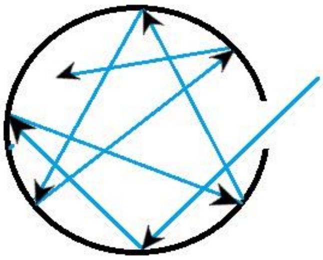


A typical black body.

UNIT 11

THEORY OF RADIATION

Structure

STUDY GUIDE

The blackbody radiation presented huge challenge to theoretical physicists of late nineteenth and early twentieth century. Lord Rutherford described it as one of the two darkest clouds on the horizon of theoretical physics. All efforts based on classical theory assumed that the energy of a system could be taken as continuous variable but these failed to explain the experimental results in entirety. It required the genius of Planck to provide satisfactory explanation of observed results for all wavelengths. He made a drastic deviation from classical concept about energy of a system in that it should not be treated as a continuous variable. He proposed that energy can change only in concrete steps in units of what is now known as Planck's constant. You will learn to obtain expression for Planck's formula and show that all other laws of radiation are contained in it.

The derivations given in this unit require good knowledge of geometrical series, calculus and acquaintance with special functions. So, you are advised to re-read Block 1 on kinetic theory of gases before studying this unit. To make the unit self-contained and for completeness, we have given all mathematical steps. But you will enjoy the subject more if you solve these steps by yourself. Therefore, keep a pen/pencil as well as a notebook ready with you. Also, answer SAQs and solve TQs or other numerical problems to gain greater proficiency.

"An experiment is a question which science poses to Nature and a measurement is the recording of Nature's answer."

Max Planck

11.1 INTRODUCTION

In your school physics, you have learnt that all bodies emit thermal radiation. And the intensity, wavelength and rate of emission depend on temperature. For instance, at room temperature, most of the energy is radiated in the far infra-red region, whereas at 6000K, which corresponds to the temperature of the outer surface of the Sun, it lies in the visible region. You have also learnt that the mode of energy transmission from the Sun to the Earth is ***radiation***. In fact, radiation is the main mechanism for energy transfer in our solar system, interstellar space and the galaxies. It implies that energy transfer by radiation does not require intervening medium to participate actively.

It is now well accepted that thermal radiations are electromagnetic in nature. Moreover, these produce a sensation of warmth. An enclosure maintained at a constant temperature can be imagined to be filled with electromagnetic radiation, which is in thermal equilibrium with its walls. The electromagnetic radiation in a cavity is called **blackbody radiation** corresponding to a well-defined temperature. In the beginning, the laws of thermodynamics in conjunction with the law of equipartition of energy were used to study the behaviour of blackbody radiation. However, these efforts proved only partly successful.

In this unit, you will get the correct insight into the nature of blackbody radiation and its spectral distribution. We begin by discussing some important terms and concepts related to blackbody radiation in Sec. 11.2. This is followed by a discussion of spectral distribution of radiant energy in Sec. 11.3. Planck proposed the concept of quanta as carriers of energy in emission or absorption of blackbody radiation and explained all observed results available then rather well. In Sec. 11.4, you will learn how to derive Planck's formula of blackbody radiation following the approach used by Planck. (For the number of modes per unit volume in the frequency range ν to $\nu+d\nu$, he used the expression obtained by Rayleigh and Jeans.) In Sec. 11.5, you will learn that all other laws of radiation (Wien's law, Rayleigh-Jeans law and Stefan-Boltzmann's law) are contained in Planck's law.

Expected Learning Outcomes

After studying this unit, you should be able to:

- explain the concepts of blackbody radiation, spectral distribution and energy density;
- discuss Planck's theory of black body radiation;
- obtain Planck's formula for spectral distribution of black body radiation; and
- derive Rayleigh-Jeans law, Wien's law and Stefan's law from Planck's law.

11.2 DEFINITIONS AND CONCEPTS

To discuss distribution of energies in blackbody radiation, we have to introduce some basic definitions:

Spectral energy density (u_λ) is defined with respect to a particular wavelength λ as a measure of the energy per unit volume per unit range of wavelength. It means that $u_\lambda d\lambda$ denotes the energy per unit volume in the wavelength range from λ to $\lambda + d\lambda$. Therefore, a sum of spectral energy densities for all wavelengths from 0 to ∞ per unit volume gives the total energy density:

$$u = \int_0^\infty u_\lambda d\lambda \tag{11.1}$$

Note that the total energy density is measured in units of Jm^{-3}.

Spectral emissive power (e_λ) of a body corresponding to wavelength λ is a measure of energy radiated per second per unit surface area per unit wavelength. Therefore, $e_\lambda d\lambda$ denotes the energy emitted by unit area in one second in the wavelength range from λ to $\lambda + d\lambda$. A sum of spectral emissive powers for all wavelengths from 0 to ∞ gives total emissivity:

$$e = \int_0^\infty e_\lambda d\lambda \tag{11.2}$$

Note that emissivity is measured in $\text{Jm}^{-2}\,\text{s}^{-1}$ or Wm^{-2}.

Spectral absorptivity (a_λ) denotes the fraction of incident energy of a particular wavelength absorbed by unit surface area of a body in one second.

