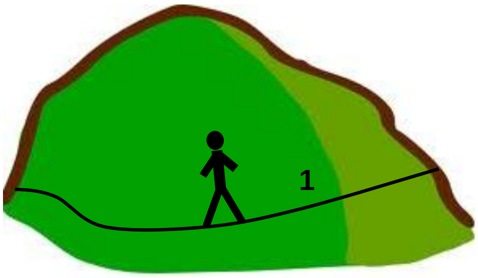



**How would you determine the path of steepest ascent on a hill?**

UNIT 1

# SCALAR FIELDS AND THEIR GRADIENT

## Structure



## STUDY GUIDE

In this unit you will study about scalar fields and their gradient. Before studying this unit, you should revise Units 1 and 2 of the Physics Elective BPHCT-131 entitled Mechanics. You should also be familiar with the basic concepts of differential calculus which you have studied in school. These concepts are also explained in the Mathematics Elective BMTC-131. For calculating the gradient you must know how to obtain the partial derivatives of functions. You can revise this from the Appendix of this unit. Partial derivatives are also explained in the Mathematics course BMTC-132, which you may revise. A brief summary of the basic concepts of vector algebra and derivatives of vector functions is provided in Appendix A1 of this block. You may like to go through it before studying this unit.

***"And, believe me, if I were again beginning my studies, I should follow the advice of Plato and start with mathematics."***

***Galileo Galilei***

## 1.1 INTRODUCTION

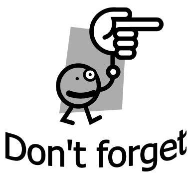
In the first two units of Block 1 of the first course in Physics entitled Mechanics (BPHCT-131), you have studied the basic concepts of vector algebra. You have learnt how to add, subtract and multiply vector quantities using both the geometric and algebraic representation of vectors. You have seen examples of vectors, their sum and difference, as well as scalar and vector products in physics. In Unit 2 you have also studied the preliminary concepts of vector differential calculus. You now know about vector functions or vector valued functions and how to obtain their derivatives with respect to a scalar variable. You have also learnt how to obtain the derivatives of vector and scalar products of vectors. In this unit, we further study vector differential calculus. In Sec. 1.2, we introduce the concept of **scalar fields**. In Sec. 1.3, you will learn about the **gradient** and **directional derivative** of a scalar field.

You may wish to know: Why do you need to learn these concepts? To understand this, consider the example of a bar, plate or a cylinder that is heated non-uniformly. So, its temperature at different points is different. The temperature distribution of the bar/plate/cylinder is represented mathematically by a **scalar field**. If we now wish to know the **rate of change** of the temperature within the object, **in any given direction**, we need to determine the **directional derivative** of this temperature distribution. For this, we must know the **gradient of the scalar field**. In Sec. 1.3.1, you will learn how to determine the gradient of a scalar field. Then you will learn how to determine the directional derivative of the field in Sec. 1.3.2.

### Expected Learning Outcomes

After studying this unit, you should be able to:

- explain the concept of scalar fields and give examples in physics;
- determine the gradient of a scalar field; and
- determine the directional derivative of a scalar field.




**IN YOUR WRITTEN WORK, ALWAYS USE AN ARROW ABOVE THE LETTER YOU USE TO DENOTE A VECTOR, E.G., $\vec{r}$. USE A CAP ABOVE THE LETTER YOU USE TO DENOTE A UNIT VECTOR, E.G., $\hat{r}$.**

## 1.2 SCALAR FIELDS

In Block 1 of the course BPHCT-131 entitled Mechanics you have learnt about vector functions, which can depend on one or more variable. When we are describing physical quantities, one obvious variable with which many physical quantities change is time. However, you have also studied about many physical quantities which have different values at different points in space. **A function that describes a physical quantity at different points in space is called a field**. As physical quantities are either scalar or vector in nature, we can have both **scalar** and **vector fields**. In this section, we focus on **scalar fields**.

For example, the density of air (in the Earth's atmosphere) is a scalar quantity that changes with the altitude above the sea level. Similarly, the atmospheric pressure is also a scalar having different values at different altitudes. It is also different at different points around the Earth. The temperature of an unevenly heated plate is a function of both space coordinates and time. All these are examples of **scalar** fields.

Let us discuss the concept of a scalar field in more detail.

## 1.2.1 Defining a Scalar Field

**A scalar field is a function that assigns a unique scalar to every point in a given region.** So you can say that it is essentially a scalar function of space coordinates. As you know from your school mathematics, every point in space may be specified by the Cartesian coordinates of the point $(x, y, z)$. So we can write the scalar function or scalar field as $f = f(x, y, z)$. This means that for every point $(x, y, z)$ in space, there exists a unique scalar quantity given by $f(x, y, z)$.

You have studied in Unit 2 of BPHCT-131 that every point in space can be denoted by the position vector

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

So we could also say that corresponding to every point represented by a position vector $\vec{\mathbf{r}}$ we have a unique scalar quantity $f(\vec{\mathbf{r}})$.

The gravitational potential energy of an object near the surface of the Earth is a simple example of a scalar field. Suppose we take the *xy* plane to lie on the surface of the Earth and the *z*-axis to point upwards. Then the gravitational potential energy of the object is given by

$$\phi(x, y, z) = mgz \qquad (1.1a)$$

where *m* is the mass of the object. In this case the scalar field depends only on the *z* coordinate and is, therefore, a one-dimensional scalar field.

Temperature, pressure and density are examples of some other physical quantities which are scalars and can be represented by scalar fields. For example, the temperature on a perfectly flat hot plate can be described by

$$T(x, y) = \frac{250}{x^2 + y^2 + 1} \qquad (1.1b)$$

*T* (*x*, *y*) gives us the temperature at any point (*x*, *y*) on the surface of the hot plate (Fig. 1.1). Since the surface of the plate is two-dimensional, the value of *T* depends only on *x* and *y*. This is an example of a two-dimensional scalar field.

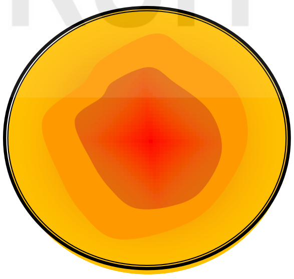


**Fig. 1.1**: **Temperature on a flat hot plate heated at the centre is a two-dimensional scalar field. At the centre the temperature is very high. The temperature is lower as we move away from the centre of the hot plate.**

Another example of a scalar field is the **electric potential** in free space at a distance *r* from a point charge *q*. It is given by

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \text{ volt} \qquad (1.1c)$$

where *q* is measured in coulomb and *r* in m. Here $\varepsilon_0$ is the permittivity of free space. If we consider the origin of the Cartesian coordinate system to be located at the charge we can also write the electric potential at a distance *r* as

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(x^2 + y^2 + z^2)^{1/2}}\right] \quad (1.1d)$$

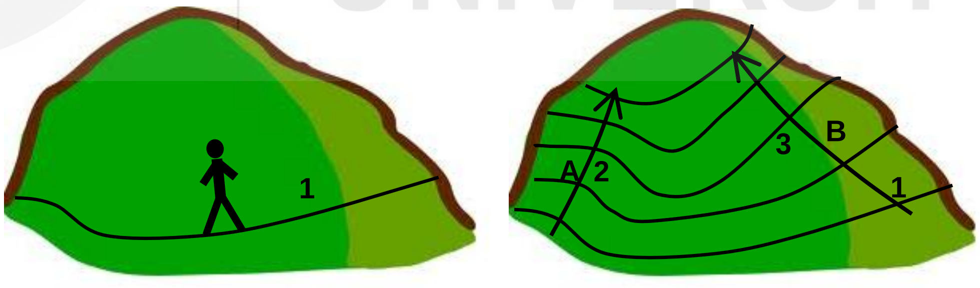
This function describes a scalar field in three-dimensions. In this unit, we are interested in finding out about the rate at which these fields change in space.

