10.1 INTRODUCTION

In Unit 9 we mentioned that the conformal mapping and the harmonic functions have wide range of applications in physics and engineering. They play a vital role in solving a large number of physical problems in electrostatics, hydrodynamics, aerodynamics, heat transfer, vibrations and acoustics. All these physical problems can be formulated mathematically in terms of Laplace equation. Considering the Dirichlet’s problem of the steady-state temperature distribution in a thin, homogeneous semi-infinite solid plate in the \( xy \)-plane, it was illustrated how the harmonic functions are used in finding the solutions of such boundary value problems in a simply connected domain. In this unit, we shall discuss the physical problems of heat conduction, fluid flow and electrostatic potential in a region free of charges in two-dimension.

In Sec. 10.2, we begin with the problem of fluid flow. The existence of complex potential, velocity potential and stream functions is proved for an irrotational and incompressible fluid flow in a simply connected domain. It has been shown that the velocity potential is a harmonic function and the flow of the fluid is always along the streamlines. In the electrostatics, the notion of electric field intensity and its connection with the electrostatic potential is also established in this section.

The problems of heat conduction are discussed in Sec.10.3. The expression for the steady-state temperature function \( T(x, y) \) satisfying the Laplace equation in a certain region \( D \) of the \( z \)-plane is determined. The considered domain is mainly the upper half-plane and one of its subset in the form of a quadrant. Moreover, \( T(x, y) \) also satisfies certain boundary conditions on the boundary \( C \) of the region \( D \). The main crux in solving these problems lies in transforming the given problem in \( xy \)-plane onto the \( uv \)-plane through a transformation \( w = f(z) \) that is analytic in \( D \) and conformal along the boundary \( C \). The solution of the problem in \( uv \)-plane is again transformed into the solution of the original problem in \( xy \)-plane by Theorem 6 of Unit 9. Further, Theorem 7 of Unit 9 ensures that the boundary conditions are preserved.
Objectives
After studying this unit, you should be able to

• state the properties of an irrotational and incompressible fluid flow in a simply connected domain;
• obtain the relation between the velocity vector, complex potential, velocity potential and stream function;
• determine the streamlines and equipotentials for a fluid flow;
• determine the electrostatic potential in a given domain;
• show that the steady-state temperature \( T(x, y) \) in a thin, homogeneous semi-infinite plate (solid) in the \( xy \)-plane satisfies the Laplace equation;
• determine the isotherms and flow of lines within a solid;
• obtain the expressions for the steady temperatures \( T(x, y) \) in a thin semi-infinite plate, whose faces are insulated, under the prescribed boundary conditions; and
• obtain the expressions for the steady temperatures \( T(x, y) \) in a thin plate in the form of a quadrant, whose one edge is insulated and other is kept at fixed temperature, under the prescribed boundary conditions.

10.2 APPLICATIONS OF HARMONIC FUNCTIONS

This section introduces you to the applications of harmonic functions in two-dimensional fluid flow and electrostatic potential. We first take up the problem of fluid flow.

Fluid Flow
Fluid flow is two-dimensional if the fluid (water, air etc.) moves in planes parallel to the \( xy \)-planes and the motion and physical properties of the fluid in each plane be the same as it is in the \( xy \)-plane. Firstly, we introduce you to some basic terms related to the fluid flow in hydrodynamics.

Let \( V \) denote the velocity of a particle of the fluid at any point \((x, y)\) where the vector function \( V \) is represented as a complex number

\[
V = p + iq
\]

\( p \) and \( q \) are the \( x \) and \( y \) components of \( V \). The fluid is said to be incompressible (constant density) if \( \text{div} \, V = 0 \), i.e. \( p_x = -q_y \) and the flow is said to be irrotational if \( q_x = p_y \) (that is, \( \text{curl} \, V = 0 \)).

The divergence of the vector field

\[
f(x, y) = P(x, y)i + Q(x, y)j
\]

is a measure of the rate of change of the density of the fluid at a point. It is denoted by \( \text{div} \, f \) (or \( \nabla \cdot f \)) and is a scalar function given as

\[
\nabla \cdot f = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}
\]

The Curl of the vector field

\[
f(x, y) = P(x, y)i + Q(x, y)j
\]

measures the rotation in a fluid flow. It is denoted by \( \text{curl} \, f \) (or \( \nabla \times f \)) and is a vector function given as

\[
\nabla \times f = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k
\]

Hence \( p \) and \( q \) are the \( x \) and \( y \) components of \( V \). The fluid is said to be incompressible (constant density) if \( \text{div} \, V = 0 \), i.e. \( p_x = -q_y \) and the flow is said to be irrotational if \( q_x = p_y \) (that is, \( \text{curl} \, V = 0 \)).

Let the fluid flow under the following assumptions.

Assumptions:

i) The velocity \( V \) varies with only \( x \) and \( y \) coordinates so that the flow of fluid is two-dimensional.

ii) The velocity function \( V(x, y) \) and its partial derivatives of first order are continuous at each point interior to a region of flow.

iii) The velocity \( V \) does not vary with time (that is, the fluid flow is in a steady-state).
iv) No sources or sink are present in the fluid. That is, fluid cannot leave or enter the region of flow.

v) The fluid is incompressible and free from viscosity.

Consider a simply connected domain \( D \) in which the flow is irrotational. Then the velocity vector \( V(x, y) = p(x, y) + iq(x, y) \) satisfies

\[
p_x = -q_y \quad \text{and} \quad q_x = p_y
\]

throughout \( D \). Consequently, the function \( f(z) = p - iq \) satisfies the Cauchy-Riemann equations in \( D \). Also, \( p \) and \( q \) have continuous first-order partial derivatives in \( D \). By the sufficient condition of differentiability, \( f \) is analytic in \( D \). Since \( D \) is simply connected, \( f \) has an antiderivative function say, \( \Phi \). We thus have, \( \Phi'(z) = f(z) \) for all \( z \in D \). If we write \( \Phi = \phi + i\psi \), then

\[
p - iq = f(z) = \Phi'(z) = \phi_x + i\psi_x = \phi_x - i\phi_y
\]

(because \( \phi \) and \( \psi \) satisfy the Cauchy Riemann equations).

On comparing the real and imaginary parts in Eqn. (1), we get

\[
p = \phi_x \quad \text{and} \quad q = \phi_y,
\]

Since an analytic function has derivatives of all orders, therefore the same is true for the function \( \Phi \). Thus Eqn. (2) gives

\[
p_x = \phi_{xx} \quad \text{and} \quad q_y = \phi_{yy}.
\]

As \( p_x = -q_y \), it follows that \( \phi_{xx} + \phi_{yy} = 0 \) i.e., \( \phi \) satisfies the Laplace equation.