If a body absorbs all radiations incident on it, $a_\lambda = 1$, then the body is said to be a **perfect blackbody**. This nomenclature is based on the colour that we see due to selective absorption of light. Do you know that the text of this unit appears black because letters in it absorb all light falling on them? Why does a flower have colour or why does the paper of your unit appear white?

Note that e and e_λ characterise the properties of a body as emitter whereas a_λ describes the properties of the body as an absorber of radiation. However, these three physical quantities depend on temperature and the nature of the surface of the body.

When radiation of a particular wavelength λ is incident on a body, it may be partially reflected, partially absorbed and partially transmitted. But a blackbody absorbs all radiations incident on it. Then we can write

$$r_\lambda + a_\lambda + t_\lambda = 1$$

where r_λ, a_λ and t_λ, respectively, characterise energy reflection, absorption and transmission coefficients of the body corresponding to wavelength λ. If $r_\lambda = t_\lambda = 0$, then $a_\lambda = 1$. That is, the body is perfectly black for a given wavelength. In practice, no surface or body satisfies this ideal definition strictly. Even lamp black and platinum black respectively absorb nearly 96% and 98% of visible light. So, a_λ is always less than unity.

11.3 SPECTRAL DISTRIBUTION OF RADIANT ENERGY

In your school physics, you have learnt Stefan's law of blackbody radiation. It states that **the rate of emission of radiant energy by unit area of a perfect blackbody is directly proportional to the fourth power of its absolute temperature.** Mathematically, we express it as

$$E = \sigma T^4 \quad (11.3)$$

where σ is called **Stefan's constant** and has value $5.672 \times 10^{-8}\,\text{Jm}^{-2}\text{K}^{-4}\text{s}^{-1}$. Stefan's law in the above form refers to the amount of heat emitted by the body by virtue of its temperature, irrespective of what it receives from the surroundings. Therefore, it is natural to extend the scope of this law to represent the exchange of heat and be stated as follows:

For a blackbody at absolute temperature T surrounded by another blackbody at absolute temperature T_0, the amount of net heat lost by the blackbody at higher temperature per unit time can be expressed as

$$E = \sigma(T^4 - T_0^4) \quad (11.4)$$

This law is known as **Stefan-Boltzmann law.**

Note that Stefan-Boltzmann law relates total energy density of black body radiation with temperature; it does not give any information about the distribution of energy in different parts of the spectrum.

Now refer to Fig. 11.1, which shows observed results of spectral energy density of a black body at different temperatures.

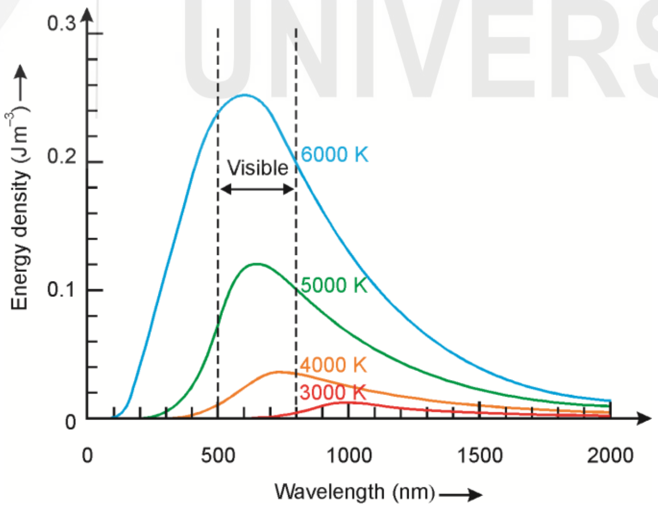


Fig. 11.1: **Plot of spectral energy density of a black body with wavelength at different temperatures.**

You will note in Fig. 11.1 that:

- For a given wavelength λ, u_λ increases with temperature.
- For each temperature, the spectral energy density plot shows a maximum. It shifts to shorter wavelengths with increase in temperature.

- The spectral energy density becomes zero as wavelength tends to either zero or infinity.

To explain experimental results, Wien and Rayleigh and Jeans used thermodynamic reasoning with the principle of equipartition of energy, wherein energy is considered a continuous variable. However, they could not explain the results satisfactorily in the entire range of the spectrum. In fact, their efforts succeeded either in the higher or in the lower energy regions. This raised doubts about the applicability/utility of the principle of equipartition of energy to understand the physics of blackbody radiation.

Planck then conjectured, albeit heuristically, that **emission and absorption of radiation is a discontinuous process**. To derive Planck's formula, we have preferred discussion of developments in chronological order as this approach is more informative and learner-friendly. It will give you a feel of how scientists handle difficult unknown situations, particularly when their results do not conform to experimental results. (This law was later derived by Indian physicist Prof. S.N. Bose by treating radiation as an assembly of photons, which obey Bose-Einstein statistics. You will learn about it in Block 4.)