In order to understand physically the rate of change of scalar fields, it is a good idea to learn how to represent them pictorially. So our next question is: **How do we represent scalar fields in diagrams**? It is possible to represent scalar fields visually using what are called **contour curves** or **contour surfaces**. Let us see how.

## 1.2.2 Representations of a Scalar Field

Have you heard about contour lines? Do you remember the maps you studied in your school geography courses? How do we show the heights of places on a map? Recall from school geography courses that these are shown using **contour lines** or **contour curves**. On a map, **contour lines or contour curves connect those points which are at the same height (elevation) above a fixed level**. This is usually the sea level. Remember that **contour lines never cross each other**. If you walk along a contour line you neither gain nor lose elevation. Contour lines are useful because they tell us about the **shape** of the land surface.

Imagine that you are standing at some point on a hill at a certain height (Fig. 1.2a). Study Fig. 1.2b. It shows the contour curve 1 joining all points of that height on the hill. In fact, **each contour curve joins all the points on the hill which are at the same height**. Suppose we write the height at a point ($x$, $y$) as $z$ ($x$, $y$). Then the contour curves in Fig 1.2b join the points $z(x, y)$ = constant. The contour curves are then a pictorial representation of the scalar function $z(x, y)$.




(a) (b)

**Fig.1. 2**: **a) Imagine you are standing somewhere on a hill; b) contour map of the hill. A plot of several contour lines is called a contour map.**

We have already described the scalar field for the gravitational potential energy in Eq. (1.1a). You know that all points in any given contour curve in Fig. 1. 2b depict points which lie at the same height above the ground. So as per Eq. (1.1a), it is also the curve on which the gravitational potential energy of the object would be constant. Thus, the set of contour curves shown in Fig. 1. 2b can be used to depict the gravitational potential energy field given by Eq. (1.1a).

In general,

**A contour curve is defined as a curve in two-dimensions on which the value of the scalar field** $f(x, y)$ **is constant**:

$$f(x, y) = C \qquad (1.2a)$$

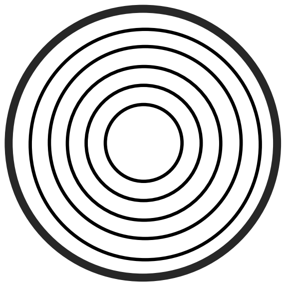


**Fig. 1.3**: **Contour Map for the temperature of the hot plate. Each contour line corresponds to one value of the temperature defined in Eq. (1.1b).**

A plot of several contour curves is called a **contour plot** or a **contour map**.

The contour map for the temperature of the hot plate is shown in Fig. 1.3.

We now ask: What else can we determine from the contour map of a scalar field? Study Fig. 1.2b once again. Look at the regions around the points *A* and *B* in it. Note that the contour curves around *A* are close together and those around *B* are far apart. What does this tell us?

Imagine that you are climbing the hill along a steep path (path 2 in Fig. 1.2b). Since the height above the sea level changes rapidly along this path, the contour curves (the curves of equal height above the ground) lie close to each other. On the other hand, if you walk along path 3, where the contour curves are far apart, the height above sea level changes comparatively slowly. So, the spacing of the contour curves on a contour map indicates how rapidly the function is changing: If the contour curves lie close to each other, the scalar field changes rapidly in that region. If these are far apart, the change in the scalar field is slower.

So far we were talking about scalar functions in two-dimensions.

You may now ask: How do we represent scalar fields in three-dimensions? One way of doing this is to define **contour surfaces** as follows:

**Contour surfaces are the surfaces on which the value of a three-dimensional scalar field is constant**.

So, if a scalar field is defined by the function $\phi(x, y, z)$, the contour surface would be the collection of all points $(x, y, z)$ for which the value of $\phi$ is constant, say *C*. The contour surface is defined by the equation

$$\phi(x, y, z) = C \qquad (1.2b)$$

We get different contour surfaces for different values of *C*. A collection of such contour surfaces would then be a representation of the scalar field $\phi(x, y, z)$.

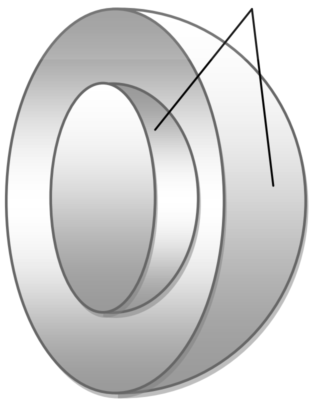



**Fig. 1.4**: **Contour surfaces of constant electric potential are surfaces of concentric spheres. The radii of the spheres are defined by Eq. (1.3b).**

For example, for the scalar field described by Eq. (1.1d), the contour surfaces are the surfaces of constant electric potential given by $V(x, y, z) = V_0$ and are described by the equation

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2 + y^2 + z^2)^{1/2}} = V_0 \Rightarrow x^2 + y^2 + z^2 = R^2 \qquad (1.3a)$$

where *R* is a constant. You can see that the contour surface given by Eq. (1.3a) is a sphere whose radius is given by the equation:

$$R = \frac{q}{4\pi\varepsilon_0 V_0} \qquad (1.3b)$$

For different values of $V_0$, $R$ is different. The contour surfaces (for different values of $V_0$) are surfaces of concentric spheres, as shown in Fig. 1.4.

Any contour curve on which a potential is constant is called an **equipotential curve**. The contour surfaces described by Eq. (1.3a) are called **equipotential surfaces**. Similarly, contour curves or surfaces for the temperature field are called **isothermals**.

Now that we have defined a scalar field and represented it pictorially, we would like to know: **What is the rate of change of a scalar field in a given direction?** For example, if you are standing somewhere on a hill, you may wish to find out the direction of the fastest way down (or the steepest slope) so that you may avoid it! You would of course like to take the direction of least slope to climb down, lest you fall! Recall from school calculus that the slope also gives us the rate of change. We use this fact to arrive at the concept of the gradient operator and the directional derivative. You will learn in the next section that the **gradient of a scalar field gives the maximum rate of change of the function**. So let us define the gradient of a scalar field. Then we shall use it to explain the concept of directional derivative.

## 1.3 GRADIENT OF A SCALAR FIELD AND THE DIRECTIONAL DERIVATIVE

In your school calculus course, you have learnt about the concept of the slope of a function. Suppose you heat a thin rod at one of its ends. Then the temperature $T$ of the rod would be different at different points along its length. Suppose you wish to know the rate at which $T$ changes along the rod. Let us treat the rod as a one-dimensional object and choose the $x$-axis to be along the rod. So, $T = T(x)$ is a function of $x$.