Now, we are ready to define various terms associated with the fluid flow:

- The function \( \phi(x, y) \) is called the velocity potential and it satisfies the Laplace equation throughout \( D \). Hence \( \phi \) is a harmonic function in \( D \).
- The analytic function \( \Phi = \phi + i\psi \) is called the complex potential of the flow. The function \( \psi \) which is called the stream function is the harmonic conjugate of the velocity potential \( \phi \).
- The level curves, \( \phi(x, y) = \text{constant} \) and \( \psi(x, y) = \text{constant} \), of \( \phi \) and \( \psi \) are called the equipotentials and streamlines of the flow, respectively. If the analytic function \( \Phi = \phi + i\psi \) happens to be conformal (that is, \( \Phi' \neq 0 \)), then the level curves of \( \phi \) and \( \psi \) are orthogonal (see Corollary 1, Unit 9).

Since the anti-derivative of a function is unique up to an additive constant, any two stream functions or velocity potentials differ by an additive constant. Let us take up some examples to have better understanding of the various terms defined above.

**Example 1:** Let \( \Phi(z) = \phi(x, y) + i\psi(x, y) \) be the complex potential of an irrotational flow in a simply connected domain \( D \). Show that the velocity vector \( V \) and its speed \( |V| \) are given by \( V(x, y) = \text{grad}\phi(x, y) \) and \( |V| = |\Phi'(z)| \).

**Solution:** Using Eqn. (2), we can write the velocity vector as

\[
V(x, y) = p(x, y) + iq(x, y) = \phi_x(x, y) + i\phi_y(x, y) = \text{grad}\phi(x, y).
\]
By Eqn. (1), \( \Phi'(z) = p - iq = \overline{V} \). Thus the speed of the flow is
\[ |V| = |\overline{V}| = |\Phi'(z)|. \]

***

Using your knowledge of vector calculus you can easily show that the gradient of a function at a point is perpendicular to the level curve of the function at that point (see E1)). Therefore, \( \nabla \Phi \) is always perpendicular to the level curves of \( \Phi \). It then follows from Example 1 that the velocity vector is normal to the equipotentials. Also, if \( \Phi'(z) \neq 0 \), then the streamlines and equipotentials are orthogonal. Consequently, it follows that the velocity vector is tangent to the streamlines. Hence the flow of the fluid is always along the streamlines.

**Example 2:** Find the velocity vector, velocity potential and stream function for the flow whose complex potentials are given by: i) \( \Phi(z) = z \) and

ii) \( \Phi(z) = z^2 \). Also, determine the streamlines of the flow.

**Solution:**

i) Since \( \Phi(z) = z = x + iy \), the velocity potential is \( \phi(x, y) = x \) and
the stream function is \( \psi(x, y) = y \). By Example 1, the velocity vector is
\( V = \nabla \phi = 1 \). The streamlines are \( \psi(x, y) = \text{constant} \), which represents the horizontal lines \( y = \text{constant} \).

ii) In this case, \( \Phi(z) = z^2 = (x^2 - y^2) + 2xyi \), so that
\( \phi(x, y) = x^2 - y^2, \psi(x, y) = 2xy \) and \( V(x, y) = 2(x - iy) \). The streamlines are the hyperbolas \( 2xy = \text{constant} \).

***

We now take up another application of harmonic functions and consider the problem of electrostatics in two-dimension.

**Electrostatic Potential**

A two-dimensional electrostatic force field is produced by a system of charged wires, plates and cylindrical conductors that are perpendicular to the \( z \)-plane. This induces an **electric field intensity** \( E(x, y) \) at a point \((x, y)\), a vector, visualized as the force acting on a unit positive charge placed at that point. The **electrostatic potential** is a scalar function \( V(x, y) \) that satisfies the identity:
\[ E(x, y) = -\nabla V(x, y) = -V_x(x, y) - iV_y(x, y). \]

Thus the \( x \) and \( y \) components of the electric field intensity are \( -V_x(x, y) \) and
\( -V_y(x, y) \), respectively. If we take an extra assumption that there are no charges within a domain, then \( V \) is a harmonic function of two variables \( x \) and \( y \) :
\[ V_{xx}(x, y) + V_{yy}(x, y) = 0. \]

The level curves \( V(x, y) = \text{constant} \) are called **equipotentials**. If \( U \) is a harmonic conjugate of \( V \), then \( V + iU \) is an analytic function and the level curves \( U(x, y) = \text{constant} \) are called the **flux lines**. Thus the electrostatic potential is a harmonic function in domains free of charges and electric field intensity is always along the flux lines. Further, if the analytic function \( V + iU \) is conformal at a point, that is, it has a non-zero derivative at that point, then the level curves of \( U \) and \( V \) are orthogonal and the electric field intensity
Let us consider an example illustrating the method of finding the electrostatic potential in a given domain.

**Example 3:** Find the electrostatic potential $V(x, y)$ in the domain between two infinite coaxial cylinders $r = a$ and $r = b$, which are kept at the potential $V_1$ and $V_2$, respectively.

**Solution:** The analytic function $f(z) = \log z = u(x, y) + iv(x, y)$ maps the annular region between the circles $r = a$ and $r = b$ in the $z$-plane onto the infinite strip $\ln a < u < \ln b$ in the $w$-plane, where $u(x, y) = \ln |z|$ and $v(x, y) = \text{Arg } z$ (see Fig. 1). The electrostatic potential $V(u, v)$ in $w$-plane satisfy the boundary values

$$V(\ln a, v) = V_1 \text{ and } V(\ln b, v) = V_2 \quad \forall v$$

We are looking for a solution $V(u, v)$ that takes on constant values along the vertical lines $u = \ln a$ and $u = \ln b$ with $V(u, v)$ being a function of $u$ alone. That is,

$$V(u, v) = \phi(u) \quad (\ln a < u < \ln b) \quad \forall v.$$ 

Since $V$ satisfies the Laplace equation $V_{uu}(u, v) + V_{vv}(u, v) = 0$, we have

$$\phi''(u) = 0$$

and hence

$$\phi(u) = \alpha u + \beta$$

where $\alpha$ and $\beta$ are constants. The boundary conditions

$$V(\ln a, v) = \phi(\ln a) = V_1 \text{ and } V(\ln b, v) = \phi(\ln b) = V_2$$

gives

$$V(u, v) = V_1 + \frac{V_2 - V_1}{\ln b - \ln a}(u - \ln a).$$

Since $u = \ln |z|$, the electrostatic potential $V(x, y)$ is given by

$$V(x, y) = V_1 + \frac{V_2 - V_1}{\ln b - \ln a}(\ln |z| - \ln a).$$

***
In Fig.1 you can see that equipotential curves $V(x, y) = \text{constant}$ are concentric circles centred at the origin and the lines of flux (dotted lines) are portion of rays emanating from the origin.

You may note here that in Example 3 the expression for the electrostatic potential $V$ has been determined in the $z$-plane by making use of an analytic transformation $w = f(z)$. In fact, here we have used Theorems 6 and 7 of Unit 9. This technique will be further illustrated in the next section when we study the steady temperature distribution.

