5. We take the curl of curl $\vec{\mathbf{B}}$ and get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{B}}) = \vec{\nabla}\times \mu_0\,\varepsilon_0\left(\frac{\partial\vec{\mathbf{E}}}{\partial t}\right) = \mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathbf{E}}) \qquad \text{(i)}$$

We now make use of the following vector identity for a vector field $\vec{\mathbf{F}}$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}}$$

Using this vector identity in Eq. (i), we can write it as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{B}}) - \nabla^2\vec{\mathbf{B}} = \mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathbf{E}}) \qquad \text{(ii)}$$

Then using $\vec{\nabla}\times\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}$ from Eq. (16.22), we can write Eq. (ii) as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{B}}) - \nabla^2\vec{\mathbf{B}} = -\mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} \qquad \text{(iii)}$$

Since $\vec{\nabla}.\,\vec{\mathbf{B}} = 0$ from Eq. (16.23), Eq. (iii) becomes

$$\nabla^2\vec{\mathbf{B}} = \mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2}$$

6. We have obtained the conditions $E_0 = vB_0$ and $B_0 = \dfrac{vE_0}{c^2}$ for the electric and magnetic fields of SAQ 4.

Since both these conditions are satisfied at the same time, we substitute the expression $E_0 = vB_0$ in the second condition and get

$$B_0 = \frac{vE_0}{c^2} = \frac{v^2 B_0}{c^2} \quad\Rightarrow\quad \frac{v^2}{c^2} = 1 \quad\Rightarrow\quad v = \pm c$$

If we substitute this result in the first condition, we get $|B_0|\,c = |E_0|$.

Terminal Questions

1. From the definition of the capacitance of a parallel plate capacitor given in Unit 11 of Block 3, we have

$$C = \frac{q}{V} \quad \text{or} \quad q = CV$$

where q is the charge on the capacitor plates and V, the potential difference across them. Now from Example 16.1, we know that

$$i_d = i = \frac{dq}{dt} \qquad \text{or} \qquad i_d = C\frac{dV}{dt}$$

Substituting the numerical values of the capacitance of the parallel plate capacitor and displacement current in the above expression, we get

$$\frac{dV}{dt} = \frac{i_d}{C} = \frac{1.0\,\text{A}}{5.0\,\text{nF}} = 2.0\times10^8\,\text{Vs}^{-1}$$

2. Substituting the expressions for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ given in the problem in Maxwell's equations (16.21 to 16.24), we get

i) $\vec{\nabla}.\vec{\mathbf{E}} = \vec{\nabla}.(E_0\,\hat{\mathbf{z}}\cos kx\cos ky\cos\omega t) = E_0\frac{\partial}{\partial z}\cos kx\cos ky\cos\omega t = 0$

ii) $$\vec{\nabla}\times\vec{\mathbf{E}} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & E_0\cos kx\cos ky\cos\omega t\end{vmatrix}$$

$$= E_0\left(\hat{\mathbf{x}}\frac{\partial}{\partial y}\cos kx\cos ky\cos\omega t - \hat{\mathbf{y}}\frac{\partial}{\partial x}\cos kx\cos ky\cos\omega t\right)$$

$$= E_0(-\hat{\mathbf{x}}\,k\cos kx\sin ky\cos\omega t) + \hat{\mathbf{y}}\,k\sin kx\cos ky\cos\omega t)$$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \frac{\partial}{\partial t}[B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\sin\omega t]$$

$$= \omega B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

From Eq. (16.22), we have

$$-kE_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

$$= -\omega B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

from where we get the condition that

$$kE_0 = \omega B_0 \quad \text{or} \quad \frac{E_0}{B_0} = \frac{\omega}{k} \qquad \text{(i)}$$

iii) $\vec{\nabla}.\vec{\mathbf{B}} = \vec{\nabla}.[B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\sin\omega t]$

$$= B_0\left(\frac{\partial}{\partial x}\cos kx\sin ky - \frac{\partial}{\partial y}\sin kx\cos ky\right)\sin\omega t$$

$$= B_0(-k\sin kx\sin ky + k\sin kx\sin ky)\sin\omega t = 0$$

iv) $$\vec{\nabla}\times\vec{\mathbf{B}} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_0\cos kx\sin ky\sin\omega t & -B_0\sin kx\cos ky\sin\omega t & 0\end{vmatrix}$$

or $$B_0\,\hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}\sin kx\cos ky - \frac{\partial}{\partial y}\cos kx\sin ky\right)\sin\omega t$$

$$= \frac{E_0}{c^2}\hat{\mathbf{z}}\frac{\partial}{\partial \mathrm{t}}(\cos kx\cos ky\cos\omega t)$$

or $$-B_0\,\hat{\mathbf{z}}\,k\left(\cos kx\cos ky + \cos kx\cos ky\right)\sin\omega t$$

$$= -\frac{E_0\,\omega}{c^2}\hat{\mathbf{z}}\,(\cos kx\cos ky\sin\omega t)$$

or $$2kB_0 = \frac{E_0\,\omega}{c^2}$$

or $$B_0 = \frac{E_0}{2c^2}\frac{\omega}{k} = \frac{B_0}{2c^2}\frac{\omega^2}{k^2} \qquad \left(\because\ \frac{E_0}{B_0} = \frac{\omega}{k}\right)$$