Then the question is: **If you change $x$ by a small amount $dx$, by what amount does $T(x)$ change?** Let this change in $T(x)$ be denoted by $dT$. From school calculus, you know that the answer is

$$dT = \left(\frac{dT}{dx}\right)dx \tag{1.4}$$

So if $x$ is changed by an amount $dx$, $T$ changes by an amount $dT$ given by Eq. (1.4). The derivative $\frac{dT}{dx}$ is the proportionality factor. It gives **the rate of change in $T(x)$ along the $x$-direction**.

In this example, the temperature is a function of just one variable $x$ measured along the length of the rod.

Now imagine a closed room with a fire lit in the fireplace (or a room heater) in one corner. The temperature in the room is described by the scalar field $T(x, y, z)$. Standing at a point in the room, say at the centre, we could ask at what rate the temperature changes as we move away from the point in the $x$-direction, say by a distance $\Delta x$. From your school calculus course, you know that the rate of change of a two or three-dimensional function is described by **partial derivatives** of the function. Thus, the partial derivative of $T(x, y, z)$ with respect to $x$ gives its rate of change in the $x$-direction. Can we then find out the direction in which the rate of change of temperature is maximum? The answer is, yes, we can, in terms of the partial derivatives of the scalar field $T(x, y, z)$. At this stage, you may like to refresh your knowledge of partial derivatives. For this, you should **study the Appendix to this unit before studying further**.

Let us now answer the question: How do we express the **maximum rate of change of any scalar field**? To do so, we define the **gradient** of a scalar field using the concept of partial derivatives.

## 1.3.1 The Gradient of a Scalar Field

Let us begin by considering a two-dimensional scalar field $f(x, y)$. Then we shall generalize the results to $f(x, y, z)$. Consider two points $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ in the region in which the scalar field $f(x, y)$ is defined. Suppose the position vectors of the points are $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}+\Delta\vec{\mathbf{r}}$, respectively. Then the change in $f(x, y)$ as one goes a small distance from the point $P$ to point $Q$ is given by

Do the steps from Eq. (1.5a) to Eq. (1.7c) yourself on a separate piece of paper as you study them. You will understand this concept better.

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x, y) \tag{1.5a}$$

To get to the final result, we add and subtract a term $f(x, y+\Delta y)$ and write:

$$\Delta f = [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + [f(x, y+\Delta y) - f(x, y)] \tag{1.5b}$$

Then we divide and multiply the two terms on the right hand side of Eq. (1.5b) by $\Delta x$ and $\Delta y$, respectively, and write:

$$\Delta f = \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x + \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y \tag{1.5c}$$

Next we consider the terms on the right-hand side of Eq. (1.5c) and take the limits of $\Delta f$ as $\Delta x \to 0$ and $\Delta y \to 0$:

$$\lim_{\Delta x\to 0, \Delta y\to 0} \Delta f = \lim_{\Delta x\to 0, \Delta y\to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x + \lim_{\Delta x\to 0, \Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y \tag{1.5d}$$

Using the definition of partial derivative in Eq. (1) of the Appendix to this unit, we can write

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x = \frac{\partial f(x, y+\Delta y)}{\partial x}\, dx$$

and $$\lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y = \frac{\partial f(x, y)}{\partial y}\, dy$$

Using these results, we can write Eq. (1.5d) as

$$\lim_{\Delta x\to 0, \Delta y\to 0} \Delta f = \lim_{\Delta y\to 0} \frac{\partial f(x, y+\Delta y)}{\partial x} dx + \lim_{\Delta x\to 0} \frac{\partial f(x, y)}{\partial y}\, dy$$

$$= \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y}\, dy \tag{1.5e}$$

The left hand side of the Eq. (1.5e) defines a function *df* called the **total differential** of *f*. Thus, the total differential of a two-dimensional scalar field $f(x, y)$ or *f* (to keep the writing of the equation simple) is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{1.6a}$$

We can generalize this result to $f(x, y, z)$ for which the total differential *df* is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{1.6b}$$

The significance of $df$ is that it is a good approximation to $\Delta f$ if $dx (= \Delta x), dy = (\Delta y)$ and $dz (= \Delta z)$ represent small changes in $x, y$ and $z$, respectively. So $df$ becomes a better and better approximation of $\Delta f$ as $dx, dy$ and $dz$ become smaller.

We are now ready to introduce the concept of the gradient. We can rewrite Eq. (1.6b) in a more convenient form as follows:

$$df = \left( \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} \right) . (dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}}) \tag{1.7a}$$

You must verify by simplifying Eq. (1.7a) that it is indeed the same as Eq. (1.6b). You can also identify that

$$dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \equiv d\vec{\mathbf{r}} \tag{1.7b}$$

is the change in the position vector. The vector

$$\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} = \text{grad } f \equiv \vec{\nabla} f \tag{1.7c}$$

is defined as the **gradient of the scalar field** *f*.

Using Eqs. (1.7b and c), we can write Eq. (1.7a) as

$$df = (\vec{\nabla} f) \,.\, d\vec{\mathbf{r}} \tag{1.8}$$

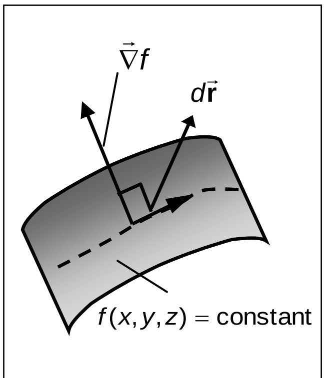



**Fig. 1.5**: **The gradient of a scalar field is perpendicular to the surface on which the scalar field is constant.**

The symbol $\nabla$ is pronounced as 'del' and $\vec{\nabla} f$ as grad *f*. Eq. (1.7c) tells us that **the gradient of a scalar field is a vector**. Note that $\vec{\nabla} f$ *is a vector such that the change df in f for an arbitrarily small change* $d\vec{\mathbf{r}}$ *is given by* Eq. (1.8). Let us interpret the physical meaning of this result.

Suppose the angle between $\vec{\nabla} f$ and $d\vec{\mathbf{r}}$ is given by $\theta$. Then we can write Eq. (1.8) as

$$df = |\vec{\nabla} f|\, |d\vec{\mathbf{r}}| \cos\theta = |\vec{\nabla} f|\, dr \cos\theta \tag{1.9a}$$

What happens when $\theta$ is 90°, that is $\vec{\nabla} f$ is **perpendicular** to $d\vec{\mathbf{r}}$ ? Eq. (1.9a) tells us that *df* = 0 which means that *f* is constant. In other words, the **value of the scalar field *f* is constant along the direction perpendicular to its gradient** (Fig. 1.5). We can express this result as follows:

> The vector $\vec{\nabla} f$ is perpendicular (normal) to the curve or surface
>
> $f = \text{constant}$

Don't forget

We now keep *dr* constant and find the change *df* in various *directions* by changing $\theta$. Then we ask: For what value of $\theta$ (or in which direction) is the change maximum? From Eq. (1.8), you can immediately say that *df* is maximum when $\theta$ is 0°. This means that for a fixed *dr*, the change *df* is **maximum** when you move in the same direction as $\vec{\nabla} f$. In addition, the rate of change of *f* with respect to *r* is given by

$$\frac{df}{dr} = \left|\vec{\nabla} f\right| \cos\theta \qquad (1.9b)$$

So the **maximum rate of increase of the scalar field *f* in space is along the direction of the gradient of the field $\vec{\nabla} f$ and its magnitude is**

$$\left(\frac{df}{dr}\right)_{max} = \left|\vec{\nabla} f\right| \qquad (1.10)$$

> The magnitude of $\vec{\nabla} f$ gives the **maximum rate of change** of the scalar field in space.

