UNIT 6 MATRICES AND DETERMINANTS

Structure

6.0 OBJECTIVES

After reading this unit you will be able to:

- Explain the concept of a matrix;
- Perform operations of matrix addition, subtraction and multiplication;
- Describe the properties of some very useful special matrices;
- Define the concept of a determinants;
- Introduce the notions of minor and cofactor and, evaluate a higher (more than second) order determinant; and
- Compute the inverse of a matrix.

6.1 INTRODUCTION

In mathematics, we are often concerned with operations involving blocks of numbers and variables. Matrix Algebra is a very powerful but convenient way to handle such situations. As the term matrix suggests, it refers to an array (combination of rows and columns).The building blocks of matrix algebra are arrays of numbers and variables. Let us now be familiar with the concepts and notations of matrix algebra by considering the matrix representation of the following simultaneous equation system

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{12}x_2 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

In the above equation system, there are m equations involving n variables represented by *x*'s, m x n coefficients represented by *a*'s and n constants represented by *b*'s. The coefficients and the constants together form the parameters of the equation system. Let us now see how these parameters and variables can be represented by different matrices or arrays. If **A** is the array for coefficients, **x** is the array for variables and **b** is the array for constants, then

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

In **A,** there are m rows and n columns. The first row consists of the coefficients attached to the variables in the first equation of the given equation system. The second row consists of the coefficients figuring in the second equation and so on. Finally, the last row consists of the coefficients attached to the variables in the last equation of the system.

The array **x** has n rows and one column. In the first row, we have the variable x_1, in the second row we have x_2 and similarly in the last row, we have the variable x_n. The array **b** also consists of m rows and one column. In the first row, we have the constant b_1 of the first equation, the second row has the constant b_2 of the second equation and in the same fashion, the last row has the constant b_m of the last equation. Thus, the three matrices **A**, **x** and **b** contain all the ingredients of the simultaneous equation system under consideration. The number of rows and columns of a particular matrix constitutes its dimension or order. We have seen that the matrix **A** consists of m rows and n columns. Accordingly, the dimension of **A** is said to be (m x n) (pronounced as m-by-n) or, we say that A is an (m x n) matrix. Similarly, **x** is (n x 1) and **b** is (m x 1) matrix. A matrix, in which the number of rows and columns are equal, is called a square matrix.

6.2 MATRIX OPERATIONS

6.2.1 Addition and Subtraction of Matrices

Suppose we have two matrices A = [a_{ij}] and B = [b_{ij}] where a_{ij} and b_{ij} are the typical elements of A and B respectively. The matrix A + B is defined as A + B = [a_{ij} + b_{ij}]. What this means is that in order to get the matrix A + B we take the elements in the first row and first column of A and B and add them together to get the element in the first row and first column of the matrix A + B. To get the element in the second row and first column of A + B, we add a_{21} and b_{21} to get $a_{21}+b_{21}$. Now notice that in each case we are adding two numbers. And we know how to do that! In general, to get the element in the ith row and the jth column of A + B, we add a_{ij} and b_{ij} to get $a_{ij}+b_{ij}$. The only problem here is that unless the matrices A and Bare compatible in a particular sense, they cannot be added together to get A+B. Why? Suppose:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -5 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 9 & 2 \end{bmatrix}$$

We want A + B. If we apply our rule we get:

$$\text{A + B} = \begin{bmatrix} 1 & + & 0 & 2 & + & 2 & * & + & 3 \\ 3 & + & 5 & 0 & + & 9 & * & + & 2 \\ -5 & + & * & 7 & + & * & * & + & * \end{bmatrix}$$

Where the (*) symbol represents a missing element. Consider the element in the first row and third column of A+B. To get this we have to add a_{13} and b_{13}. We know b_{13}=3. But A is a (3×2) matrix. So it does not have an element a_{13}! We find that the matrix A+B in this case has several (*) elements for similar reasons. And hence, A+B is meaningless in this case. The point then is that for A+B to be defined, we require that A and B have the same dimension, ie., the same number of rows and the same number of columns.

