Then

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}}$$

which simplifies to the quadratic equation in e^{iw}:

$$(e^{iw})^2 - (2iz)\, e^{iw} - 1 = 0.$$

On solving for e^{iw}, we get

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where $(1 - z^2)^{1/2}$ is a multi-valued function. Taking logarithm on both the sides, we get

$$w = \sin^{-1} z = -i \log (iz + (1 - z^2)^{1/2}).$$

The inverse sine function is multi-valued and can be made single-valued and analytic by choosing the specific branches of the square root and the logarithmic function.

E9) Consider the analytic function $f(z) = i/z = u(x, y) + iv(x, y)$ where

$$u(x, y) = \frac{y}{x^2 + y^2} \text{ and } v(x, y) = \frac{x}{x^2 + y^2}$$

in the domain $D_z = \{z = x + iy : x > 0, y > 0\}$. If $z = x + iy \in D_z$, then $u(x, y) > 0$ and $v(x, y) > 0$, so that f maps D_z onto the domain $D_w = \{w = u + iv : u > 0, v > 0\}$. For the boundary behaviour, note that

- If $z = iy$, then $f(z) = 1/y$ so that the segment $0 < y < 1$ and the line $y > 1$ on the y-axis are mapped onto the line $v = 0, u > 1$ and line segment $v = 0, 0 < u < 1$, respectively.
- If $z = x, x > 0$, then $f(z) = i/x$, which maps the x-axis onto the v-axis (see Fig. 12).

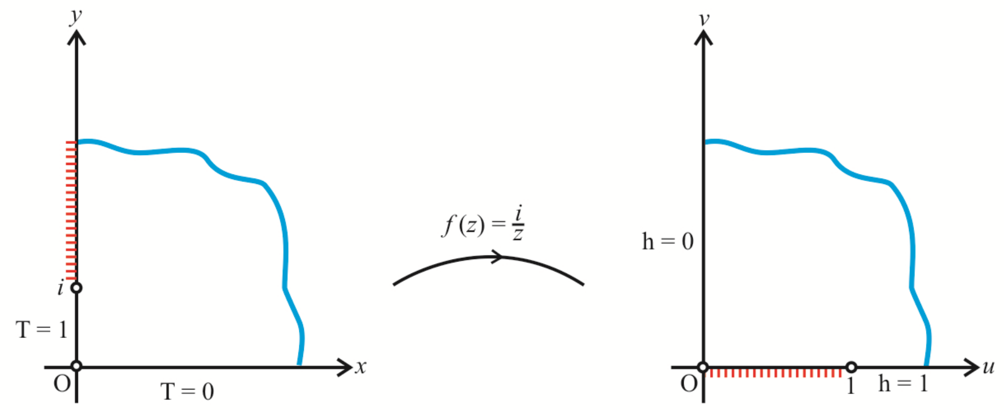

Fig. 12

Next, we need to construct a harmonic function $h(u, v)$ in the w-plane such that

$$h_v(u, 0) = 0 \qquad (0 < u < 1) \text{ and } h(u, 0) = 1 \qquad u > 1$$

$$h(0, v) = 0 \qquad (v > 0)$$

$$0 \le h(u, v) \le 1.$$

But such a harmonic function is already constructed in Sec. 10.3 (ref. Eqn. (12)). Thus we may take

$$h(u,v) = \frac{2}{\pi}\sin^{-1}\left(\frac{\sqrt{(u+1)^2+v^2}-\sqrt{(u-1)^2+v^2}}{2}\right) \quad (0 \le \sin^{-1} t \le \pi/2).$$

defined in D_w. Therefore, by using Theroems 6, 7 of Unit 9 and E7), the temperature function

$$T(x,y) = h(u(x,y), v(x,y))$$

$$= \frac{2}{\pi}\sin^{-1}\left(\frac{\sqrt{(x^2+y^2+y)^2+x^2}-\sqrt{(x^2+y^2-y)^2+x^2}}{2(x^2+y^2)}\right)$$

is a harmonic function in D_z with the required prescribed boundary conditions.

– x –