Don't forget

In the same way, when $d\vec{\mathbf{r}}$ is in the direction opposite to $\vec{\nabla} f$, then $\theta$ is 180° in Eq.(1.8) and $\frac{df}{dr} = -|\vec{\nabla} f|$. This is then the direction in which the **rate of decrease of the field *f* is maximum**.

We have arrived at the **definition of the gradient of a scalar field and learnt its physical meaning**. Let us put these results together.

Recap

## GRADIENT OF A SCALAR FIELD

The **gradient of a scalar field** *f* is defined as follows:

$$\text{grad } f \equiv \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad (1.11a)$$

For a scalar field $f(x, y)$ in two-dimensions,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} \qquad (1.11b)$$

The **magnitude** of $\vec{\nabla} f$ gives the maximum rate of change of the scalar field in space.

The **direction** of $\vec{\nabla} f$ is **perpendicular** to the curves or surfaces $f = \text{constant}$.

**You should note that:** $f(x, y, z)$ is a scalar field, because it assigns a **scalar quantity** to every point $(x, y, z)$ in space. On the other hand, the **gradient** $\vec{\nabla} f$ of the scalar field $f$ **assigns a vector quantity to every point in space**. Therefore, $\vec{\nabla} f$ is a **vector field**.

Let us further understand the concept of gradient with the help of examples from physics and calculate its value.

In physics, many vector quantities can be expressed as the gradient of a scalar field. For example, we can express the gravitational force as $\vec{\mathbf{F}} = -\vec{\nabla} V$, where $V$ is the gravitational potential energy. The electric field $\vec{\mathbf{E}}$ due to a static charge distribution is the gradient of the electric potential: $\vec{\mathbf{E}} = -\vec{\nabla}\phi.$ Let us take up an example to calculate the gradient of a potential field.

## EXAMPLE 1.1: GRADIENT OF A SCALAR FIELD

The potential that represents an inverse square force is $V(x, y, z) = \dfrac{k}{(x^2 + y^2 + z^2)^{1/2}}$ where $k$ is a constant. Using the definition $\vec{\mathbf{F}} = -\vec{\nabla} V,$ calculate the components of this force.

**SOLUTION ■** We use the definition $\vec{\mathbf{F}} = -\vec{\nabla} V$ and calculate the gradient of $V(x, y, z)$ from Eq. (1.11a).

Thus, we have

$$(F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}} + F_z \hat{\mathbf{k}}) = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right)$$

or
$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

So we have to calculate the above partial derivatives of $V(x, y, z)$.

Now
$$F_x = -\frac{\partial V}{\partial x} = -k\frac{\partial}{\partial x}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$= -k\left[-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] \quad \text{(read the margin remark)}$$

$$= \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}$$

Similarly,
$$F_y = -\frac{\partial V}{\partial y} = \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}$$

and
$$F_z = -\frac{\partial V}{\partial z} = \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}$$

Let

$$x^2 + y^2 + z^2 = t$$

then

$$\frac{\partial}{\partial x}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right] = \left[\frac{\partial}{\partial t}\left(\frac{1}{t^{1/2}}\right)\left(\frac{\partial t}{\partial x}\right)\right]$$

$$= -\frac{1}{2}\left(\frac{1}{t^{3/2}}\right) \times \frac{\partial}{\partial x}[x^2 + y^2 + z^2]$$

$$= -\frac{1}{2t^{3/2}} \,.\, 2x$$

$$= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \qquad \vec{\mathbf{F}} = -\vec{\nabla} V = \frac{k}{(x^2+y^2+z^2)^{3/2}}\,(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) \qquad \text{(i)}$$

$$= \frac{k\,\vec{\mathbf{r}}}{r^3}$$

$$= \frac{k}{r^2}\hat{\mathbf{r}} \qquad \text{since } \vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$

where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $r = (x^2+y^2+z^2)^{1/2}$.

Do the steps of Example 1.1 yourself on a separate piece of paper.

Example 1.1 illustrates an important application of the gradient. In physics, we use scalar and vector field functions to represent various physical quantities. They are often related in this way: *a vector field is the scalar multiple of the gradient of some scalar field function*. This also suggests that we can construct a vector field from a scalar field by taking its gradient. Notice that in this section, we have used the term 'vector field'. You will learn more about it in Unit 2.

Let us take another example for calculating the gradient of a scalar field.

## *EXAMPLE 1.2*: GRADIENT OF A SCALAR FIELD

Determine the gradient of an arbitrary function $f(r)$ where $r = (x^2+y^2+z^2)^{1/2}$. Use this to determine the gradient of the function $\phi(x,y,z) = \dfrac{k}{(x^2+y^2+z^2)}$.

**SOLUTION ■** We use Eq. (1.11a) with $\phi = f(r)$:

$$\vec{\nabla}\phi = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad \text{(i)}$$

Using the chain rule from calculus we can write

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}\,\frac{\partial r}{\partial x}$$

$$= \left(\frac{\partial f}{\partial r}\right)\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{1/2}\right] \quad [\because\ r = (x^2+y^2+z^2)^{1/2}]$$

$$= \frac{\partial f}{\partial r}\cdot\left[\frac{1}{2}\cdot\frac{2x}{(x^2+y^2+z^2)^{1/2}}\right]$$

$$= \frac{x}{r}\,\frac{\partial f}{\partial r} \qquad (\because\ r = (x^2+y^2+z^2)^{1/2}) \qquad \text{(ii)}$$

$$\frac{\partial}{\partial x}[(x^2+y^2+z^2)^{1/2}]$$

$$= \frac{1}{2}\left[(x^2+y^2+z^2)^{\frac{1}{2}-1}\right]\times(2x)$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{\partial f}{\partial r} \cdot \left[\frac{1}{2} \cdot \frac{2y}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{y}{r}\frac{\partial f}{\partial r} \qquad \text{(iii)}$$

and $$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r} \cdot \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{\partial f}{\partial r} \cdot \left[\frac{1}{2} \cdot \frac{2z}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{z}{r}\frac{\partial f}{\partial r} \qquad \text{(iv)}$$

Substituting (ii), (iii) and (iv) in (i) we get

$$\vec{\nabla} f(r) = \frac{x}{r}\frac{\partial f}{\partial r}\hat{\mathbf{i}} + \frac{y}{r}\frac{\partial f}{\partial r}\hat{\mathbf{j}} + \frac{z}{r}\frac{\partial f}{\partial r}\hat{\mathbf{k}} = \frac{1}{r}\frac{\partial f}{\partial r}[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}]$$

or $$\vec{\nabla} f(r) = \frac{\partial f}{\partial r}\left[\frac{\vec{\mathbf{r}}}{r}\right] = \frac{\partial f}{\partial r}\hat{\mathbf{r}} \qquad \text{(v)}$$

where $\hat{\mathbf{r}}$ is the unit vector along $\vec{\mathbf{r}}$.