Before we proceed further, you may try the following exercises.

E1) Show that the gradient vector $\nabla \phi(x_0, y_0)$ at a point $(x_0, y_0)$ is perpendicular to the level curve $\phi(x, y) = c_0$ where $c_0 = \phi(x_0, y_0)$.

E2) Consider an irrotational and incompressible fluid flow with complex potential $\Phi(z) = z + \frac{1}{z}$. Show that the velocity vector at the point $z = e^{i\theta}$, $\theta \in [0, 2\pi]$ on the unit circle is given by $1 - \cos(2\theta) - i\sin(2\theta)$.

E3) Consider the fluid flow in the channel bounded by the hyperbolas $xy = 1$ and $xy = 4$ in the first quadrant whose complex potential is given by $\Phi(z) = z^2$. Find the formula for the speed. Also, find the point on the boundary at which the speed attains a minimum value.

We shall now take up in the next section, the applications of conformal mappings to the steady-state heat conduction problems in two-dimension.

10.3 APPLICATIONS OF CONFORMAL MAPPING

Let us start with the derivation of the equation governing the steady-state temperature distribution $T(x, y)$ in a thin, homogeneous semi-infinite plate (solid) in the $xy$-plane. We shall show that the temperature function $T(x, y)$ satisfies the Laplace equation at each interior point of the solid.

The important concept in the steady temperature is the notion of flux, which is used in many physical phenomena. In the mathematical theory of heat conduction, the flux across a surface within a solid body at a point on that surface is defined as the quantity of heat flowing in a specified direction normal to the surface per unit time per unit area at that point. It is governed by the Fourier’s law:

$$\Phi = -K \frac{dT}{dN}$$

where $\Phi$ is the flux, $dT/dN$ is the normal derivative of the temperature $T$ at a point on the surface and the constant $K > 0$ denotes the thermal conductivity of the material of the solid (which is assumed to be homogeneous). Note that

$$\frac{dT}{dN} = (\text{grad } T).N$$  (3)
where $N$ is the unit vector normal to the surface. Here, we make the following remark:

**Remark 1:** The total flux across a line segment of length $L$ is given by $-K((\text{grad} \, T).N) \, L$ where $N$ is the unit vector perpendicular to the line segment.

Further, consider the following assumptions:

**Assumptions:**

i) The points in the solid are assigned rectangular coordinates in three-dimensional space.

ii) The temperature $T$ varies with only $x$ and $y$ coordinates so that the flow of heat is two-dimensional. Moreover, the temperature function $T(x,y)$ and its partial derivatives of the first and the second order are continuous at each point interior to that solid.

iii) The temperature $T$ does not vary with time (that is, the flow is in a steady-state).

iv) No heat sink or sources are present in the solid.

With the above assumptions, we now move on to the derivation of the equation.

Consider an element in the interior to the solid in the shape of a rectangular prism of unit height perpendicular to the $xy$-plane with base $\Delta x$ by $\Delta y$ in the plane (see Fig. 2).

**Fig. 2**

Let $\Phi_1$ denotes the net rate of heat loss from the element through left and right faces. Then

$$\Phi_1 = \Phi_R - \Phi_L$$

where $\Phi_L$ and $\Phi_R$ are the flux across the left-hand and right-hand faces of the element (of length $\Delta y$), respectively. By the Fourier’s law, Eqn.(3) and Remark 1, it follows that

$$\Phi_L = -K(T_s(x,y)i + T_s(x,y)j)(i+0j)\Delta y = -KT_s(x,y)\Delta y$$
Applications of Analytic Functions

and

\[ \Phi = -K(T_x(x + \Delta x, y)i + T_y(x + \Delta x, y)j) \Delta y = -K T_x(x + \Delta x, y) \Delta y \]

where \( i \) and \( j \) are the unit vectors along \( x \) and \( y \) axis, respectively. Thus

\[ \Phi_1 = \Phi - \Phi_2 = -K(T_x(x + \Delta x, y) - T_x(x, y)) \Delta y \]

\[ = -K \left( \frac{T_x(x + \Delta x, y) - T_x(x, y)}{\Delta x} \right) \Delta x \Delta y. \]

If \( \Delta x \) is very small, then \( \Phi_1 = -K T_{xx}(x, y) \Delta x \Delta y \).

Similarly, if \( \Phi_2 \) denotes the net rate of heat loss from the element through the upper and the lower faces perpendicular to the \( xy \)-plane, then a similar analysis shows that \( \Phi_2 = -K T_{yy}(x, y) \Delta x \Delta y \). Since the temperatures within the elements are steady and the heat enters or leaves the element only through the four faces, we must have \( \Phi_1 + \Phi_2 = 0 \) and hence,

\[ T_{xx}(x, y) + T_{yy}(x, y) = 0. \]

Thus the temperature function satisfies the Laplace equation at each interior point of the solid. Moreover, Assumption ii) implies that \( T(x, y) \) is a harmonic function of \( x \) and \( y \) in the interior of the solid body. The surfaces \( T(x, y) = \) constant are called the isotherms within the solid. The isotherms can be interpreted as curves in the \( xy \)-plane. In this case, \( T(x, y) \) can be considered as the temperature at a point \( (x, y) \) in a thin sheet in that plane with the faces of the sheet thermally insulated. Thus the level curves \( T(x, y) = \) constant, are the isotherms of the function \( T \). If the domain in which \( T(x, y) \) is defined is simply connected, then a conjugate harmonic function \( S(x, y) \) exists and the function \( T(x, y) + iS(x, y) \) is analytic. The level curves \( S(x, y) = \) constant, are called the lines of flow. If the analytic function \( T(x, y) + iS(x, y) \) happens to be conformal, then the level curves of \( S \) and \( T \) are orthogonal (see Corollary 1, Unit 9).

We now determine the expressions for the steady temperature \( T(x, y) \) in a thin semi-infinite plate whose faces are insulated and whose edge \( y = 0 \) is kept at \( T = 1 \) on the line segment \(-1 < x < 1 \) and \( T = 0 \) for \( |x| > 1 \).

**Steady Temperature in a Half Plane:** Let us consider the following Dirichlet boundary value problem for the upper half of \( z \)-plane for finding an expression for the steady temperature \( T(x, y) \) in thin semi-infinite plate:

\[ T_{xx}(x, y) + T_{yy}(x, y) = 0 \quad (-\infty < x < \infty, y > 0) \]

\[ T(x, 0) = \begin{cases} 
1, & |x| < 1; \\
0, & |x| > 1.
\end{cases} \]

\[ |T(x, y)| < M \quad \text{for some } M > 0. \]

To solve this problem, we make use of Theorems 6 and 7 of Unit 9. Firstly, let us write the algorithm for finding the solution of this problem.