11.4 PLANCK'S LAW

Planck presented the following formula for energy density empirically to fit the experimental results on blackbody spectrum:

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3}\left(\frac{h\nu}{\exp(h\nu/k_BT)-1}\right)d\nu \tag{11.5}$$

We can rewrite it as:

$$u_\nu d\nu = n_\nu \varepsilon_\nu d\nu = \frac{8\pi\nu^2\varepsilon_\nu}{c^3}d\nu \tag{11.6}$$

where

$$\varepsilon_\nu = \left(\frac{h\nu}{\exp(h\nu/k_BT)-1}\right) \tag{11.6a}$$

is the average energy of an oscillator, and

$$n_\nu d\nu = \frac{8\pi\nu^2}{c^3}d\nu \tag{11.6b}$$

defines the number of modes per unit volume in the frequency range ν to $\nu + d\nu$. (The calculation of the number of modes is given in Appendix 11A) Eq. (11.6b) can also be written as

$$N_\nu d\nu = \frac{8\pi V\nu^2}{c^3}d\nu \tag{11.6c}$$

to define the number of modes in volume *V* in the frequency range ν to $\nu + d\nu$.

You should now go through the following example.

EXAMPLE 11.1: Calculation of Number of Modes

Calculate the number of modes of oscillations in a chamber of volume $100\,\text{cm}^3$ in the frequency range $4.02\times10^{14}\text{Hz}$ to $4.03\times10^{14}\text{Hz}$.

SOLUTION ■ It is given that $V = 100\ \text{cm}^3 = 10^{-4}\text{m}^3$, $\nu = 4.02\times10^{14}\text{Hz}$, $d\nu = 0.01\times10^{14}\text{Hz}$ and $c = 3\times10^8\text{ms}^{-1}$. On substituting these values in the expression given in Eq. (11.6c), we get

$$N_\nu d\nu = \frac{8\times3.14\times(4.02\times10^{14}\text{Hz})^2\times(10^{-4}\text{m}^3)\times(0.01\times10^{14}\text{Hz})}{(3\times10^8\text{ms}^{-1})^3} = 1.5\times10^{13}$$

By substituting $\varepsilon_\nu = k_BT$ in Eq. (11.6), you will obtain Rayleigh-Jeans law, which was derived by them based on the law of equipartition. (It suggests that ε_ν is the average energy of a mode of oscillation in Planck's theory.) But Planck was convinced about the inappropriateness of the classical theories and he made a drastic deviation. He postulated that

- the exchange of energy between matter (walls) and radiation (cavity) could take place only in bundles of a certain size; and
- the quantum of exchange is directly proportional to its frequency. That is, the energy of an oscillator having frequency ν could only be an integral multiple of $h\nu$, where h is a constant.

These postulates marked a fundamental departure from the contemporary ideas. The constant h is now known as *Planck's constant*. Its value is 6.62618×10^{-34} Js. (Planck was awarded Nobel Prize in Physics in 1918 for his work on blackbody radiation.)

Before proceeding further, it would be appropriate to clarify the significance of Planck's postulates with an example. Suppose two litre of milk is to be distributed between two persons. Since milk is an infinitely indivisible entity, you can divide it between two persons in an infinite number of ways. Next you are asked to distribute milk in units of a litre. Now both the persons can receive either 0, 1 or 2 litre meaning thereby that the number of ways reduces to three. The number of ways will be five if the unit (quantum) of distribution is half-a-litre. From this example, you can convince yourself how discretisation introduces a drastic change. Planck achieved similar result in the case of blackbody radiation by introducing the concept of energy quanta in energy exchange.

Planck argued that blackbody radiation chamber be considered to be filled up not only with radiation but also with a perfect gas, whose molecules exchanged energy via resonators of molecular dimensions. (Matter-radiation interaction was necessary to introduce the notion of temperature.) The resonators were assumed to absorb energy from the radiation and transfer the same wholly or partially to gas molecules when they collided with them. This helped to establish thermodynamic equilibrium. (You may think that the process is somewhat roundabout but this was the only one possible and consistent with accepted ideas at that time.)

Let us now suppose that the total number of Planck resonators is N and their total energy is E. The average energy of Planck resonators is given by

$$\frac{h\nu}{[\exp(h\nu / k_B T) - 1]} \text{ rather than } k_B T \text{ .}$$

Before proceeding further, go through the following example.

EXAMPLE 11.2: MEAN ENERGY

An oscillator vibrates with frequency 1.51×10^{14} Hz at T = 1800 K. Compare the values of its average energy by treating it as (a) a classical oscillator and (b) Planck's oscillator. Take $h = 6.62 \times 10^{-34}$ Js^{-1}, and $k_B = 1.38 \times 10^{-23}$ JK^{-1}.

SOLUTION ■ (a) The average energy of a classical oscillator is given by

$$\bar{\varepsilon} = k_B T = (1.38 \times 10^{-23}\,\text{JK}^{-1}) \times (1800\,\text{K})$$

$$= 2.48 \times 10^{-20}\,\text{J}$$

(b) The average energy of Planck's oscillator is given by

$$\bar{\varepsilon} = \frac{h\nu}{e^{h\nu / k_B T} - 1} = \frac{k_B T (h\nu / k_B T)}{e^{h\nu / k_B T} - 1}$$

We note that $\frac{h\nu}{k_B T} = \frac{(6.62 \times 10^{-34}\,\text{Js}) \times (1.51 \times 10^{14}\,\text{s}^{-1})}{(1.38 \times 10^{-23}\,\text{JK}^{-1}) \times (1800\,\text{K})}$

$$= \frac{9.99 \times 10^{-20}\,\text{J}}{2.48 \times 10^{-20}\,\text{J}} = 4.03$$

Hence, $\bar{\varepsilon} = \frac{(2.48 \times 10^{-20}\,\text{J}) \times (4.03)}{e^{4.03} - 1} = \frac{9.99 \times 10^{-20}\,\text{J}}{53.6} = 1.81 \times 10^{-20}\,\text{J}$

Note that the average energy of Planck's oscillator is less than that of a classical oscillator.