In physics, we encounter many scalar functions of *r* and hence Eq. (v) is an important result.

To determine the gradient of the scalar field $\phi(x,y,z) = \dfrac{k}{(x^2+y^2+z^2)}$, we write it as $\phi = \dfrac{k}{r^2} = f(r)$. Using Eq. (v), we can write:

$$\vec{\nabla}\phi = \vec{\nabla} f(r) = \frac{\partial}{\partial r}\left(\frac{k}{r^2}\right)\hat{\mathbf{r}} = -\left(\frac{2k}{r^3}\right)\hat{\mathbf{r}} = -\frac{2k}{r^3}\hat{\mathbf{r}}$$

## *SAQ 1* – Gradient of a scalar field

a) Calculate the gradient of the following scalar fields:

(i) $\phi(x,y) = \ln(x^2+y^2)$

(ii) $\phi(x,y,z) = xy + yz + zx$

b) The height of a hill is given by $z = 50 - x^2y^2$. Calculate the maximum rate of change (also called the steepest ascent) in the height of the hill at the point (1, 2). What is its direction?

The gradient of a scalar field is important in physics, because we use it to express the relationship between a conservative force and a scalar potential. A conservative force field is related to a scalar potential *V* as

$$\vec{\mathbf{F}} = -\vec{\nabla} V \qquad (1.12)$$

You have studied about conservative forces in your first semester course BPHCT-131 entitled “Mechanics”. You know that forces like the electrostatic force, the gravitational force and the spring force are conservative forces. Each of these forces can be related to a corresponding potential function. The negative sign in this equation is important. **It says that the force is in the direction of the negative gradient of the potential**. This tells us, for example, that the gravitational force is directed from the point of higher gravitational potential to lower gravitation potential, like from the top of a building to the ground. In SAQ 2 you will solve problems on some applications of the gradient of a scalar field.

*SAQ 2* – **Applications of the gradient**

a) Determine the unit vector normal to the curve $x^2 + 4y^2 = 1$ at the point (1, − 1).

b) Obtain a unit vector normal to the surface $x + 2y - z + 5 = 0$ at any point.

So far we have defined the gradient of a scalar field which tells us the direction of maximum space rate of change of the scalar field. We can determine the rate of change of the scalar field **in any direction** by defining the **directional derivative of the scalar field** *f* in terms of the gradient of the scalar field. This is what we do now.

## 1.3.2 The Directional Derivative of a Scalar Field

Let us find the rate of change of *f* (*x*, *y*, *z*) with distance *s*, at a given point $P\ (x_0, y_0, z_0)$ in the field and in a given direction (Fig. 1.6). This is called the **directional derivative** (*df* / *ds*) of the function with distance *s*. Let $\hat{\mathbf{s}}$ be the unit vector in the direction in which we want to find the rate of change of the scalar field. Let the unit vector $\hat{\mathbf{s}}$ be defined by

$$\hat{\mathbf{s}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} \qquad (1.13a)$$

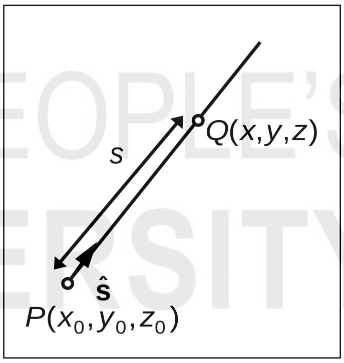



**Fig. 1.6**: **Directional derivative.**

Let us start from the point *P* and go a distance $s(\geq 0)$ along the direction of the unit vector $\hat{\mathbf{s}}$ to reach the point *Q* (*x*, *y*, *z*). We can write the displacement vector $\overrightarrow{\mathbf{PQ}} = s\,\hat{\mathbf{s}}$ as,

$$\vec{\mathbf{s}} = (x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - z_0)\,\hat{\mathbf{k}} \qquad (1.13b)$$

We can also write $\vec{\mathbf{s}}$ in terms of the unit vector $\hat{\mathbf{s}}$ as

$$\vec{\mathbf{s}} = sa\,\hat{\mathbf{i}} + sb\,\hat{\mathbf{j}} + sc\,\hat{\mathbf{k}} \qquad (1.13c)$$

Comparing Eqs. (1.13b and c), we get

$$x - x_0 = sa; \qquad y - y_0 = sb; \qquad z - z_0 = sc$$

or

$$x = x_0 + sa; \qquad y = y_0 + sb; \qquad z = z_0 + sc \qquad (1.13d)$$

What do Eqs. (1.13d) tell us? You can see that the variables $x = x(s)$, $y = y(s)$ and $z = z(s)$ are all functions of a single variable *s*. Eqs. (1.13d) are called the **parametric equations** of the line *PQ* which passes through the points $(x_0, y_0, z_0)$ and $(x, y, z)$ with the **parameter** as $s$. Remember that here $s$ is the distance between the points $P$ and $Q$ measured along the line $PQ$. So if we substitute $x$, $y$ and z in terms of $s$, the function $f(x, y, z)$ becomes a function of a single variable, the **parameter** $s$.

We can now write the directional derivative $(df/ds)$ using the following chain rule from calculus for the derivative of a function $f(x, y, z)$ where $x$, $y$ and $z$ are functions of $s$:

In Eqs. (1.14a to c), note that since $f$ is a function of $x$, $y$ and $z$, the derivative of $f$ with respect to $x$ is a partial derivative. Since $x$ is a function of only $s$, the derivative of $x$ with respect to $s$ is an ordinary derivative.

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \qquad (1.14a)$$

From Eq. (1.13d), we have

$$\frac{dx}{ds} = a; \qquad \frac{dy}{ds} = b; \qquad \frac{dz}{ds} = c \qquad (1.14b)$$

So Eq. (1.14a), which gives the expression for the directional derivative of $f$, now becomes:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b + \frac{\partial f}{\partial z}c \qquad (1.14c)$$

We now make use of the expression for the gradient of a scalar field given by Eq. (1.11a):

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Then we take the scalar or dot product of the vectors $\vec{\nabla} f$ and $\hat{\mathbf{s}}$ and get:

$$\vec{\nabla} f \,.\, \hat{\mathbf{s}} = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) . \, (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}})$$

$$= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b + \frac{\partial f}{\partial z}c \qquad (1.14d)$$

which is the same as the right hand side of Eq. (1.14c). So we can write

$$\frac{df}{ds} = \vec{\nabla} f \,.\, \hat{\mathbf{s}} \qquad (1.15)$$

In other words, **the directional derivative of a scalar field at a point <u>in a given direction</u> is the scalar product of the gradient of the scalar field at that point and the unit vector along the given direction**.