In a similar manner one can define what one means by subtraction. Given any two matrixes A and B, the matrix A-B = [a_{ij}-b_{ij}]. Check that for this definition to be meaningful, A and B must have the same dimensions.

6.2.2 Multiplication of Matrices

This is a bit more complicated. So let us begin with a series of examples.

Example 1: Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Now if you asked a mathematician to give you the product matrix AB, he would tell you immediately: AB = $\begin{bmatrix} 17 \\ 39 \end{bmatrix}$. What strikes one immediately is that multiplying a (2×2) matrix by a (2×1) matrix yields a (2×1) matrix. Suppose we want to complicate things for him. Let us give him two matrices:

Example 2: A = $\begin{bmatrix} -5 & 1 & 0 \\ 0 & 2 & -7 \\ 3 & 0 & 5 \end{bmatrix}_{(3\times 3)}$ and B = $\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 5 & 1 \end{bmatrix}_{(3\times 2)}$

and ask for the product matrix AB. He may take a little more time but in the end he is going to claim:

$$AB = \begin{bmatrix} -5 & 7 \\ -35 & 7 \\ 28 & 5 \end{bmatrix}_{(3\times 2)}$$

But suppose we ask him to give us the product matrix BA. He is going to look at it for a second and will claim that this does not exist! All this is quite troublesome and there has to be some method in his madness. If we ask him to explain he is going to say:

$\mathbf{a_i b_j} = [a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ldots.. + a_{in}\, b_{nj}]$

Let us check this. We turn to Example 1.

Look at A. It has 2 rows, so I = 1, 2; and two columns so j=1, 2. The matrix B has two rows, so i = 1, 2 and one column, so j = 1.

Let us try to compute the element in the first row and first column position of AB. If we follow his rule, then, we get:

$a_{11}\, b_{11} + a_{12}\, b_{21} = 1\times 5 + 2\times 6 = 17$

Then we want the element in the second row, first column position. This, according to his rule should be:

$a_{21}\, b_{11} + a_{22}\, b_{21} = 3\times 5 + 4\times 6 = 39$

Suppose we wanted to carry on and get the term in the first row second column position. Then we would have to compute the sum: $a_{11}\, b_{12} + a_{12}\, b_{22}$. But b_{12} and b_{22} do not exist. So we cannot compute this sum. Now we try to compute the element in the second row and second column position. Once again you end up with the same problem. The essential point is that if you have two matrices A and B, where A is of dimension $(m\times n)$ and B is of dimension $(p\times q)$ then AB is defined if and only if n = p and AB is of dimension $(m\times q)$.

Let us return to the introduction of this unit to check why.

$$\begin{bmatrix} 3 & 1 \\ 7 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 20 \end{bmatrix} \qquad (1)$$

is equivalent to:

$3x_1 + x_2 = 0$ (i)

$7x_1 - 2x_2 = 5$ (ii)

$2x_1 + 5x_2 = 20$ (iii)

$$\begin{bmatrix} 3x_1 & + & x_2 \\ 7x_1 & - & 2x_2 \\ 2x_1 & + & 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 20 \end{bmatrix}$$

But two vectors are the same if they are equal element by element. So,

$3x_1 + x_2 = 0$ (i)

$7x_1 - 2x_2 = 5$ (ii)

$2x_1 + 5x_2 = 20$ (iii)

Check Your Progress 1

1) Let $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix}$

Find i) A+ B

ii) A – B

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

2) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

Can you find A + B? Justify your answer.

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

3) Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Find AB and BA. Is AB = BA?

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

4) Given two matrices:

A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Show that AB $\neq$ BA.

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

6.3 SOME SPECIAL MATRICES

Now that we know what we mean by matrix multiplication, we can ask the following question. Given a matrix A, does there exist a matrix, let us call it I, such that:

IA = AI = A …..(4)

If such an I exists, then it would form the counterpart of the number 1 in arithmetic. This is so, because, we know that for any number x

1. x = x. 1 = x.