You should now answer an SAQ.

SAQ 1 – Mean energy

Obtain expression for mean energy of a Planck's oscillator in the limit $\nu \to 0$.

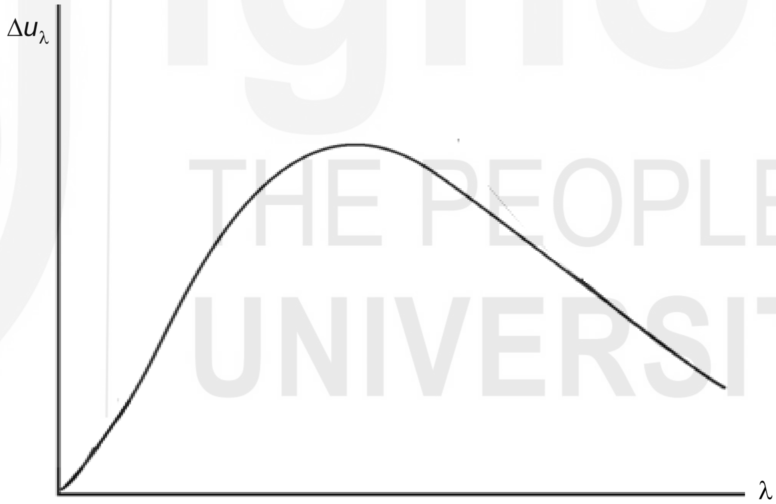
While answering SAQ 1 you have noted that in the limit $\nu \to 0$, the mean energy of a Planck's oscillator is $k_B T$. The implication of this result is that when $h\nu$ is small compared to k_BT, the discrete nature of energy does not show up.

In terms of wavelength, we can express Planck's formula using the relation $\nu = c/\lambda$ in Eq. (11.5). Note that $|d\nu| = \left|-\frac{c}{\lambda^2}d\lambda\right|$ and we use the fact that $u_\lambda d\lambda$ corresponds to $u_\nu d\nu$. Hence, Planck's law in terms of wavelength can be expressed as

$$u_\lambda d\lambda = \frac{8\pi h}{c^3}\left(\frac{c}{\lambda}\right)^3\left(\frac{1}{\exp(hc/\lambda k_BT)-1}\right)\left|-\frac{c}{\lambda^2}d\lambda\right|$$

$$= \frac{8\pi hc}{\lambda^5}\left(\frac{1}{\exp(hc/\lambda k_BT)-1}\right)d\lambda \qquad (11.7)$$

Now refer to Fig. 11.2. It shows a plot of Planck's law based on Eq. (11.7). Since Planck's law explained the observed results of blackbody radiation for all wavelengths available then, the validity of the concept of discreteness of energy was established. In fact, this revolutionary idea, led to the birth of a new branch of physics known as **quantum mechanics**.



11.2: **Plot of Planck's law based on Eq. (11.7).**

11.5 DEDUCTIONS FROM PLANCK'S LAW

We now show that Planck's law provides us with the most general description of blackbody radiation. That is, you are justified to think that all other laws of blackbody radiation are its special cases. We first show that Rayleigh-Jeans law and Wien's law are its limiting cases in the region of longer and shorter wavelengths, respectively.

11.5.1 Rayleigh-Jeans Law

To begin with, we obtain the expression for Rayleigh-Jeans law. For $\lambda >> hc/k_BT$, the exponential term in Eq. (11.7) can be approximated as

$$\exp(hc/\lambda k_B T) \approx 1 + \frac{hc}{\lambda k_B T} + \ldots$$

so that $\exp(hc/\lambda k_B T) - 1 = \dfrac{hc}{\lambda k_B T}$

Hence, for $\lambda >> hc/k_B T$, Eq. (11.7) reduces to

$$u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \times \left(\frac{\lambda k_B T}{hc}\right) d\lambda = \frac{8\pi k_B T}{\lambda^4} d\lambda \qquad (11.8)$$

This is Rayleigh-Jeans law.

11.5.2 Wien's Law

For $\lambda << hc/k_B T$, the exponential term in Eq. (11.7) will be significantly greater than unity. Therefore, we can ignore 1 in comparison to the exponential in Eq. (11.7). Then Eq. (11.7) reduces to

$$u_\lambda d\lambda = \left(\frac{8\pi hc}{\lambda^5}\right) \exp(-hc/\lambda k_B T)\, d\lambda \qquad (11.9)$$

This is Wien's law.