To reconcile Eqs. (1.15) and (1.16), note that $\frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$ is the unit vector along $\vec{\mathbf{a}}$.

Note that if the direction is specified by an arbitrary vector $\vec{\mathbf{a}}$ which is not the unit vector in that direction, the expression for the directional derivative becomes

$$\frac{df}{ds} = \vec{\nabla} f \,.\, \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \qquad (1.16)$$

Let us now work out an example on calculating the directional derivative of a scalar function using Eq. (1.15 or 1.16).

## EXAMPLE 1.3: DIRECTIONAL DERIVATIVE

Determine the directional derivative of the scalar field $\phi = xy^2z^3$ in the direction $(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ at the point (3, 1, – 1).

**SOLUTION** ■ To determine the directional derivative we must use either Eq. (1.15) or Eq. (1.16). Note that the direction is not given in terms of a unit vector (check for yourself that the vector $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is not a unit vector). Hence, we will use Eq. (1.16). In the first step we determine the gradient of $\phi$.

We can find the $\vec{\nabla}\phi$ using Eq. (1.11a) as follows:

$$\vec{\nabla}\phi = \frac{\partial(xy^2z^3)}{\partial x}\hat{\mathbf{i}} + \frac{\partial(xy^2z^3)}{\partial y}\hat{\mathbf{j}} + \frac{\partial(xy^2z^3)}{\partial z}\hat{\mathbf{k}}$$

$$= y^2z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + 3xy^2z^2\,\hat{\mathbf{k}}$$

Then from Eq. (1.16), the directional derivative in the given direction is

$$\frac{d\phi}{ds} = (y^2z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + 3xy^2z^2\,\hat{\mathbf{k}}).\frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{3}$$

$$= \frac{1}{3}y^2z^3 - \frac{4}{3}xyz^3 + 2xy^2z^2$$

At the point (3, 1, – 1), $x = 3$, $y = 1$, $z = -1$ and hence, $\frac{d\phi}{ds} = \frac{29}{3}$ (read the margin remark).

Substituting $x = 3$, $y = 1$ and $z = -1$ in Eq. (i) we can write,

$$\frac{d\phi}{ds} = \frac{1}{3}.1^2.(-1)^3 - \frac{4}{3}(3).(1).(-1)^3 + 2.(3).(1)^2.(-1)^2 = -\frac{1}{3} + 4 + 6 = \frac{29}{3}$$

You may now like to work out the following SAQ.

### SAQ 3 - Directional derivative

Obtain the directional derivative of the scalar field $V = x^2 + \cos y - xz$ at the point $(2, \pi/6, -1)$ in the direction $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$.

Let us now summarise what we have learnt in this unit.

## 1.4 SUMMARY

| Concept | Description |
|---|---|
| *Scalar field* | ■ A function $f$ which associates a unique scalar with each point in a given region is called a **scalar field function** or a **scalar field**. The curves $f(x, y)$ = constant or the surfaces $f(x, y, z)$ = constant are called the **contour curves** and **contour surfaces**, respectively. |

***Gradient of a scalar field***

- The **gradient** of a scalar field is defined as

$$\vec{\nabla} f = \hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

It is a vector such that the change *df* in *f* for an arbitrarily small change $d\vec{\mathbf{r}}$ is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(\vec{\nabla} f\right).d\vec{\mathbf{r}}$$

***Properties of the gradient***

- The gradient of a scalar field is **normal** to the curves of constant value or surfaces of constant value of the scalar field.

The gradient of a scalar field at a point gives the direction of **maximum rate of change of the scalar field**.

***Directional derivative of a scalar field.***

- The **directional derivative** of a scalar field *f* in the **direction** specified by the unit vector $\hat{\mathbf{s}}\left(\equiv \frac{\vec{\mathbf{s}}}{s}\right)$ is

$$\frac{\partial f}{\partial s} = \hat{\mathbf{s}}.\vec{\nabla} f = \frac{\vec{\mathbf{s}}}{s}\,.\,\vec{\nabla} f$$

It is the projection of $\vec{\nabla} f$ on $\hat{\mathbf{s}}$.

## 1.5 TERMINAL QUESTIONS

1. Determine the gradient of the following scalar fields:

   a) $\phi(x,y,z) = e^{xyz}$.

   b) $f = y\,\sin z - xy$ at the point $(1, 2, \pi/6)$

   In which direction is the scalar field of part (b) decreasing most rapidly?

2. Determine the gradient for the temperature field given by $T(x, y, z) = 2x^2 + xyz + y^2 + 273$ at the point (– 1, 2, 1). What is the direction of heat flow?

3. Determine a unit vector normal to the scalar field $F = e^x \cos y$.

4. If $\vec{\mathbf{A}} = 2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ and $\phi = 2x^2yz^3$, determine $(\vec{\mathbf{A}}\,.\,\vec{\nabla})\phi$.

5. Determine the unit vector normal to the surface $(x-2)^2 + (y+1)^2 + z^2 = 9$ at the point (2, 1, 4).

6. Calculate the gradient of the scalar field $r = (x^2 + y^2 + z^2)^{1/2}$.

7. Given a potential energy function $V(r) = \alpha r^2$, where $r = (x^2 + y^2 + z^2)^{1/2}$ and $\alpha$ a constant, calculate the force field.

8. Determine the force field in three dimensions for a potential energy function given by:

$$V = \frac{e^{-\alpha r}}{r}$$

9. Determine the directional derivative of the scalar field $\phi = x^2 y + xz$ at the point (1, 2, 1) in the direction of the vector $\vec{\mathbf{C}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

10. Obtain the directional derivative of the scalar field $\phi = x^2 + y^2 + z^2$ at the point (3, 0, 1) in the direction $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

# 1.6 SOLUTIONS AND ANSWERS

## Self-Assessment Questions

1. a) i) Since $\phi(x, y)$ is two-dimensional scalar field, we use Eq. (1.11b). For $\phi = \ln(x^2 + y^2)$, we have

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}[\ln(x^2 + y^2)]\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}[\ln(x^2 + y^2)]\,\hat{\mathbf{j}}$$

$$= \frac{2x}{x^2 + y^2}\,\hat{\mathbf{i}} + \frac{2y}{x^2 + y^2}\,\hat{\mathbf{j}}$$

ii) Using Eq. (1.11a) with $\phi = xy + yz + zx$, we get,

$$\phi = \frac{\partial}{\partial x}(xy + yz + zx)\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}(xy + yz + zx)\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}(xy + yz + zx)\,\hat{\mathbf{k}}$$

$$= (y + z)\,\hat{\mathbf{i}} + (x + z)\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$$

b) The magnitude of the maximum rate of change is the magnitude of the gradient, as given by Eq. (1.10). Using Eq. (1.11b),

$$\vec{\nabla} z = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)(50 - x^2 y^2) = -\,2x\,y^2\,\hat{\mathbf{i}} - 2yx^2\,\hat{\mathbf{j}}$$

At the point (1, 2), it is

$$\vec{\nabla} z\Big|_{z=(1,\,2)} = -\,8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$$

$$\therefore \quad |\vec{\nabla} z| = |-8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}| = 4\sqrt{5}$$

The direction of maximum rate of change of the height, which is the steepest ascent is along the gradient of *z* at (1, 2) and is given by

$$\vec{\nabla} z\Big|_{z=(1,\,2)} = -\,8\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$$

2. a) The gradient of a scalar field *f* (*x*,*y*) is the normal to the curve $f(x, y) = c$, where *c* is a constant. So we can define the field $f(x, y)$ as

$$f(x, y) = x^2 + 4y^2 - 1$$

We determine $\vec{\nabla} f(x, y)$ which will be normal to the curve $f(x, y) = c$.