Now from the last section, we know that if A is of dimension $(m\times n)$ then for the product IA to exist, I must have m columns. Similarly, for AI to exist, I must have n rows. Now look at the product AI. This is of dimension $(m\times n)$. But we require AI = A. So it must be true that m = n. This leads us to conclude that, for the equality (4) to hold A must be a matrix of dimension $(m\times n)$, i.e., the number of rows of A must be the same as the number of columns of A. Such a matrix is called a **Square Matrix.** It follows that I must also be a square matrix of the same dimension as A. The question is: what does I look like? The matrix I, called the **Identity Matrix** has a special form: All the diagonal terms are **1,** and the off-diagonal terms are all zero:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots\ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots\ldots & 0 \\ : & : & : & : & :::::::: & : \\ : & : & : & : & :::::::: & : \\ 0 & 0 & 0 & 0 & \ldots\ldots & 1 \end{bmatrix}_{(m\times n)}$$

Another type of matrix that we encounter quite frequently, is a matrix whose elements are all zero. Such a matrix is called a **Null Matrix.** Unlike an Identity Matrix, a Null Matrix does not have to be square. Another important type of matrix is a symmetric matrix. Square matrices of this type, i.e., for which A = A' are called **Symmetric** matrices.

6.4 DETERMINANTS

6.4.1 Concept of a Determinant.

Suppose we have to solve the equations

$a_{11}x_{11} + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

By the method of cross-multiplication we get

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}; \text{and } x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

The denominator $a_{11}a_{22} - a_{12}a_{21}$ is written as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and is called a **determinant** of the second order. It is a scalar and in the two by two cases we get the scalar by cross multiplication. A determinant has as many rows as columns. This particular determinant is of the second order because there are two rows and two columns. Two things should be kept in mind about determinants. First, a determinant is only of a square matrix. A non-square matrix cannot have a determinant. Secondly, a determinant is a scalar. It is a single number.

6.4.2 Minors and Co-factors

Consider the following square matrix:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now consider the element a_{11}=1. Its location in the matrix is at the intersection of the first row and first column of A. If we wipe out the first row and the first column of A we are left with the matrix $M_{11} = \begin{bmatrix} 5 & 6 \\ 2 & 5 \end{bmatrix}$. The determinant of this matrix is called the **minor** of the element a_{11} and is written as $| M_{11} |$. Here $| M_{11}| = 13$. In general, the minor of the element a_{ij} of A is obtained by deleting the ith row and the jth column of A and then computing the determinant of the resulting matrix. So for computing the minor of the element a_{23} of A, we compute wiping out the second row and the third column of A.

$$| M_{23} | = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}.$$ Then $| M_{23} | = 2-4 = -2$.

Definition: (*Cofactor of a_{ij}*): Let A = $[a_{ij}]$ be any square matrix. If i + j is even then the **cofactor** of the element $a_{ij} = | M_{ij} |$. If I + j is odd then the cofactor of the element a_{ij} is

$- | M_{ij} |$. The cofactor of a_{ij} is written as $| c_{ij} |$.

Now in our example, $| M_{11}| = 13$. Notice that for the element a_{11}, i = 1 and j = 1, so i + j = 2. Hence, from our definition, $| c_{11}| = | M_{11}| = 13$. For the element a_{23}, i = 2. and j = 3, so i + j = 5, is odd. Then $| c_{23} | = -| M_{23} | = -(-2) = 2$.

Definition: (*Cofactor Matrix of A*): Let A = $[a_{ij}]$ be any square matrix. If we replace each element a_{ij} by its cofactor, the resultant matrix, designated by C = $[| c_{ij} |]$, is called the **cofactor matrix** of A.