11.5.3 Stefan's Law

By integrating Eq. (11.7) for photons of all wavelengths, we obtain the expression for total energy density:

$$u(T) = \int_0^\infty u_\lambda d\lambda = 8\pi hc \int_0^\infty \frac{d\lambda}{\lambda^5 [\exp(hc/\lambda k_B T) - 1]} \qquad (11.10)$$

To evaluate this integral, we introduce a change of variable and define $x = \dfrac{hc}{\lambda k_B T}$ so that $\lambda = \dfrac{hc}{x k_B T}$ and $d\lambda = -\dfrac{hc}{x^2 k_B T} dx$. Note that the limits of integration will change as $-\infty$ to 0. Using these results in Eq. (11.10), we get

$$u(T) = 8\pi hc \int_{-\infty}^{0} \frac{\left(-\dfrac{hc}{x^2 k_B T}\right) dx}{\left(\dfrac{hc}{x k_B T}\right)^5 [\exp(x) - 1]}$$

If we now change the limits of integration as 0 to ∞, the negative sign will be automatically absorbed. Hence, we can write

$$u(T) = \frac{8\pi k_B{}^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3 dx}{\exp(x) - 1}$$

The procedure to solve the integral in this expression is quite involved.

You should just remember that it has the value $\Gamma(4)\,\zeta(4) = \pi^4/15$. (Here $\Gamma(4) = 6$ is gamma function of order 4 and $\zeta(4) = \pi^4/90$ is zeta function of order 4.) So, we can write the expression for total energy density at temperature T as

$$u(T) = \frac{8\pi^5 k_B^4}{15h^3c^3}T^4$$

or $$u(T) = aT^4 \tag{11.11}$$

where $a = \dfrac{8\pi^5 k_B^4}{15h^3c^3} = 7.56\times10^{-16}\ \text{Jm}^{-3}\ \text{K}^{-4}$.

The interior of the Sun can be assumed to consist of photon gas at constant temperature 3×10^6 K. It means that the energy density radiated by the Sun is given by

$$u = (7.56\times10^{-16}\,\text{Jm}^{-3}\,\text{K}^{-4})\times(3\times10^6\text{K})^4$$

$$= 6.1\times10^{10}\,\text{Jm}^{-3}$$

The volume of the Sun is known to be nearly equal to $1.4\times10^{27}\ \text{m}^3$. It means that the total energy of photons inside the Sun is

$$E = uV = 8.6\times10^{37}\ \text{J}$$

If photons are assumed to **effuse** through a small cavity-like opening in the surface of the Sun, the net rate of flow of radiation per unit area of the opening will be given by

$$R = \frac{1}{4}\,u\,c = \frac{2\pi^5 k_B^4}{15h^3c^2}T^4$$

We can rewrite it as

$$R = \sigma T^4 \tag{11.12}$$

where $\sigma = \dfrac{2\pi^5 k_B^4}{15h^3c^2} = 5.672\times10^{-8}\ \text{Jm}^{-2}\ \text{K}^{-4}\,\text{s}^{-1}$ is Stefan's constant.

There is another law of radiation: Wien's displacement law: $\lambda_{max}T = 2.897\times10^{-6}$ mK . We can use it to calculate the temperature of the surface of celestial bodies.

Though obtaining this expression from Planck's law involves cumbersome mathematics, we have put it as a TQ. You should try to obtain this expression rather than looking at the solution at the first attempt itself.

Let us now summarise what you have learnt in this unit.

11.6 SUMMARY

Concept | **Description**

Blackbody

- A **blackbody** absorbs all radiations incident on it, regardless of their frequency. A small hole in a large enclosure or cavity is a practical approximation to an ideal black body.

Spectroscopic analysis of blackbody radiation

- Spectroscopic analysis of black body radiation shows that
  - for a given wavelength λ, u_λ increases with temperature;
  - for each temperature, the spectral energy density versus wavelength curve shows a maximum, which shifts to shorter wavelengths as temperature increases; and
  - the energy density goes to zero as $\lambda \to 0$ or as $\lambda \to \infty$.

Planck's hypothesis

- According to Planck:
  - The exchange of energy between matter (walls) and radiation (cavity) takes place in bundles of a certain size; and
  - The quantum of exchange is directly proportional to its frequency. That is, the energy of an oscillator having frequency ν is an integral multiple of $h\nu$, where h is a constant. It is now referred to as Planck's constant and its value is 6.67×10^{-34} Js.

Planck's law

- According to Planck's law, the energy density of blackbody radiation is given by

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3}\left(\frac{h\nu}{\exp(h\nu/k_BT)-1}\right)d\nu$$

In terms of wavelength, we can express it as

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{\exp(hc/\lambda k_BT)-1}\right)d\lambda$$

Wien's law

- For $\lambda << hc/k_BT$, Planck's law reduces to Wien's law:

$$u_\lambda d\lambda = \left(\frac{8\pi hc}{\lambda^5}\right)\exp(-hc/\lambda k_BT)\,d\lambda$$

Rayleigh-Jeans law

- For $\lambda >> hc/k_BT$, Planck's law reduces to Rayleigh-Jeans law:

$$u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\times\left(\frac{\lambda k_BT}{hc}\right)d\lambda = \frac{8\pi k_BT}{\lambda^4}d\lambda$$

Stefan's law

- Stefan's law gives total energy density of all photons in blackbody spectrum. It states that total rate of emission of radiant energy per unit area is related to energy density as fourth power of temperature:

$$E = \sigma T^4$$

where σ is known as Stefan's constant. Its value is $5.672 \times 10^{-8}\,\text{Jm}^{-2}\text{K}^{-4}\text{s}^{-1}$.