Using Eq. (1.11b), we get

$$\vec{\nabla} f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)(x^2 + 4y^2 - 1) = (2x\,\hat{\mathbf{i}} + 8y\,\hat{\mathbf{j}})$$

At the point (1, – 1), the value of the gradient is

$$\vec{\nabla} f(1, -1) = 2\,\hat{\mathbf{i}} - 8\,\hat{\mathbf{j}}$$

This vector is normal to the curve $x^2 + 4y^2 = 1$ at (1, – 1). The unit vector normal to the curve at the point (1, –1) is

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f(1,-1)}{|\vec{\nabla} f(1,-1)|} = \frac{2\,\hat{\mathbf{i}} - 8\,\hat{\mathbf{j}}}{\sqrt{2^2 + 8^2}} = \frac{\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{17}}$$

b) For this question, as in SAQ 2(a) we first determine the gradient to the surface $f(x, y, z) = x + 2y - z + 5$ using Eq. (1.11a):

$$\vec{\nabla} f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x + 2y - z + 5) = (\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \qquad \text{(i)}$$

Note that $\vec{\nabla} f$ which is normal to the surface $f(x, y, z)$, given by Eq. (i) is a constant vector. The unit vector normal at any point on this surface is then

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}})}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{1}{\sqrt{6}}\,(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

3. We use Eq. (1.15) with the unit vector $\hat{\mathbf{s}} = \frac{1}{\sqrt{3}}\,(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ and

$f = V(x, y, z) = x^2 + \cos y - xz.$

We first determine the gradient $\vec{\nabla} V$:

$$\vec{\nabla} V = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + \cos y - xz)$$

$$= (2x - z)\,\hat{\mathbf{i}} - \sin y\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$$

The gradient at the point $(2, \pi/6, -1)$ is,

$$\vec{\nabla} V\,(2, \pi/6, -1) = 5\,\hat{\mathbf{i}} - \sin\pi/6\,\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$= 5\,\hat{\mathbf{i}} - \frac{1}{2}\,\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

From Eq. (1.15), the directional derivative at that point, along $\frac{1}{\sqrt{3}}\,(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ is:

$$\frac{dV}{ds} = \left(5\,\hat{\mathbf{i}} - \frac{1}{2}\,\hat{\mathbf{j}} - 2\hat{\mathbf{k}}\right).\left(\frac{1}{\sqrt{3}}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(5 - \frac{1}{2} + 2\right) = \frac{13}{2\sqrt{3}}$$

## Terminal Questions

1. a) We use Eq. (1.11a) with $\phi = e^{xyz}$ and get

$$\vec{\nabla}\phi = \frac{\partial}{\partial x} e^{xyz}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y} e^{xyz}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z} e^{xyz}\hat{\mathbf{k}}$$

$$= yz\,e^{xyz}\,\hat{\mathbf{i}} + xz\,e^{xyz}\,\hat{\mathbf{j}} + xy\,e^{xyz}\hat{\mathbf{k}}$$

b) We first find the gradient at the point $(1, 2, \pi/6)$:

$$\vec{\nabla} f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(y \sin z - xy)$$

$$= -y\,\hat{\mathbf{i}} + (\sin z - x)\,\hat{\mathbf{j}} + y\,\cos z\,\hat{\mathbf{k}}$$

$$\vec{\nabla} f\left(1, 2, \frac{\pi}{6}\right) = -2\,\hat{\mathbf{i}} + \left(\sin\frac{\pi}{6} - 1\right)\hat{\mathbf{j}} + 2\,\cos\frac{\pi}{6}\,\hat{\mathbf{k}}$$

$$= -2\,\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \sqrt{3}\,\hat{\mathbf{k}}$$

The direction in which $f$ decreases most rapidly is **opposite** to the gradient vector at (1, 2, $\pi$ / 6).

This direction is then along the vector $\vec{\mathbf{a}}$

$$\vec{\mathbf{a}} = -\,\vec{\nabla} f\left(1, 2, \frac{\pi}{6}\right) = 2\,\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} - \sqrt{3}\,\hat{\mathbf{k}}$$

2. Using Eq. (1.11a) with $f = T$, we get

$$\vec{\nabla} T = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(2x^2 + xyz + y^2 + 273)$$

$$= (4x + yz)\,\hat{\mathbf{i}} + (xz + 2y)\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$$

$$\vec{\nabla} T\,(-1, 2, 1) = (-4 + 2)\,\hat{\mathbf{i}} + (-1 + 4)\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$$

$$= -2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$$

Heat would flow from higher to lower temperature regions. Hence, it would flow along the direction in which the temperature decreases most rapidly at $(-1, 2, 1)$. Therefore, the direction of heat flow is along $-\vec{\nabla} T$. Therefore, it is along $(2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})$.

3. The vector normal to the scalar field $F$ is $\vec{\nabla} F$. Using Eq. (1.11b) with $f = F$, we get

$$\vec{\nabla} F = \hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y$$

The unit normal vector is then:

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} F}{|\vec{\nabla} F|} = \frac{\hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y}{\sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}}$$

$$= \frac{\hat{\mathbf{i}}\,e^x \cos y - \hat{\mathbf{j}}\,e^x \sin y}{e^x} = \hat{\mathbf{i}}\cos y - \hat{\mathbf{j}}\sin y$$

4. We first determine the scalar product of $\vec{\mathbf{A}}$ and the del operator $\vec{\nabla}$ :

$$(\vec{\mathbf{A}}.\vec{\nabla}) = (2yz\,\hat{\mathbf{i}} - x^2 y\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}).\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$$

$$= \left(2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)$$

Note that this is now a differential operator which can act on the field $\phi$:

$$(\vec{\mathbf{A}}\,.\,\vec{\nabla})\phi = \left(2yz\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(2x^2 y z^3)$$

$$= 2yz\frac{\partial}{\partial x}(2x^2 y z^3) - x^2 y\frac{\partial}{\partial y}(2x^2 y z^3) + xz^2\frac{\partial}{\partial z}(2x^2 y z^3)$$

$$= 8xy^2 z^4 - 2x^4 y z^3 + 6x^3 y z^4$$

5. Using Eq. (1.11a), we first find the gradient of the scalar function $f(x, y, z) = (x-2)^2 + (y+1)^2 + z^2 - 9$:

$$\vec{\nabla} f = \left[ \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right] \left[ (x-2)^2 + (y+1)^2 + z^2 - 9 \right]$$

$$= 2(x-2)\,\hat{\mathbf{i}} + 2(y+1)\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$$

At the point (2, 1, 4), the gradient vector is $\vec{\nabla} f\,(2, 1, 4) = 4\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}}$