In our example,

$| c_{11} | = 13 \quad | c_{12} | = -3 \quad | c_{13} | = -4$

$| c_{21} | = -2 \quad | c_{22} | = -3 \quad | c_{23} | = 2$

$| c_{31} | = -8 \quad | c_{32} | = 6 \quad | c_{33} | = -1$

so that $C = \begin{bmatrix} 13 & -3 & -4 \\ -2 & -3 & 2 \\ -8 & 6 & -1 \end{bmatrix}$

6.4.3 Computation of a Third-order Determinant

A determinant of the third order is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The rule used above is known as expansion by co-factors.

6.4. 4 Properties of Determinants

There are several rules for manipulating determinants. These should be learned from mathematical textbooks some of which are mentioned in the reading list. We mention here two properties which help us to establish a method of solution of simultaneous equations.

1) $$\begin{vmatrix} ka_{11} & a_{12} & \dots & a_{1n} \\ ka_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ ka_{n1} & a_{n2} & \dots. & a_{nn} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots. & a_{nn} \end{vmatrix}$$

   i.e., multiplying a row or a column of a determinant by any number multiplies the value of determinant by the same number.

2) Adding k times a row or column of a determinant to another row or column respectively of the determinant yields the original determinant, that is, the value of the determinant is unchanged.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ka_{11} & a_{22}+ka_{12} & a_{23}+ka_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

   We may show this in a simpler way with a 2 x 2 determinant.

   Suppose the determinant is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The value is $ad - bc$

   If we add k times the top row to the bottom row, we get

   $\begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = a(d + kb) - b(c + ka) = ad - bc$, which is the original determinant i.e., the value of determinant is unchanged when a multiple of the element of a column (or row) is added to corresponding element of another column (or row).

Check Your Progress 2

1) i) Is an identity matrix always a square matrix?

   ii) Can a square matrix be of dimension $m \times n, m \neq n$?

   iii) Can a null matrix have one negative element?

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

2) Are the following identity matrices?

i) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

3) What is a determinant? Explain the concept of minor and cofactor.

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

6.5 INVERSE OF A MATRIX

In arithmetic we know that for any number, say 7, its inverse is $\frac{1}{7}$. What does this mean? It means that $\frac{1}{7} \times 7 = 7 \times \frac{1}{7} = 1$, i.e., if we multiply $\frac{1}{7}$ by 7 we get 1, and if we multiply 7 by $\frac{1}{7}$ we get 1. Is there a counterpart for matrices? More formally, give a matrix A, does there exist a matrix, called the inverse of A and written as A^{-1}, such that $A^{-1} A = A A^{-1} = 1$.

Notice that for A^{-1} to exist, A must be a square matrix and further, A^{-1} must also be a square matrix with the same dimension as A. As an exercise, check this assertion. Further, if A^{-1} exists, it is unique.

Definition (non-singularity of a matrix): A square matrix is called a non-singular matrix if its inverse exists. If A does not have an inverse, it is called a singular matrix.

The next question is, given a square matrix A, how do we know that it has an inverse? The condition for A to have an inverse is that its determinant does not vanish, i.e. determinant of A, written as | A |, is not equal to zero.

Now that we know what we mean by the inverse of a matrix, we would like to know why acquiring this bit of knowledge is of use to us. It turns out that there is an immediate application. We began this unit by looking at systems of linear equations. Equation (2) summarises such a system in matrix form:

$$\mathbf{Ax} = \mathbf{b} \qquad \text{..... (2)}$$

Suppose we want to find a unique solution to this equation. When does such a solution exist? We shall find that the answer to this question is that a unique solution exists if A^{-1} exists, i.e., if $|A| \neq 0$. Let us look at three examples.