Stefan-Boltzmann law states that when a blackbody at absolute temperature T *is* surrounded by another blackbody at absolute temperature T_0, the amount of net heat lost by the blackbody at higher temperature per unit time can be expressed as

$$E = \sigma(T^4 - T_0^4) \ .$$

11.7 TERMINAL QUESTIONS

1. Calculate the number of modes in a chamber of volume 1m^3 in the frequency range $0.6 \times 10^{14}\,\text{Hz}$ to $0.61 \times 10^{14}\,\text{Hz}$.

2. Calculate the average energy of a Planck oscillator of frequency $0.6 \times 10^{14}\,\text{Hz}$ at 2000 K. How does it compare with the energy of a classical oscillator?

3. Calculate the number of modes of vibration in a 100 cm^3 chamber in the wavelength region (a) 500.0 nm – 500.2 nm and (b) frequency range $1.5 \times 10^{14}\,\text{Hz}$ to $1.51 \times 10^{14}\,\text{Hz}$.

4. Calculate the number of photons in 1 cm^3 cavity containing black-body radiation at 1000 K.

5. Using (Eq. 11.7), obtain Wien's displacement law.

11.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. In the limit $\nu \to 0$, the Planck factor reduces to $k_B T$:

$$\frac{h\nu}{[\exp(h\nu / k_B T) - 1]} \to \frac{h\nu}{\left(1 - \frac{h\nu}{k_B T} + \ldots - 1\right)} = k_B T \ .$$

Terminal Questions

1. Using Eq. (11.6a), we can write

$$N_\nu d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

$$V = 1\text{m}^3;\ \nu = 0.6\times 10^{14}\,\text{Hz}\ \ d\nu = 0.1\times 10^{14}\,\text{Hz}\qquad c = 3\times 10^8\,\text{ms}^{-1}.$$

$$N_\nu d\nu = \frac{8\times 3.14\times 1}{(3\times 10^8)^3}\times(0.6\times 10^{14})^2\times(0.1\times 10^{14})$$

$$= \frac{25.12\times 0.36\times 10^{28}\times 0.01\times 10^{14}}{27\times 10^{24}}$$

$$= \frac{0.090432\times 10^{42}}{27\times 10^{24}}$$

$$= \frac{90.432\times 10^{15}}{27}$$

$$= 3.35\times 10^{15}$$

2. We recall that average energy of photons is

$$\bar{\in} = \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right)-1}$$

$$= \frac{6.6\times 10^{-34}\times 0.6\times 10^{14}}{\exp\left(\frac{6.6\times 10^{-34}\times 0.6\times 10^{14}}{1.3\times 10^{-23}\times 2000}\right)-1}$$

$$= \frac{3.96\times 10^{-20}}{\exp\left(\frac{3.96\times 10^{-20}}{2.6\times 10^{-20}}\right)-1} = \frac{3.96\times 10^{-20}}{e^{1.523}-1}$$

$$\bar{\in} = \frac{3.96\times 10^{-20}}{4.59-1} = \frac{3.96\times 10^{-20}}{3.59} \approx 1.10\times 10^{-20}\,\text{J}$$

The energy of a classical oscillator is

$$= k_B T = 1.38\times 10^{-23}\times 2000$$

$$= 2.76\times 10^{-20}\,\text{J}$$

$\therefore$ The energy of a Planck's oscillator is nearly half of classical oscillator.

3. a) The number of modes per unit volume in the wavelength region λ to $\lambda + d\lambda$ is given by

$$N(\lambda) = \frac{8\pi V}{\lambda^4}d\lambda$$

Therefore, the number of modes within wavelength range λ to $\lambda + d\lambda$ in a chamber of volume *V* is

$$N(\lambda) = \frac{8 \times 3.14 \times (100\,\text{cm}^3)}{(5 \times 10^{-5}\,\text{cm})^4} \times (2 \times 10^{-8}\,\text{cm})$$

$$= \frac{50.24 \times 10^{-6}}{625 \times 10^{-20}}$$

$$= 8.014 \times 10^{12}$$

b) The number of modes in the frequency range ν to $\nu + d\nu$ in chamber of volume *V* is given by

$$N(\nu) = V\,\frac{8\pi\nu^2}{c^3}\,d\nu$$

Here $\nu = 1.5 \times 10^{14}$ Hz, $d\nu = 0.01 \times 10^{14}$ Hz, $c = 3 \times 10^{10}\text{cm s}^{-1}$ and $V = 100\ \text{cm}^3$. Hence on substituting the values, we get

$$N(\nu) = \frac{8 \times 3.14 \times (100\,\text{cm}^3) \times (1.5 \times 10^{14}\,\text{s}^{-1})^2}{(3 \times 10^{10}\,\text{cms}^{-1})^3} \times (0.01 \times 10^{14}\,\text{s}^{-1})$$

$$= \frac{56.52 \times 10^{42}\,\text{cm}^3\text{s}^{-3}}{27 \times 10^{30}\,\text{cm}^3\text{s}^{-3}}$$

$$= 2.09 \times 10^{12}$$

4. According to Planck, the energy density of the radiation in the frequency range ν and $\nu + d\nu$ is given by

$$u_\nu d\nu = \frac{8\pi h}{c^3}\,\frac{\nu^3}{\exp(h\nu / k_B T - 1)}\,d\nu$$

Since $u_\nu d\nu = h\,\nu\,dn_\nu$, the number density of photons in the frequency range ν and $\nu + d\nu$ is

$$dn_\nu = \frac{8\pi}{c^3}\,\frac{\nu^2 d\nu}{\exp\left(\dfrac{h\nu}{k_B T}\right) - 1}$$