**Step 1:** To define an analytic function \( f(z) = u(x, y) + iv(x, y) \) that maps \( D_z = \{ z : \text{Im} z > 0 \} \) in the \( z \)-plane onto a domain \( D_w \) in the \( w \)-plane.
Moreover, \( f \) should be conformal along the boundary \( y = 0 \) except at the points \((\pm 1, 0)\) at which \( f \) is not defined. Conformality of \( f \) is essential on the boundary to apply Theorem 7 of Unit 9.

**Step 2:** To investigate the mapping properties of the boundary points of \( D_z \) and \( D_w \).

**Step 3:** To construct a bounded harmonic function \( h(u, v) \) on the domain \( D_w \) in the \( w \)-plane and discuss its behaviour on the boundary of \( D_w \).

**Step 4:** Apply Theorem 6 of Unit 9 to conclude that the function \( h(u(x, y), v(x, y)) \) is harmonic in the domain \( D_z \). Also, the boundary conditions for the two harmonic functions are the same on the corresponding parts of the boundary by Theorem 7 of Unit 9.

Let us now proceed step-wise to solve the Dirichlet problem given by Eqns.(4), (5) and (6).

**Step 1:** Consider the analytic function
\[
f(z) = \log\left(\frac{z-1}{z+1}\right) = u(x, y) + iv(x, y)
\]
defined in the upper half-plane \( \text{Im} z \geq 0 \) (except at the points \( z = \pm 1 \)), where
\[
u(x, y) = \ln\left(\frac{z-1}{z+1}\right) \quad \text{and} \quad v(x, y) = \text{Arg}\left(\frac{z-1}{z+1}\right).
\]
Here the principal branch of the logarithmic function is used. Clearly, the function \( f \) is conformal along the boundary \( \text{Im} z = 0(z \neq \pm 1) \) since
\[
f'(z) = \frac{2}{z^2-1} \neq 0.
\]
To find the image domain, note that \( f \) is the composition of the functions
\[
g(z) = \frac{z-1}{z+1} \quad \text{and} \quad h(z) = \log z.
\]
The function \( g \) maps the upper half-plane \( y > 0 \) in the \( z \)-plane onto the upper half-plane \( v > 0 \) in the \( w \)-plane. The function \( h \) maps the upper half-plane \( y > 0 \) in the \( z \)-plane onto the horizontal strip \( 0 < v < \pi \) in the \( w \)-plane, where \( z = x+iy \) and \( w = u+iv \) (see E4)). Consequently, \( f \) maps the domain \( D_z \) onto the domain \( D_w = \{w = u+iv : 0 < v < \pi\} \).

**Step 2:** To study the behaviour on the boundary, if \( z = x \in \mathbb{R} \setminus \{\pm 1\} \), then
\[
f(x) = \log\left(\frac{x-1}{x+1}\right).
\]
If \( |x| > 1 \), then the point \((x-1)/(x+1)\) is always positive so that
\[
f(x) = \ln\left(\frac{x-1}{x+1}\right)
\]
lies on the line \( v = 0 \) in \( w \)-plane. In fact, \( f(x) < 0 \) if \( x > 1 \), \( f(x) > 0 \) if \( x < -1 \) and \( f'(x) > 0 \) so that \( f \) is an increasing function of \( x \). If \( |x| < 1 \), then the point \((x-1)/(x+1)\) is always negative which is mapped onto a point...
Applications of Analytic Functions

\[ f(x) = \ln \left( \frac{1-x}{1+x} \right) + i\pi \]

on the line \( v = \pi \) in \( w \)-plane. In fact, \( f'(x) < 0 \) in this case, so that \( f \) is a decreasing function of \( x \). All these cases are shown pictorially in Fig. 3. The points \( A, B, C \) and \( D \) in the \( z \)-plane are mapped onto \( A', B', C' \) and \( D' \) in the \( w \)-plane, respectively.

Fig. 3

Step 3: We need to construct a bounded harmonic function \( h(u,v) \) in the \( w \)-plane such that \( h(u,0) = 0 \) and \( h(u,\pi) = 1 \) so that the desired boundary behaviour of the resulting harmonic function in \( D_z \) is preserved. To achieve this, consider a function

\[ h(u,v) = \frac{1}{\pi} v, \quad (u,v) \in D_w \]

in the \( w \)-plane. Then

- \( h \) is harmonic in \( D_w \) because \( h \) is the imaginary part of the entire function \( w/\pi \).
- \( h \) is bounded since \( 0 \leq h(u,v) \leq 1 \) for all \( (u,v) \in D_w \).
- \( h \) satisfies

\[ h(u,v) = \begin{cases} 
0, & v = 0; \\
1, & v = \pi.
\end{cases} \]

Step 4: Since the function \( f \) is analytic in \( D_z \) that maps \( D_z \) onto \( D_w \) and the function \( h \) is harmonic in \( D_w \), by Theorem 6 of Unit 9, it follows that the temperature function

\[ T(x,y) = h(u(x,y),v(x,y)) = \frac{1}{\pi} \text{Arg} \left( \frac{z-1}{z+1} \right) \]

is a bounded harmonic function in \( D_z \). Also, Theorem 7 of Unit 9 implies that \( T = 0 \) in \( |x| > 1 \) and \( T = 1 \) in \( |x| < 1 \). Since \( 0 \leq T \leq 1 \), \( \tan^{-1} t \) has the range values satisfying \( 0 \leq \tan^{-1} t \leq \pi \), where \( t = 2y/(x^2 + y^2 - 1) \).
In this case, the isotherms \( T(x, y) = c \) \((0 < c < 1)\) are given by the curves

\[
\frac{1}{\pi} \tan^{-1}\left(\frac{2y}{x^2 + y^2 - 1}\right) = c \tag{7}
\]

\[
\Rightarrow \frac{2y}{x^2 + y^2 - 1} = \tan(\pi c)
\]

\[
\Rightarrow 2y\cot(\pi c) = x^2 + y^2 - 1
\]

\[
\Rightarrow 2y\cot(\pi c) = x^2 + y^2 - (\csc^2(\pi c) - \cot^2(\pi c))
\]

\[
\Rightarrow \csc^2(\pi c) = x^2 + (y - \cot(\pi c))^2,
\]

which represents the arcs of the circles passing through the points \((\pm 1, 0)\) and with centres on the \(y\)-axis.

Let us take up the following examples which illustrate the similar Dirichlet problems as above.

**Example 4:** Find an expression for the bounded steady temperature in a plate having the form of a quadrant \(x \geq 0\) and \(y \geq 0\) with its faces insulated and its edges having the temperatures \(T(x, 0) = 0\) and \(T(0, y) = 1\) (see Fig. 4). Also find the associated isotherms.

**Solution:** We follow the similar technique as adopted in the Dirichlet problem above. Consider the function \(f(z) = \log z = u(x, y) + iv(x, y)\) in the domain \(D_z = \{z = x + iy : x > 0, y > 0\}\), where \(u(x, y) = \ln |z|\) and \(v(x, y) = \arg z\). The function \(f\) is analytic and conformal along the boundary as \(f'(z) = 1/z \neq 0\) for all \(z \neq 0\). Also, \(f\) maps \(D_z\) onto the domain \(D_w = \{w = u + iv : 0 < v < \pi/2\}\).