The unit vector normal to the surface is,

$$\hat{\mathbf{n}} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{4\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}}}{\sqrt{80}} = \frac{1}{\sqrt{5}}(\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

6. Using Eq. (1.11a), we can write

$$\vec{\nabla} r = \frac{\partial r}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{k}} \qquad \text{(i)}$$

Let us determine each component of Eq. (i):

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r} \qquad \text{(ii)}$$

Similarly, the other two partial derivatives $\frac{\partial r}{\partial y}$ and $\frac{\partial r}{\partial z}$ are:

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2 + y^2 + z^2)^{1/2}} = \frac{y}{r} \qquad \text{(iii)}$$

and $$\frac{\partial r}{\partial z} = \frac{z}{(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{r} \qquad \text{(iv)}$$

Therefore, $$\vec{\nabla} r = \frac{x}{r}\,\hat{\mathbf{i}} + \frac{y}{r}\,\hat{\mathbf{j}} + \frac{z}{r}\,\hat{\mathbf{k}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{r} = \frac{1}{r}\,\vec{\mathbf{r}} \qquad \text{(v)}$$

where $\vec{\mathbf{r}}$ is the position vector for the point $(x, y, z)$. Can you identify what the right hand side of Eq. (v) represents? You can see that $\vec{\mathbf{r}}/r$ is just the unit vector $\hat{\mathbf{r}}$ in the direction of $\vec{\mathbf{r}}$. This brings us to a result that you will use often in physics:

$$\vec{\nabla} r = \hat{\mathbf{r}} \qquad \text{(vi)}$$

7. We use Eq. (1.12) with $V = \alpha r^2$ to determine the force field $\vec{\mathbf{F}}$:

$$\vec{\mathbf{F}} = -\vec{\nabla}(\alpha r^2)$$

To evaluate $\vec{\nabla} V$, we use the result derived for the gradient of an arbitrary scalar function $f(r)$ in Example 1.2 which is

$$\vec{\nabla} f(r) = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} \qquad \text{(i)}$$

With $f(r) = \alpha r^2$ we get

$$\vec{\mathbf{F}} = -\frac{1}{r}\,\frac{d}{dr}(\alpha r^2)\,\hat{\mathbf{r}} = -2\alpha r\,\hat{\mathbf{r}}$$

8. The force field $\vec{\mathbf{F}}$ is given by Eq. (1.12).

   Using Eq. (v) of Example 1.2 with $f(r) = V = \dfrac{e^{-\alpha r}}{r}$ we get,

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{\mathbf{r}}$$

   Since $\vec{\mathbf{F}} = -\vec{\nabla} V,$ we can write

$$\vec{\mathbf{F}} = -\left(\frac{\partial V}{\partial r}\right)\hat{\mathbf{r}} = -\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r}\right)\hat{\mathbf{r}}$$

$$= -\left[-\frac{\alpha e^{-\alpha r}}{r} - \frac{e^{-\alpha r}}{r^2}\right]\hat{\mathbf{r}} = \frac{e^{-\alpha r}}{r}\left[\alpha + \frac{1}{r}\right]\hat{\mathbf{r}}$$

9. We use Eq. (1.16) with $\vec{\mathbf{a}} = \vec{\mathbf{C}}$. The unit vector along $\vec{\mathbf{C}}$ is

$$\hat{\mathbf{C}} = \frac{\vec{\mathbf{C}}}{|\vec{\mathbf{C}}|} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{4+9+16}} = \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{29}}$$

   Using Eq. (1.11a) with $f = \phi$, we have

$$\vec{\nabla}\phi = \hat{\mathbf{i}}\,(2xy + z) + \hat{\mathbf{j}}\,(x^2) + \hat{\mathbf{k}}\,(x)$$

   At the point (1, 2, 1), the value of the gradient is

$$\vec{\nabla}\phi(1,\ 2, 1) = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

   Then the directional derivative is

$$\frac{d\phi}{ds} = \hat{\mathbf{C}}.\,(\vec{\nabla}\phi) = \left(\frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{29}}\right).(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = \frac{10}{\sqrt{29}} - \frac{3}{\sqrt{29}} + \frac{4}{\sqrt{29}} = \frac{11}{\sqrt{29}}$$

10. We first determine the gradient of $\phi(x, y, z)$ at the point (3,0,1). Using Eq. (1.11a) with $f = \phi = x^2 + y^2 + z^2,$ we get

$$\vec{\nabla}\phi = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$$

   At (3, 0, 1), the gradient of $\phi(x, y, z)$ is

$$\vec{\nabla}\phi(3,\ 0,\ 1) = 6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{k}}$$

   To find the directional derivative at the point (3, 0, 1) in the direction of $\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}},$ we use Eq. (1.16) with $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}.$ Thus

$$\frac{d\phi}{ds} = (6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{k}})\,.\,\frac{(\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

# APPENDIX PARTIAL DERIVATIVES

By definition, the partial derivative of a function $f(x, y, z)$ with respect to $x$ is

$$\frac{\partial f(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \qquad (1)$$

The function $\partial f / \partial x$ is obtained by differentiating the function $f(x, y, z)$ with respect to $x$ as in ordinary calculus, treating ***other variables*** $y$, $z$ **as constants**. You can similarly determine $\partial f / \partial y$ and $\partial f / \partial z$. The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x, y, z)$ give us, respectively, the rate of change of $f$ in the directions of $x, y$ or $z$-axes. Thus, $\frac{\partial f}{\partial x}$ gives the rate of change of $f$ with respect to $x$ at a given point in space.

Let us explain how to calculate the partial derivatives of a function $f(x, y, z)$ with respect to $x$, $y$ and $z$ ***holding other variables to be constant***.

For example, let $f(x, y, z) = 2x^2 y z^3$. Then

$$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2)\right](2yz^3) = 4xyz^3 \text{ since } y \text{ and } z \text{ are treated as constants.}$$

Similarly, for the partial derivative with respect to any other variable, we keep the remaining variables as constant. Thus,

$$\frac{\partial f}{\partial y} = \left[\frac{\partial}{\partial y}(y)\right](2x^2 z^3) = 2x^2 z^3 \quad \text{and} \quad \frac{\partial f}{\partial z} = \left[\frac{\partial}{\partial z}(z^3)\right](2x^2 y) = 6x^2 yz^2$$

You may quickly work out a couple of exercises to learn how to calculate partial derivatives of a function.

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a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y^3 + \exp(x^2 y)$.

b) For the function $u(x, y, z) = 2x + yz - xy$, evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

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The solutions are as follows:

a) $$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2 y^3) + \frac{\partial}{\partial x}\exp(x^2 y)\right] = \left[\frac{\partial}{\partial x}(x^2)\right] y^3 + 2x\exp(x^2 y)$$

$$= 2xy^3 + 2x\exp(x^2 y)$$

$$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial y}(x^2 y^3) + \frac{\partial}{\partial x}\exp(x^2 y)\right] = 3y^2 x^2 + \exp(x^2 y)$$

b) $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x + yz - xy) = 2 - y \text{ since } y \text{ and } z \text{ are treated as constants.}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x + yz - xy) = z - x$ and $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(2x + yz - xy) = y$