To obtain the expression of the total number density of photons, we integrate this expression to get

$$n = \frac{8\pi}{c^3}\int \frac{\nu^2 d\nu}{\exp\left(\dfrac{h\nu}{k_B T}\right) - 1} \qquad \text{(i)}$$

We now put $\dfrac{h\nu}{k_B T} = x$.

Then Eq. (i) can be rewritten as

$$n = \frac{8\pi}{c^3}\left(\frac{k_B T}{h}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{16\varsigma(3)\,\pi(k_B T)^3}{c^3 h^3}$$

The Riemann Zeta function $\varsigma(3) = 1.202$.

On substituting the given values

$c = 3\times 10^8$ ms^{-1}, $h = 6.6\times 10^{-34}$ Js, $k_B = 1.38\times 10^{-23}$ JK^{-1} and $T = 1000$ K, we get

$$n = \frac{16\times 1.202\times \pi(1.38\times 10^{-23})^3\,(10^3)^3}{(3\times 10^8)^3\,(6.6\times 10^{-34})^3} = 2.0456\times 10^{16}\text{m}^{-3}$$

5. The wavelength at which maximum occurs can be obtained from Eq. (11.7) using the condition

$$\left[\frac{\partial u_\lambda}{\partial \lambda}\right]_{\lambda=\lambda_{max}} = 0$$

This leads to

$$\frac{\partial u_\lambda}{\partial \lambda} = 8\pi hc\,\frac{\partial}{\partial \lambda}\left[\frac{\lambda^{-5}}{\exp(hc/\lambda k_B T) - 1}\right]$$

$$= 8\pi hc\left[\frac{-5\lambda^{-6}\left(\exp(hc/\lambda k_B T) - 1\right) - \lambda^{-5}\left(-\frac{hc}{\lambda^2 k_B T}\exp(hc/\lambda k_B T)\right)}{\left(\exp(hc/\lambda k_B T) - 1\right)^2}\right]$$

$$= \frac{8\pi hc}{\lambda^5}\frac{1}{(\exp(hc/\lambda k_B T) - 1)}\left[-\frac{5}{\lambda} + \frac{hc}{\lambda^2 k_B T}\frac{\exp(hc/\lambda k_B T)}{(\exp(hc/\lambda k_B T) - 1)}\right]$$

$$= u_\lambda\left[-\frac{5}{\lambda} + \frac{hc}{\lambda^2 k_B T}\frac{\exp(hc/\lambda k_B T)}{(\exp(hc/\lambda k_B T) - 1)}\right]$$

Suppose the value of u_λ is maximum for $\lambda = \lambda_{max}$.

Therefore, we equate the right-hand side of the above expression equal to zero and put $\lambda = \lambda_{max}$. This gives

$$\left[-\frac{5}{\lambda_{max}} + \frac{hc}{\lambda_{max}^2 k_B T}\frac{\exp(hc/\lambda_{max} k_B T)}{(\exp(hc/\lambda_{max} k_B T) - 1)}\right] = 0$$

or $$\frac{hc}{\lambda_{max} k_B T} \frac{\exp(hc/\lambda_{max} k_B T)}{[\exp(hc/\lambda_{max} k_B T) - 1]} = 5 \qquad \text{(i)}$$

We now introduce a new variable by defining $x = hc/\lambda_{max} k_B T$. Then we can rewrite Eq. (i) in an elegant form:

$$x \frac{\exp(x)}{\exp(x) - 1} = 5$$

or $$x = 5(1 - e^{-x}) \qquad \text{(ii)}$$

This is a transcendental equation and can be solved either graphically or numerically. The exact value of x is 4.965.

Hence, we can write

$$x = \frac{hc}{\lambda_{max} k_B T} = 4.965$$

or $$\lambda_{max} T = b = \frac{hc}{k_B \times 4.965} \qquad \text{(iii)}$$

This is *Wien's displacement law*. On substituting for h, c and k_B , we get

$$\lambda_{max} T = 2.897 \times 10^{-6} \text{ mK}$$

APPENDIX 11A: NUMBER OF ALLOWED MODES OF STANDING WAVES IN AN ENCLOSURE

Rayleigh considered blackbody radiation in an enclosure, a hollow cubical box of side L, say, to consist of a number of electromagnetic waves which travelled in all possible directions. As a result, these made multiple reflections at the walls of the enclosure. Their subsequent superposition led to formation of standing waves and the walls of the enclosure acted as nodes.

The standing waves in such a system are described by the wave equation

$$\nabla^2 \psi(x,y,z,t) = \frac{1}{v}\frac{\partial^2 \psi(x,y,z,t)}{\partial t^2} \qquad (11A.1)$$

where v is velocity of the standing waves.