For the behaviour of \(f\) on the boundary, consider \(z = x > 0\), then \(f(z) = \ln x\) lies on the line \(v = 0\) in the \(w\)-plane. In fact, the intervals \([0,1]\) and \([1,\infty]\) on the \(x\)-axis are mapped onto the intervals \([-\infty,0]\) and \([0,\infty]\) on the \(u\)-axis, respectively. Similarly, if \(z = iy, y > 0\), then \(f(z) = \ln y + i\pi/2\) lies on the line \(v = \pi/2\) in the \(w\)-plane.

For the construction of the bounded harmonic function \(h(u, v)\) with \(h(u, 0) = 0\) and \(h(u, \pi/2) = 1\), we may consider the function

\[
h(u, v) = \frac{2}{\pi} v.
\]

The similar reasoning shows that \(h\) is a bounded harmonic function in \(D_w\) with the desired boundary conditions. Finally, making use of Theorems 6 and 7 of Unit 9, the temperature function

\[
T(x, y) = h(u(x, y), v(x, y)) = \frac{2}{\pi} \arg z = \frac{2}{\pi} \tan^{-1}\left(\frac{y}{x}\right)
\]

is a harmonic function in \(D_z\) with \(0 \leq T \leq 1, T(x, 0) = 0\) and \(T(0, y) = 1\). The associated isotherms given by \(T(x, y) = c\) \((0 < c < 1)\) represent the lines

\[
y = x \tan \left(\frac{c\pi}{2}\right)
\]

which passes through the origin.
Example 5: Solve the Dirichlet problem for a semi-infinite strip:
\[ T_{xx}(x, y) + T_{yy}(x, y) = 0 \quad (0 < x < \pi/2, y > 0) \]
\[ T(x, 0) = 0 \quad (0 < x < \pi/2) \]
\[ T(0, y) = 1, \quad T(\pi/2, y) = 0 \quad (y > 0) \]
\[ 0 \leq T(x, y) \leq 1. \]

Solution: Consider the analytic function \( f(z) = \sin z = u(x, y) + iv(x, y) \) where
\[ u(x, y) = \sin x \cosh y \quad \text{and} \quad v(x, y) = \cos x \sinh y \]
in the domain \( D_z = \{ z = x + iy : 0 < x < \pi/2, y > 0 \} \). If \( z = x + iy \in D_z \), then
\[ u(x, y) > 0 \quad \text{and} \quad v(x, y) > 0 \quad (\text{since } \cosh y > 0 \text{ always and } \sinh y > 0 \text{ if } y > 0) \]
so that \( f \) maps \( D_z \) onto the domain \( D_w = \{ w = u + iv : u > 0, v > 0 \} \).

For the behaviour of \( f \) on the boundary, note that
- If \( z = iy, y > 0 \), then \( f(z) = i \sinh y \), so that the positive \( y \)-axis in the \( z \)-plane is mapped onto the line \( u = 0, v > 0 \).
- If \( z = \pi/2 + iy, y \geq 0 \), then \( f(z) = \cosh y \) maps the line \( x = \pi/2 \) in the first quadrant onto the line \( u \geq 1, v = 0 \) (observe that \( \cosh t \geq 1 \) for all \( t \in \mathbb{R} \) and \( \cosh t = 1 \Leftrightarrow t = 0 \)).
- If \( z = x, 0 < x < \pi/2 \), then \( f(z) = \sin x \), which maps the interval \( ]0, \pi/2[ \) on the \( x \)-axis onto the line segment \( 0 < u < 1, v = 0 \) (see Fig. 5).

Note that the points \( A, B \) and \( C \) are mapped onto \( A', B' \) and \( C' \), respectively under the function \( f \).

Next, we want to construct a harmonic function \( h(u, v) \) in the \( w \)-plane such that \( h(u, 0) = 0, h(0, v) = 1 \) and \( 0 \leq h(u, v) \leq 1 \). But we have already obtained such a harmonic function in the domain \( D_w \) in Example 4 satisfying these boundary conditions. Here we may take
\[ h(u, v) = \frac{2}{\pi} \tan^{-1} \left( \frac{v}{u} \right) \]
defined in \( D_w \). Therefore, by using Theorems 6 and 7 of Unit 9, the temperature function
\[ T(x, y) = h(u(x, y), v(x, y)) = \frac{2}{\pi} \tan^{-1} \left( \frac{\cos x \sinh y}{\sin x \cosh y} \right) = \frac{2}{\pi} \tan^{-1} \left( \frac{\tanh y}{\tan x} \right) \]

is a harmonic function in \( D_z \) with \( 0 \leq T \leq 1 \) and other required boundary conditions.

***

You may now check your understanding of determining the solution of boundary value problems while doing the following exercises:

E4) Find the images of the upper half-plane \( \text{Im} \, z > 0 \) under the mappings \( g(z) = (z-1)/(z+1) \) and \( h(z) = \text{Log} \, z \).

E5) Solve the Dirichlet problem as shown in Fig.6, for the upper-half plane \( (T_2 > T_1) \) where

\[ T_{xx}(x, y) + T_{yy}(x, y) = 0 \quad (y > 0) \]

\[ T(x, 0) = T_1 \text{ if } x > 0 \quad \text{and} \quad T(x, 0) = T_2 \text{ if } x < 0 \quad \text{and} \quad T_1 \leq T(x, y) \leq T_2. \]

E6) Find the steady-state bounded temperature function \( T(x, y) \) of the semi-infinite strip \(-\pi/2 \leq x \leq \pi/2, y \geq 0\) with the boundary conditions given in Fig.7. Also, find the associated isotherms.

---

We now discuss the problem where we determine the expression for the steady temperature in a quadrant.

**Steady Temperature in a Quadrant:** Consider the problem of finding the steady temperature in a thin plate in the form of a quadrant satisfying the following boundary conditions:

\[ T_{xx}(x, y) + T_{yy}(x, y) = 0 \quad (8) \]

\[ T_y(x, 0) = 0 \quad (0 < x < 1) \text{ and } T(x, 0) = 1 \quad (x > 1) \quad (9) \]

\[ T(0, y) = 0 \quad (y > 0), \quad 0 \leq T(x, y) \leq 1. \quad (10) \]

In this case, a segment of one edge is insulated and rest of the portions of the edges are kept at fixed temperatures. The condition \( T_y(x, 0) = 0 \) is equivalent to

\[ \frac{dT}{dN} = (\text{grad} \, T).N = (T_x(x, 0)i + T_y(x, 0)j). (0i + 1j) = 0 \]
which prescribe the value of the normal derivative of the function $T$ on the segment $0 < x < 1$, where $N$ is the unit normal to the segment and $i, j$ are the unit vectors along the $x$ and the $y$ axis, respectively. Such problems are called Neumann problems about which we mentioned in Unit 9. In Theorem 7 of Unit 9, we had shown that the values of the function prescribed along the boundary in the Dirichlet problem remains unaltered under a conformal transformation. The same is true for the values of the normal derivative of the function prescribed on the boundary in the case of the Neumann problem. You may prove it yourself while doing E7). To find the solution of the boundary value problem given by Eqns.(8)-(10), we shall make use of the similar technique as used in the case of the half-plane problem given by Eqns.(4)-(6).