System I : $2x_1 + 3x_2 = 10$

$6x_1 + 9x_2 = 20$

$$\text{or } \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

The equations in System I represent two parallel straight lines. Since straight lines do not intersect. System I does not have a solution. Now look at the A matrix for the system, i.e.,

$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ Its determinant

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 6 = 0$$

i.e., $|A| = 0$

System II: $3x_1 + x_2 = 10$

$x_1 - x_2 = 0$

$$\text{or, } \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

The system has a unique solution at the point where the 45^0 line in the (x_1, x_2) plane intersects the straight line whose equation is $x_2 = 10 - 3x_1$. The solution is $\left(2\frac{1}{2}, 2\frac{1}{2}\right)$: Now check the A matrix, A = $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$. Its determinant $|A| = -3-1 = -4 \neq 0$

System III: $x_1 + x_2 = 10$

$x_1 - x_2 = 0$

$x_1 - 5x_2 = -20$

i.e.,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -20 \end{bmatrix}$$

The system has a solution (5, 5). Now look at the A matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -5 \end{bmatrix}$$

Since A is not a square matrix one cannot talk of | A | being zero or non-zero. This allows us to conjecture that $|A| \neq 0$ is at most a sufficient condition for A**x** = **b** to have a solution.

Suppose A is a square matrix and $|A| \neq 0$ in

A**x** = **b** ....(2)

What does the solution look like? Since $|A| \neq 0$, A^{-1} exists, which implies:

$A^{-1}A\mathbf{x} = A^{-1}b$

But $A^{-1}Ax = I$ and $\mathbf{Ix} = x$ so

$\mathbf{x} = A^{-1}b$

is the solution vector. This means that if we can compute A^{-1}, we can directly find the solution vector, $\bar{x}$ by computing $A^{-1}b$. The problem then is to find a way of computing A^{-1}.

Computation of A^{-1}

Before we can describe the procedure for computing A^{-1}, we must define three related concepts. We have come across some of these these earlier.

Definition (*Transpose of a Matrix)*: Let A = $\left[a_{ij}\right]$ be any matrix. The transpose of A, written as A', is obtained by interchanging the columns and rows of A.

Examples: (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

To get A′ and construct a matrix whose first column is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Now take the second row of A and complete this matrix by inserting the vector as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as the second column. This process yields:

$$A' = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 7 \\ 2 & 9 \\ 3 & 8 \end{bmatrix}$

Interchanging columns and rows as in example (a) yields:

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 9 & 8 \end{bmatrix}$$

Note, that A is a (3×2) matrix and A′ is a (2×3) matrix in this case.

(c) $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

Check that $A' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

This is a curious case. Notice that A′ = A. Square matrices of this type, i.e., for which A = A′ are called **Symmetric** matrices.

Another concept we need is of the cofactor of a matrix. We studied this in the previous section.

Definition: (*Adjugate of Matrix A*): Let A be any square matrix. Then the adjugate of A, written as adj A, is the transpose of the cofactor matrix of A, i.e., adj A = C′ written out more explicitly:

$$\text{adj } A = C' = \begin{bmatrix} c_{11} & c_{21} & \cdots\cdots & c_{n1} \\ c_{12} & c_{22} & \cdots\cdots & c_{n2} \\ : & : & ::::: & : \\ : & : & ::::: & : \\ c_{1n} & c_{2n} & \cdots\cdots & c_{nn} \end{bmatrix} \text{ where } C = \begin{bmatrix} c_{11} & c_{21} & \cdots\cdots & c_{1n} \\ c_{21} & c_{22} & \cdots\cdots & c_{2n} \\ : & : & ::::: & : \\ : & : & ::::: & : \\ c_{n1} & c_{n2} & \cdots\cdots & c_{nn} \end{bmatrix}$$

In our example:

$$\text{adj } A = C' = \begin{bmatrix} 13 & -2 & -8 \\ -3 & -3 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

Now we have all the machinery required to define the inverse of A.

