7. a) Consider the Gaussian surfaces S_1 and S_2 each of area ΔA in the two slabs (Fig. 11.28).

Let the displacements in the two slabs be $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$, respectively.

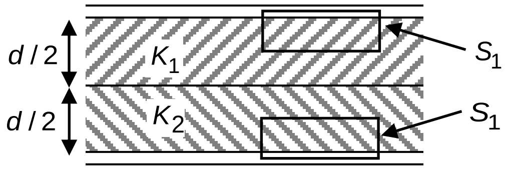

Fig. 11.28: **Diagram for answer to TQ 7.**

Applying Gauss's law to surface S_1, we get

$$\int_{S_1} \vec{\mathbf{D}}_1 . d\vec{\mathbf{S}} = Q_{en}\Big|_{S_1}$$

or $$D_1 \Delta A = \sigma \Delta A \quad \Rightarrow \quad D_1 = \sigma \qquad \text{(i)}$$

Similarly for surface S_2, $$D_2 = \sigma \qquad \text{(ii)}$$

b) Since $\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}$, we get $$E_1 = \frac{D_1}{\varepsilon_0 K_1} = \frac{\sigma}{\varepsilon_0 K_1} \qquad \text{(iii)}$$

and $$E_2 = \frac{D_2}{\varepsilon_0 K_2} = \frac{\sigma}{\varepsilon_0 K_2} \qquad \text{(iv)}$$

c) The potential difference between the plates is given by:

$$V = \int_0^d \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_0^{d/2} \vec{\mathbf{E}}_1.d\vec{\mathbf{r}} + \int_{d/2}^{d} \vec{\mathbf{E}}_2.d\vec{\mathbf{r}} = \int_0^{d/2} E_1 dr + \int_{d/2}^{d} E_2 dr$$

or $$V = E_1 r\Big|_0^{d/2} + E_2 r\Big|_{d/2}^{d} = E_1 \frac{d}{2} + E_2 \frac{d}{2}$$

Using Eqs. (iii) and (iv) in this expression, we get

$$V = \frac{\sigma}{\varepsilon_0} \frac{d}{2} \left(\frac{1}{K_1} + \frac{1}{K_2} \right) \qquad \text{(v)}$$

d) From Eq. (v), $$C = \frac{Q}{V} = \frac{\sigma A \varepsilon_0}{\frac{\sigma d}{2}\left(\frac{1}{K_1} + \frac{1}{K_2} \right)} = \frac{2A\varepsilon_0}{d} \frac{K_1 K_2}{(K_1 + K_2)}$$

8. In order to calculate the surface charge densities at $r = b$ and $r = c$, we need to calculate the polarisation for both cases. We do it as follows: Due to spherical symmetry, the electric fields and the displacements are radial for both cases. Now consider a spherical Gaussian surface of radius r such that $a < r < b$ (Fig. 11.29). Since Q is the charge enclosed by this sphere, from Gauss's law, we have

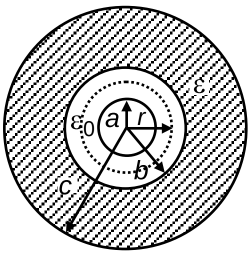

Fig. 11.29: **Diagram for answer to TQ 8.**

$$D_1 4\pi r^2 = Q \quad \Rightarrow \quad D_1 = \frac{Q}{4\pi r^2}$$

Now recall from Unit 6 that the electric field

$$\vec{\mathbf{E}}_1 = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{\mathbf{r}} \qquad a < r < b \qquad \text{(i)}$$

In the same way, we can show that for the region $b < r < c$,

$$\vec{\mathbf{E}}_2 = \frac{Q}{4\pi \varepsilon r^2} \hat{\mathbf{r}} \qquad b < r < c \qquad \text{(ii)}$$

Now, you know that for linear dielectrics,

$$\vec{\mathbf{P}} = \vec{\mathbf{D}} - \varepsilon_0 \vec{\mathbf{E}} = (\varepsilon - \varepsilon_0)\vec{\mathbf{E}} = \varepsilon_0 (K-1)\vec{\mathbf{E}}$$

Therefore, at $r = b$, $\quad \vec{\mathbf{P}}_1 = \varepsilon_0 (K-1)\vec{\mathbf{E}}_1$

Since the normal to the dielectric surface at $r = b$ is along $-\hat{\mathbf{r}}$, we have

$$\sigma_b\big|_{r=b} = -\vec{\mathbf{P}}_1 . \hat{\mathbf{r}}\Big|_{r=b} = -\varepsilon_0 (K-1)\vec{\mathbf{E}}_1 . \hat{\mathbf{r}}\Big|_{r=b}$$

$$= -\varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} . \hat{\mathbf{r}}\Bigg|_{r=b}$$

$$= -\varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon_0 b^2}$$

or $$\sigma_b\big|_{r=b} = -\frac{Q(K-1)}{4\pi b^2}$$

Notice that the unit vector normal to the dielectric's surface points *outward with respect to the dielectric sphere*, which is $+\hat{\mathbf{r}}$ at $r = c$ but $-\hat{\mathbf{r}}$ at $r = b$.

To determine σ_b at $r = c$, we follow the same method as above.

Since the normal to the dielectric surface at $r = c$ (see the margin remark) is along $\hat{\mathbf{r}}$, we have

$$\sigma_b\big|_{r=c} = \vec{\mathbf{P}}_2 . \hat{\mathbf{r}}\big|_{r=c} = \varepsilon_0 (K-1)\vec{\mathbf{E}}_2 . \hat{\mathbf{r}}\big|_{r=c}$$

$$= \varepsilon_0 (K-1)\frac{Q}{4\pi\varepsilon c^2}$$

$$= \varepsilon_0 (K-1)\frac{Q}{4\pi\varepsilon_0 K c^2} \qquad \left(\because K = \frac{\varepsilon}{\varepsilon_0}\right)$$

or $$\sigma_b\big|_{r=c} = \frac{Q(K-1)}{4\pi K c^2}$$

To determine the potential difference between the outer and inner shells, we begin from its definition

$$V = -\int_c^a \vec{\mathbf{E}} . d\vec{\mathbf{r}} = \int_a^c \vec{\mathbf{E}} . d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{E}}_1 . d\vec{\mathbf{r}} + \int_b^c \vec{\mathbf{E}}_2 . d\vec{\mathbf{r}}$$

Substituting $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$ from Eqs. (i) and (ii), we get

$$V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_0 K}\int_b^c \frac{dr}{r^2} \qquad (\because \ \varepsilon = \varepsilon_0 K)$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)\Bigg|_a^b - \frac{Q}{4\pi\varepsilon_0 K}\left(\frac{1}{r}\right)\Bigg|_b^c$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) - \frac{Q}{4\pi\varepsilon_0 K}\left(\frac{1}{c} - \frac{1}{b}\right)$$

Simplifying the expression for V, we get

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{c-b}{K\,cb} + \frac{b-a}{ab}\right]$$

and $$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{\dfrac{c-b}{K\,cb} + \dfrac{b-a}{ab}}$$