You know from Example 5, the transformation $w = \sin z$ maps the vertical strip $0 < x < \pi/2$, $y > 0$ onto the quadrant $u > 0, v > 0$. Since the function $\sin z$ is periodic with period $2\pi$, it is one-one in $0 < x < \pi/2$, $y > 0$. Also, $\sin z$ is conformal inside and on the boundary of the strip except at the point $z = \pi/2$.

By the discussion in Example 5, it follows that the inverse transformation exists and consequently the function

$$f(z) = \sin^{-1} z = u(x, y) + iv(x, y)$$

is analytic in the domain $D_z = \{z = x + iy : x > 0, y > 0\}$ (see E8)) and maps onto the domain $D_w = \{w = u + iv : 0 < u < \pi/2, v > 0\}$, where $u(x, y)$ and $v(x, y)$ satisfy

$$x = \sin u \cosh v \quad \text{and} \quad y = \cos u \sinh v. \quad (11)$$

Moreover, $f$ is conformal in $D_z$ and on the boundary except at the point $z = 1$.

For the boundary behaviour, note that the points $A, B$ and $C$ in $z$-plane are mapped onto $A', B'$ and $C'$ in the $w$-plane, respectively (see Fig. 8).

![Fig. 8](image_url)

Now, consider a function

$$h(u, v) = \frac{2}{\pi}u, \quad (u, v) \in D_w$$

in the $w$-plane. Then

- $h$ is harmonic in $D_w$ because $h$ is the real part of the entire function $2w/\pi$. 

108
• $h$ is bounded since $0 \leq h(u, v) \leq 1$ for all $(u, v) \in D_v$.

• $h$ satisfies

$$h(u, v) = \begin{cases} 0, & u = 0, \quad v > 0; \\ 1, & u = \pi/2, \quad v > 0 \\ h_u(0, 0) = 0 \quad (0 < u < \pi/2). \end{cases}$$

The last condition shows that

$$\frac{dh}{dn} = 0$$

where $\mathbf{n}$ is the unit vector normal to the segment $0 < u < \pi/2$. By Theorem 6 of Unit 9, the temperature function

$$T(x, y) = h(u(x, y), v(x, y)) = \frac{2}{\pi} \text{Re}(\sin^{-1} z) = \frac{2}{\pi} u(x, y)$$

is a bounded harmonic function in $D_v$. To obtain $u$ in terms of $x$ and $y$, we shall make use of Eqn.(11). If $0 < u < \pi/2$, then

$$\cosh^2 v - \sinh^2 v = 1$$

$$\Rightarrow \frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$$

which represents a hyperbola of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = \sin u, b = \cos u$ and the foci are given by $(\pm c, 0)$ with

$$c = \sqrt{a^2 + b^2} = 1$$

(see Fig. 9).

By the geometrical interpretation of a hyperbola (see Fig.9), if $P(x, y)$ is any point on the hyperbola in the first quadrant, then

$$PF_2 - PF_1 = 2a$$

$$\Rightarrow \sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 2 \sin u$$

Fig. 9
E7) Suppose that the transformation $w = f(z) = u(x, y) + iv(x, y)$ is conformal on a smooth arc $C$ and let $\Gamma = f(C)$. Let a function $h(u, v)$ satisfies the condition:

$$\frac{dh}{dn} = 0 \quad \text{along } \Gamma.$$

Prove that the function $H(x, y) = h(u(x, y), v(x, y))$ satisfies the corresponding condition:

$$\frac{dH}{dN} = 0 \quad \text{along } C$$

where $dh/dn$ and $dH/dN$ denote the derivatives normal to $\Gamma$ and $C$, respectively.

E8) If $z \in \mathbb{C}$, define the inverse sine function denoted by $\sin^{-1} z$. Find an explicit expression for the same and discuss its analyticity.

E9) Solve the following boundary value problem for the quadrant:

$$T_{xx}(x, y) + T_{yy}(x, y) = 0$$

$$T_x(0, y) = 0 \quad (y > 1) \quad \text{and} \quad T(0, y) = 1 \quad (0 < y < 1)$$

$$T(x, 0) = 0 \quad (x > 0)$$

$$0 \leq T(x, y) \leq 1.$$

We now end the unit by giving a summary of what we have covered in it.

### 10.4 SUMMARY

In this unit, we have covered the following:

1) If $V$ is the velocity of a particle of an irrotational and incompressible fluid flow at any point $(x, y)$ in a simply connected domain $D$, then there exists an analytic function $\Phi = \phi + i \psi$ called the complex potential. The function $\phi(x, y)$ is called the velocity potential and it is
a harmonic function in $D$. The function $\psi$ is called the **stream function** which is the harmonic conjugate of the velocity potential $\phi$.

2) The level curves $\phi(x, y) = \text{constant}$ and $\psi(x, y) = \text{constant}$ are called the **equipotentials** and the **streamlines** of the flow, respectively.

3) The velocity vector $V$, velocity potential $\phi$ and complex potential $\Phi$ are related to each other by the identities: $V(x, y) = \text{grad } \phi(x, y)$ and $|V| = |\Phi'(z)|$.

4) The steady-state temperature function $T(x, y)$ in a thin, homogeneous semi-infinite plate (solid) in the $xy$ -plane satisfies the Laplace equation at each interior point of the solid, provided no heat sink or sources are present in the solid.

5) The level curves $T(x, y) = \text{constant}$, are called the **isotherms** of the temperature function $T$. If the domain in which $T(x, y)$ is defined is simply connected, then a conjugate harmonic function $S(x, y)$ exists. The level curves $S(x, y) = \text{constant}$, are called the **lines of flow**.

6) In a two-dimensional electrostatics, the electric field intensity is negative of the electrostatic potential.

7) Suppose that the transformation $w = f(z) = u(x, y) + iv(x, y)$ is conformal on a smooth arc $C$ and let $\Gamma = f(C)$. If a function $h(u, v)$ satisfies $dh/dn = 0$ along $\Gamma$, then the function $H(x, y) = h(u(x, y), v(x, y))$ satisfies the corresponding condition $dH/dN = 0$ along $C$.