So, Eq. (16.24) gives $\frac{\omega^2}{k^2} = 2c^2$

or $\frac{\omega}{k} = \sqrt{2}c \Rightarrow \omega = \sqrt{2}ck$ and $E_0 = \frac{\omega}{k}B_0 = \sqrt{2}cB_0$

3. Since the electromagnetic wave is travelling in vacuum, the maximum electric and magnetic fields associated with the wave satisfy the relation $E_0 = cB_0$, where c is the speed of light. Therefore, the magnitude of the maximum magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{600\,\text{Vm}^{-1}}{3.0 \times 10^8\,\text{ms}^{-1}} = 2.0 \times 10^{-6}\,\text{T}$$

4. Let us first write the expressions of the resultant electric and magnetic fields:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2, \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$$

$$\vec{\mathbf{E}} = \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y - ct) + \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y + ct)$$

$$= \hat{\mathbf{z}}E_0\left[\sin\frac{2\pi}{\lambda}(y - ct) + \sin\frac{2\pi}{\lambda}(y + ct)\right]$$

$$= 2\hat{\mathbf{z}}E_0 \sin\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda} \quad \left[\because \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right]$$

$$\vec{\mathbf{B}} = \hat{\mathbf{x}}B_0 \sin\frac{2\pi}{\lambda}(y - ct) - \hat{\mathbf{x}}B_0 \sin\frac{2\pi}{\lambda}(y + ct) = -2\hat{\mathbf{x}}B_0 \cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}$$

$$\left[\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right]$$

We have to show that the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ satisfy Maxwell's equations in vacuum:

i) $$\vec{\nabla}.\vec{\mathbf{E}} = 2E_0\frac{\partial}{\partial z}\left(\sin\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}\right) = 0$$

ii) $$\vec{\nabla}\times\vec{\mathbf{E}} = \hat{\mathbf{x}}2E_0\left(\frac{\partial}{\partial y}\sin\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}\right) = \hat{\mathbf{x}}2E_0\frac{2\pi}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = -2\hat{\mathbf{x}}B_0\left(\frac{\partial}{\partial t}\cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}\right) = -2\hat{\mathbf{x}}B_0\frac{2\pi c}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

From Eq. (16.22), we have

$$\hat{\mathbf{x}}2E_0\frac{2\pi}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda} = \hat{\mathbf{x}}2B_0\frac{2\pi c}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

or $E_0 = B_0 c$ (i)

iii) $$\vec{\nabla}.\vec{\mathbf{B}} = -2B_0\frac{\partial}{\partial x}\left(\cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}\right) = 0$$

iv) $$\vec{\nabla}\times\vec{\mathbf{B}} = 2B_0\,\hat{\mathbf{z}}\left(\frac{\partial}{\partial y}\cos\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}\right) = -\hat{\mathbf{z}}\,2B_0\,\frac{2\pi}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

and $$\frac{\partial\vec{\mathbf{E}}}{\partial t} = -2\,\hat{\mathbf{z}}\,E_0\,\frac{2\pi c}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

From Eq. (16.24), we have

$$-\hat{\mathbf{z}}\,2B_0\,\frac{2\pi}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda} = -\frac{1}{c^2}\,2\,\hat{\mathbf{z}}\,E_0\,\frac{2\pi c}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

or $E_0 = B_0\,c$

So, the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ satisfy Maxwell's equations in vacuum if $E_0 = B_0\,c$.

5. We can write the electric field in vector form as

$$\vec{\mathbf{E}} = \hat{\mathbf{z}}\,(3.0\ \text{Vm}^{-1})\sin(x - 10^8\,t)$$

The magnitude of the magnetic field is obtained from Eq. (16.31a): $E_0 = B_0\,c$, where $E_0 = 3.0\ \text{Vm}^{-1}$. So,

$$B_0 = \frac{E_0}{c} = \frac{3.0\,\text{Vm}^{-1}}{3.0\times 10^8\,\text{m}\,\text{s}^{-1}} = 1.0\times 10^{-8}\ \text{T}$$

and $$\left|\vec{\mathbf{B}}\right| = (1.0\times 10^{-8}\ \text{T})\sin(x - 10^8\,t)$$