Since the walls of the enclosure act as nodal points, we can say that the amplitude ψ of the waves will be zero at x, y, $z = 0$ and x, y, $z = L$. Then we take the solution of Eq. (11A.1) to be of the form

$$\psi(x,y,z,t) = C\exp(-i\omega t)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{\ell\pi z}{L}\right) \qquad (11A.2)$$

where (m,n,ℓ) are integers and ω is the angular frequency of the wave. Note that each combination of (m,n,ℓ) defines a mode of oscillation of the waves in the enclosure.

On combining Eqs. (11A.1) and (11A.2) and simplifying the resultant expression, we get

$$\frac{\pi^2}{L^2}(m^2 + n^2 + \ell^2) = \frac{\omega^2}{v^2}$$

or $$m^2 + n^2 + \ell^2 = \frac{\omega^2}{\pi^2}.\frac{L^2}{v^2} = \left(\frac{2\nu L}{v}\right)^2 = \left(\frac{2L}{\lambda}\right)^2 \qquad (11A.3)$$

where $\lambda = v/\nu$ defines the wavelength of the standing waves of frequency ν.

It may be remarked here that Eq. (11A.3) gives the number of allowed modes of vibration inside the enclosure for different, positive and integral values of m,n and ℓ. The total number of modes of vibration will be specified by the total number of possible sets (m,n,ℓ).

If we now put $\frac{2L}{\lambda} = p,$ Eq. (11A.3) can be rewritten as

$$m^2 + n^2 + \ell^2 = p^2 \qquad (11A.4)$$

Geometrically, this result suggests that p is the radius of a sphere in (m,n,ℓ) space and the number of allowed modes can be obtained by plotting m,n,ℓ and counting the number of points corresponding to positive integral values. These will lie in the positive octant of a sphere of radius p, as shown in Fig. 11A.1. (In other octants, at least one value, either m,n or ℓ will be negative.) Note that in Fig. 11A.1, the allowed set of values of m, n and ℓ form a mesh of small cubes.

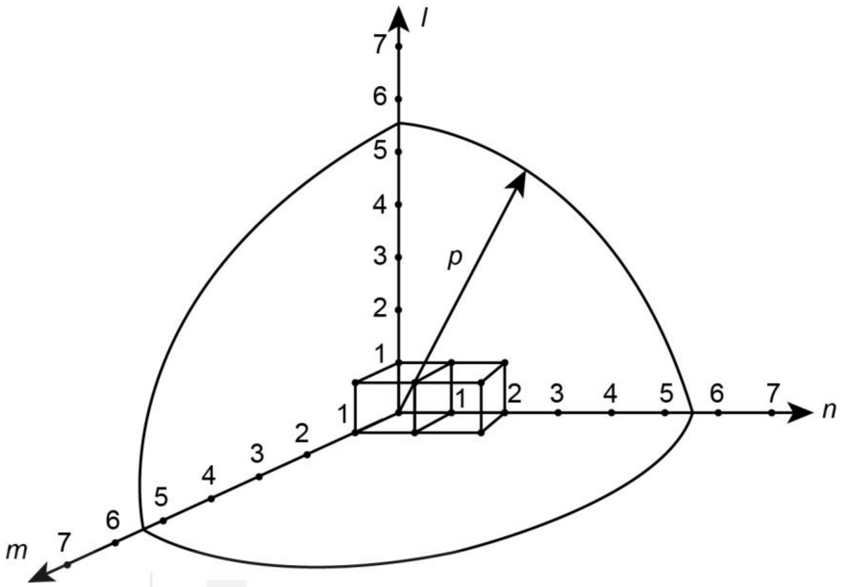


Fig. 11A.1: **Calculation of the number of allowed modes of standing waves in an enclosure filled with blackbody radiation.**

For sufficiently large values of *p*, each point will correspond in one unit cube in this octant. Therefore, the total number of allowed modes of vibration will be equal to the volume of the octant and we can write

$$N = \frac{1}{8}\left(\frac{4\pi}{3}p^3\right) = \frac{1\times 4}{8\times 3}\pi\left(\frac{2L}{\lambda}\right)^3 = \frac{4\pi L^3}{3\lambda^3}$$

Hence, the number of modes of wavelengths between λ and $\lambda + d\lambda$ is obtained by differentiating this expression for total number of modes. Thus,

$$|N_\lambda d\lambda| = \left|\frac{4\pi V}{3}(-3\lambda^{-4})\,d\lambda\right| = \frac{4\pi V}{\lambda^4}d\lambda \qquad (11A.5)$$

Here $V = L^3$ is volume of the enclosure.

You may recall that we are dealing with electromagnetic waves, which are transverse in nature and for a given value of wave vector, there will be two independent polarisation states. We, therefore, have to multiply Eq. (11A.5) by two. That is, the correct number of allowed modes will be twice as many:

The fact that the number of allowed modes was to be multiplied by two was pointed out by Jeans.

$$N_\lambda d\lambda = \frac{8\pi V}{\lambda^4}d\lambda \qquad (11A.6)$$

You can easily convince yourself that for blackbody radiation, the number of modes in the frequency range ν and $\nu + d\nu$ can be expressed as

$$N_\nu d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu \qquad (11A.7)$$