10.5 SOLUTIONS/ANSWERS

E1) Let $x = g(t), y = h(t)$ where $x_0 = g(t_0), y_0 = h(t_0)$ are the parametric equations for the curve $\phi(x, y) = c_0$. Then the derivative of $\phi(x(t), y(t)) = c_0$ with respect to $t$ is

$$\frac{d\phi}{dx} \frac{dx}{dt} + \frac{d\phi}{dy} \frac{dy}{dt} = 0$$

$$\Rightarrow \left( \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j \right) (x'(t) i + y'(t) j) = 0$$

This shows that if $\gamma'(t) = x'(t) i + y'(t) j \neq 0$ at $t = t_0$, then

$$\nabla \phi(x_0, y_0). \gamma'(t_0) = 0$$

That is, $\nabla \phi$ is perpendicular to the level curves at $(x_0, y_0)$.

E2) By Example 1.
so that
\[ V(e^{i\theta}) = 1 - e^{-2i\theta} = 1 - \cos(2\theta) + i \sin(2\theta) = 1 - \cos(2\theta) - i \sin(2\theta) \]

E3) Speed, \( |V(1)| = |\Phi'(z)| = 2 |z| \). Note that the speed is minimum when \( |z| \) is minimum. That is, \( z \) is a point on the boundary which is nearest to the origin. So, we must have \( z = 1 + i \) (see Fig. 10). Thus minimum speed is \( |V(1)| = 2(1 + i) = 2\sqrt{2} \).

E4) If \( \text{Im} z > 0 \), then
\[
\text{Im} g(z) = \text{Im} \left( \frac{z-1}{z+1} \right) = \text{Im} \left( \frac{(z-1)(\overline{z}+1)}{(z+1)(\overline{z}+1)} \right)
= \text{Im} \left( \frac{|z|^2 + 2i \text{Im} z - 1}{|z|^2 + 2i \text{Im} z + 1} \right) = \frac{2 \text{Im} z}{|z|^2 + 1} > 0.
\]
Hence \( g \) maps \( \text{Im} z > 0 \) into the domain \( \text{Im} w > 0 \). If \( z = re^{i\theta} \) with \( r > 0 \) and \( 0 < \theta < \pi \), then
\[ h(z) = \log z = \ln r + i\theta \]
\[ \Rightarrow Re h(z) = \ln r \quad \text{and} \quad \text{Im} h(z) = \theta \]
\[ \Rightarrow -\infty < Re h(z) < \infty \quad \text{and} \quad 0 < \text{Im} h(z) < \pi. \]
Thus \( h \) maps \( \text{Im} z > 0 \) onto the horizontal strip \( 0 < \text{Im} w < \pi. \)

E5) The function \( f(z) = \log z = u(x, y) + iv(x, y) \) is analytic in \( D_z = \{z : \text{Im} z > 0\} \) and conformal along the boundary \( \text{Im} z = 0 \) (except at \( z = 0 \)), where \( u(x, y) = \ln |z| \) and \( v(x, y) = \arg z \). Moreover, \( f \) maps \( D_z \) onto the domain \( D_w = \{w = u + iv : 0 < v < \pi\} \) (see E4)). For the boundary behaviour, if \( z = x > 0 \), then \( f(z) = \ln x \) lies on the line \( v = 0 \) in the \( w \)-plane. Similarly, if \( z = x < 0 \), then \( f(z) = \ln |x| + i\pi \) lies on the line \( v = \pi \) in the \( w \)-plane. For the construction of the bounded harmonic function \( h(u, v) \) with \( h(u, 0) = T_1 \) and \( h(u, \pi) = T_2 \), we may consider the function
\[ h(u, v) = T_1 + \frac{T_2 - T_1}{\pi} v. \]
Then
\[ \cdot \ h \text{ is harmonic in } D_w \text{ because } h \text{ is the imaginary part of the entire function } T_1 + (T_2 - T_1) \frac{v}{\pi}. \]
\[ \cdot \ h \text{ is bounded since } T_1 \leq h(u, v) \leq T_2 \text{ for all } (u, v) \in D_w. \]
\[ \cdot \ h \text{ satisfies } h(u, v) = \begin{cases} T_1, & v = 0; \\ T_2, & v = \pi. \end{cases} \]

Therefore, by making use of Theorems 6 and 7 of Unit 9, the temperature function
\[ T(x, y) = h(u(x, y), v(x, y)) = T_1 + \frac{T_2 - T_1}{\pi} \tan^{-1} \left( \frac{y}{x} \right) \]
is a harmonic function in \( D_z \) with \( T_1 \leq T \leq T_2 \) and other required boundary conditions. Here \( \tan^{-1} t \) has range values satisfying \( 0 \leq \tan^{-1} t \leq \pi \).

E6) The given boundary value problem is

\[
T_{xx}(x, y) + T_{yy}(x, y) = 0 \quad \left( -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0 \right)
\]

\[
T\left(-\frac{\pi}{2}, y\right) = T\left(\frac{\pi}{2}, y\right) = 0 \quad (y > 0)
\]

\[
T(x, 0) = 1 \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)
\]

and \( T(x, y) \) is bounded.

Consider the analytic function \( f(z) = \sin z = u(x, y) + iv(x, y) \) where

\[
u(x, y) = \sin x \cosh y \quad \text{and} \quad v(x, y) = \cos x \sinh y
\]

in the domain \( D_z = \{z = x + iy : -\pi/2 < x < \pi/2, y > 0\} \). If \( z = x + iy \in D_z \), then \( \cos x > 0 \) and \( \sinh y > 0 \) so that \( v(x, y) > 0 \) (in fact, if \( 0 < x < \pi/2, y > 0 \) then \( u(x, y) > 0 \) and if \( -\pi/2 < x < 0, y > 0 \) then \( u(x, y) < 0 \)) so that \( f \) maps \( D_z \) onto the domain \( D_w = \{w = u + iv : v > 0\} \). For the boundary behaviour, note that

- If \( z = -\pi/2 + iy, y > 0 \), then \( f(z) = -\cosh y \) so that the line \( x = -\pi/2 \) in the second quadrant is mapped onto the line \( u < -1, v = 0 \).

- If \( z = \pi/2 + iy, y > 0 \), then \( f(z) = \cosh y \) maps the line \( x = \pi/2 \) in the first quadrant onto the line \( u > 1, v = 0 \).

- If \( z = x, -\pi/2 < x < \pi/2, \) then \( f(z) = \sin x \) maps the interval \( ]-\pi/2, \pi/2[ \) on the x-axis onto the line segment \(-1 < u < 1, v = 0\).