Definition: (*Inverse of A*): Let A be any square matrix such that $|A| \neq 0$. Then, the inverse of A, written as

$$A^{-1} = \frac{C'}{|A|} = \text{adj } A \frac{1}{|A|}$$

Notice that if $|A| = 0$ then A^{-1} is undefined, i.e., A singular. In our example,

$|A| = a_{11} |M_{11}| - a_{12} |M_{12}| + a_{13} |M_{13}|$

$= a_{11} |c_{11}| + a_{12} |c_{12}| + a_{13} |c_{13}|$

$= 1(13) + 2(-3) + 4(-4)$

$= 13 - 6 - 16 = -9 \neq 0$

Hence,

$$A^{-1} = \text{adj } A \frac{1}{|A|} = -\frac{1}{9}\begin{bmatrix} 13 & -2 & -8 \\ -3 & -3 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

We end this section with an example to show that if A is non-singular then $\bar{x} = A^{-1}b$.

Consider the system of linear equations:-

$3x_1 + x_2 = 20$

$x_1 - x_2 = 0$

i.e., $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

Using the usual method of solving simultaneous equations yields the solution $\bar{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

What does our method yield?

Step 1: Find $|A|$. $|A| = 3(-1) - -1(1) = -4 \neq 0$

So A is non-singular.

Step 2: Compute C, $C = \begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}$

Step 3: Compute Adj C. C' = $\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}$ = Adj C

Notice, C is symmetric.

Step 4: Compute A^{-1}. $A^{-1} = \frac{C'}{A} = \frac{\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}}{-4} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$

Step 5: Compute $\bar{x}$. $\bar{x} = A^{-1}b = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{4}.20 + & \frac{1}{4}.0 \\ \frac{1}{4}.20 + & \left(-\frac{3}{4}\right).0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

which is exactly what we had found. The only difficulty is that the computation may seem quite tedious compared to the standard method taught in school level algebra. It has to be remembered that for large systems that standard method turns out to be terribly difficult and time-consuming. In such cases the method used here is much easier.

Check Your Progress 3

1) Does the matrix inverse exist for the following system of equations?

   $4x_1 + 6x_2 = 5$

   $12x_1 - 18x_2 = 10$

   Do these straight lines intersect or are they parallel?

   ……………………………………………………………………………

   ……………………………………………………………………………

   ……………………………………………………………………………

   ……………………………………………………………………………

   ……………………………………………………………………………

2) Is | A | $\neq 0$ a(i) necessary or (ii) sufficient or a (iii) necessary and sufficient condition for A**x** = **b** to have a solution?

   ……………………………………………………………………………

   ……………………………………………………………………………

   ……………………………………………………………………………

3) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Find its transpose.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

4) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

When does A become symmetric?

…………………………………………………………………………………

5) Let (i) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

a) Find their adjugate.

b) Hence, find their inverse.

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

6.6 LET US SUM UP

In this unit, you have been introduced to matrix algebra. You have learnt the concept of a matrix and how to perform various operations like addition, subtraction and multiplication on matrices. In this connection, some special matrices have also been presented. The concept of a determinant has been introduced and its various properties have been discussed. The ideas of minor and cofactor have been considered and the evaluation of a higher (more than second) order determinant has been explained. Next, the concept of the adjoint of a matrix has been introduced and the procedure for the evaluation of the inverse of a matrix has been discussed. Finally, the inverse matrix method and Cramer's rule for the solution of a simultaneous equation system have been presented.

6.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsection 6.2.1 and answer.

2) See subsection 6.2.1 and answer.

3) See subsection 6.2.2 and answer.

4) See subsection 6.2.2 and answer.

Check Your Progress 2

1) i) No, because all off-diagonal elements are not zero.

   ii) No, because all off-diagonal elements are not zero.

   iii) Yes, because all off-diagonal elements are zero and atleast one diagonal element is not zero.

   iv) No, because the given matrix is not square.

2) i) No, because all diagonal elements are not 1.

   ii) No, because diagonal elements are not 1.

3) See section 6. 4 and answer.

Check Your Progress 3

1) See section 6.5 and answer.

2) See section 6.5 and answer.

3) See section 6.5 and answer.

4) See section 6.5 and answer.

5) See section 6.5 and answer.