{ \alpha \begin{bmatrix} 1 \\ (1-\lambda) \\ (1-\lambda)^2 \end{bmatrix} \middle| \alpha \in \mathbf{C}, \lambda = \frac{5-\sqrt{21}}{2} \right\}$$

$$W_{\frac{-3-\sqrt{21}}{2}} = \left\{ \alpha \begin{bmatrix} 1 \\ (1-\lambda) \\ (1-\lambda)^2 \end{bmatrix} \middle| \alpha \in \mathbf{C}, \lambda = \frac{5+\sqrt{21}}{2} \right\}.$$

E15) For example, both $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and I_2 have the same characteristic polynomial, $(t-1)^2$. However, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalisable.

E16) Check that $A - I_3 \neq \mathbf{0}$, but $(A - I_3)^2 = \mathbf{0}$.