The points \( A, B, C \) and \( D \) are mapped onto the points \( A', B', C' \) and \( D' \), respectively (see Fig. 7). Next, our aim is to construct a harmonic function \( h(u, v) \) in the \( w \)-plane such that \( h(u, 0) = 0 \) if \( |u| > 1, h(u, 0) = 1 \) if \( |u| < 1 \) and \( 0 \leq h(u, v) \leq 1 \). But such a boundary value problem is already solved is Sec.10.3. Hence, let us take

\[
h(u, v) = \frac{1}{\pi} \tan^{-1}\left(\frac{2v}{u^2 + v^2 - 1}\right) \quad (0 \leq \tan^{-1} t \leq \pi) \quad \text{[ref. Eqn.(7)]}
\]

defined in \( D_w \). Therefore, by using Theorems 6 and 7 of Unit 9, the temperature function

\[
T(x, y) = h(u(x, y), v(x, y)) = \frac{1}{\pi} \tan^{-1}\left(\frac{2\cos x \sinh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y - 1}\right) \quad (0 \leq \tan^{-1} t \leq \pi)
\]

is a harmonic function in \( D_z \) with \( 0 \leq T \leq 1 \) and other desired boundary conditions. We may simplify the expression for \( T \) by using the identities for trigonometric and hyperbolic functions:

\[
\frac{2\cos x \sinh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y - 1} = \frac{2\cos x \sinh y}{\sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y - 1}
\]
Applications of Analytic Functions

\[
\begin{align*}
2\cos x \sinh y &= \frac{\sinh^2 x - \cos^2 x}{\sinh^2 x} \\
2\cos x / \sinh^2 y &= \frac{1 - \cos^2 x / \sinh^2 x}{1 - \tan^2 \alpha} \\
&= \tan(2\alpha)
\end{align*}
\]

where \( \tan \alpha = \cos x / \sinh y \). Hence

\[
T(x, y) = \frac{2}{\pi} \alpha = \frac{2}{\pi} \tan^{-1} \left( \frac{\cos x}{\sinh y} \right) \quad (0 \leq \tan^{-1} t \leq \pi/2)
\]

The isotherms are given by \( T(x, y) = c \) \((0 < c < 1)\) which depicts

\[
\cos x = \tan \left( \frac{\pi c}{2} \right) \sinh y
\]

which passes through the points \((\pm \pi/2, 0)\).

E7) Let \( \frac{dh}{dn} = 0 \) along \( \Gamma \). Then

\[
\frac{dh}{dn} = (\text{grad} h) \cdot n = 0
\]

where \( n \) is the unit vector normal to \( \Gamma \) at \((u, v)\). Since dot product of two vectors is zero if and only if they are orthogonal, \( \text{grad} h \) and \( n \) are orthogonal at \((u, v)\). Thus \( \text{grad} h \) is tangent to \( \Gamma \) at \((u, v)\) because tangent and normal vectors are orthogonal. Again, use the fact that the gradient of a function at a point is perpendicular to the level curve of the function at that point. Using this, we obtain that \( \text{grad} h \) is always perpendicular to the level curve of \( h \) passing through \((u, v)\). Since \( \text{grad} h \) is tangent to \( \Gamma \), it follows that \( \Gamma \) is perpendicular to the level curve \( h(u, v) = c \).

Under the transformation \( w = f(z) \), the level curve \( H(x, y) = c \) in the \( z \)-plane is mapped onto the level curve \( h(u, v) = c \) in the \( w \)-plane. Since \( \Gamma \) is orthogonal to the level curve \( h(u, v) = c \) and \( f \) is conformal, \( C \) is orthogonal to the level curve \( H(x, y) = c \). Again, by the similar reasoning stated above, \( \text{grad} H \) is orthogonal to the level curve \( H(x, y) = c \). This implies that \( \text{grad} H \) is tangent to \( C \) at \((x, y)\). If \( N \) denotes the unit vector normal to \( C \) at \((x, y)\), then \( \text{grad} H \) is orthogonal to \( N \) and hence

\[
\frac{dH}{dN} = (\text{grad} H) \cdot N = 0.
\]

Fig.11a) and b) illustrates the problem.

E8) If \( z \in \mathbb{C} \), the inverse sine function denoted by \( \sin^{-1} z \) is defined by the equation \( \sin^{-1} z = w \iff z = \sin w \).

In order to determine the formula for \( \sin^{-1} z \), suppose that \( w = \sin^{-1} z \).
Then
\[ z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}} \]
which simplifies to the quadratic equation in \( e^{iw} \):
\[ (e^{iw})^2 - (2iz)e^{iw} - 1 = 0. \]
On solving for \( e^{iw} \), we get
\[ e^{iw} = iz + (1 - z^2)^{1/2}, \]
where \((1 - z^2)^{1/2}\) is a multi-valued function. Taking logarithm on both the sides, we get
\[ w = \sin^{-1} z = -i \log (iz + (1 - z^2)^{1/2}). \]
The inverse sine function is multi-valued and can be made single-valued and analytic by choosing the specific branches of the square root and the logarithmic function.

E9) Consider the analytic function \( f(z) = i/z = u(x, y) + iv(x, y) \) where
\[ u(x, y) = \frac{y}{x^2 + y^2} \text{ and } v(x, y) = \frac{x}{x^2 + y^2} \]
in the domain \( D_z = \{ z = x + iy : x > 0, y > 0 \} \). If \( z = x + iy \in D_z \), then \( u(x, y) > 0 \) and \( v(x, y) > 0 \), so that \( f \) maps \( D_z \) onto the domain \( D_w = \{ w = u + iv : u > 0, v > 0 \} \). For the boundary behaviour, note that
- If \( z = iy \), then \( f(z) = 1/y \) so that the segment \( 0 < y < 1 \) and the line \( y > 1 \) on the \( y \)-axis are mapped onto the line \( v = 0, u > 1 \) and line segment \( v = 0, 0 < u < 1 \), respectively.
- If \( z = x, x > 0 \), then \( f(z) = i/x \), which maps the \( x \)-axis onto the \( v \)-axis (see Fig. 12).

Next, we need to construct a harmonic function \( h(u, v) \) in the \( w \)-plane such that
\[
\begin{align*}
   h_u(u, 0) &= 0 \quad (0 < u < 1) \quad \text{and} \quad h(u, 0) = 1 \quad u > 1 \\
   h(0, v) &= 0 \quad (v > 0) \\
   0 &\leq h(u, v) \leq 1.
\end{align*}
\]
But such a harmonic function is already constructed in Sec. 10.3 (ref. Eqn. (12)). Thus we may take

\[ h(u, v) = \frac{2}{\pi} \sin^{-1} \left( \frac{\sqrt{(u+1)^2 + v^2} - \sqrt{(u-1)^2 + v^2}}{2} \right) \]  

\[ (0 \leq \sin^{-1} t \leq \pi/2). \]

defined in \( D_u \). Therefore, by using Theorems 6, 7 of Unit 9 and E7), the temperature function

\[ T(x, y) = h(u(x, y), v(x, y)) \]

\[ = \frac{2}{\pi} \sin^{-1} \left( \frac{\sqrt{(x^2 + y^2 + y)^2 + x^2} - \sqrt{(x^2 + y^2 - y)^2 + x^2}}{2(x^2 + y^2)} \right) \]

is a harmonic function in \( D_z \) with the required prescribed boundary conditions.

– x –