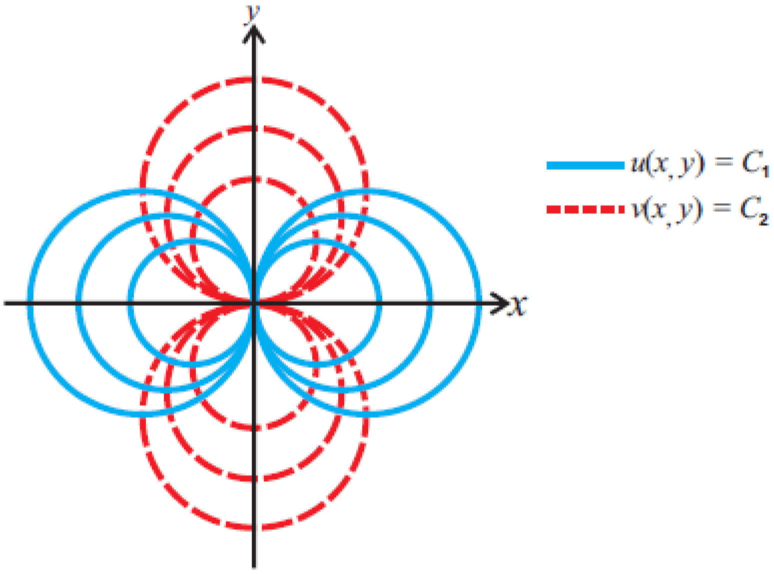
$v(x, y) = -y/(x^2 + y^2)$. The level curves of u and v passing through a point $z_0 = x_0 + iy_0 \neq 0$ are given by $x/(x^2 + y^2) = c_1$ and $-y/(x^2 + y^2) = c_2$ (see Fig. 12).



level curves of $f(z) = 1/z$

Fig. 12

As $z_0 \neq 0$, x_0 and y_0 are not both zero simultaneously. As a result, the same is true for c_1 and c_2. To prove that they intersect orthogonally, consider the following three cases:

Case 1: If $c_1 = 0$, then $x_0 = 0$, $c_2 \neq 0$ and $z_0 = -i/c_2$ lies on the imaginary axis. Consequently, the level curves reduces to the curves $x = 0$ and $-y/(x^2 + y^2) = c_2$ which intersect orthogonally at z_0.

Case 2: If $c_2 = 0$, then $y_0 = 0$, $c_1 \neq 0$ and $z_0 = 1/c_1$ lies on the real axis. Consequently, the level curves simplify to the curves $x/(x^2 + y^2) = c_1$ and $y = 0$, which intersect orthogonally at z_0.

Case 3: If c_1 and c_2 are both non-zero, then the level curves represent the circles:

$$\left(x - \frac{1}{2c_1}\right)^2 + y^2 = \frac{1}{4c_1^2}$$

and

$$x^2 + \left(y + \frac{1}{2c_2}\right)^2 = \frac{1}{4c_2^2},$$

respectively. Differentiating these equations with respect to x and obtaining the slopes of the tangent lines at z_0, we get

$$m_1 = -\left(\frac{x_0 - 1/2c_1}{y_0}\right) \text{ and } m_2 = -\left(\frac{x_0}{y_0 + 1/2c_2}\right)$$

$$\Rightarrow \quad m_1 m_2 = \frac{x_0}{y_0}\left(\frac{x_0 - 1/2c_1}{y_1 + 1/2c_2}\right).$$

Since $x_0/(x_0^2 + y_0^2) = c_1$ and $-y_0/(x_0^2 + y_0^2) = c_2$, it is a simple computation to show that $m_1 m_2 = -1$. Hence the curves intersect orthogonally at z_0.

ii) The function $f(z) = e^z$ is conformal everywhere and $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. The level curves of u and v passing through a point $z_0 = x_0 + iy_0$ are given by $e^x \cos y = c_1$ and $e^x \sin y = c_2$. Directly differentiate these equations with respect to x and obtain the slopes of the tangent lines at z_0 as

$$m_1 = \cot y \text{ and } m_2 = -\tan y$$

so that $m_1 m_2 = -1$.

E4) An arc $z = z(t)\,(a \le t \le b)$ is said to be **smooth** if the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and non-zero on the open interval $a < t < b$.

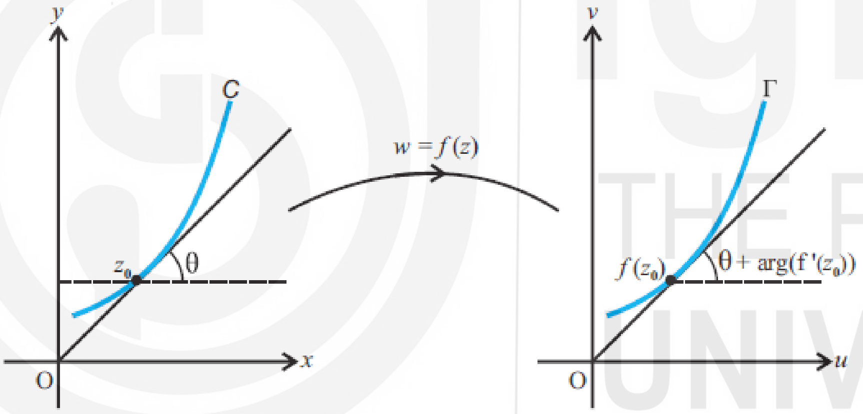


Fig. 13

Let C be the smooth arc defined by $z = z(t)\,(a \le t \le b)$ (see Fig. 13). By definition $z'(t)$ is continuous on the closed interval $a \le t \le b$ and $z'(t) \ne 0,\ a < t < b$. We have $w(t) = f(z(t))$ be the function defined at all points of C. The image of C is Γ under this transformation and $w'(t) = f'(z(t))z'(t)\,(a \le t \le b)$.
Since $z'(t) \ne 0$ and f is conformal $f'(z(t)) \ne 0$, therefore, $w'(t) \ne 0,\ a < t < b$ and $w'(t)$ is continuous. Hence, arc Γ is smooth.

E5) Given transformation is $f(z) = w = 1/z$. This function is analytic everywhere except at $z = 0$. Now $f'(z) = -1/z^2$ and at $z = i$, $f'(i) = 1$ and $\arg(f'(i)) = 0$.

E6) For the transformation $w = 1/z$. $u = \dfrac{x}{x^2 + y^2}$, $v = \dfrac{-y}{x^2 + y^2}$ and

$x = \dfrac{u}{u^2+v^2}$, $y = \dfrac{-v}{u^2+v^2}$. Now for $y = x - 1$

$$\frac{-v}{u^2+v^2} = \frac{u}{u^2+v^2} - 1$$

$$\Rightarrow \quad -v = u - u^2 - v^2 \Rightarrow u^2 + v^2 - u - v = 0$$

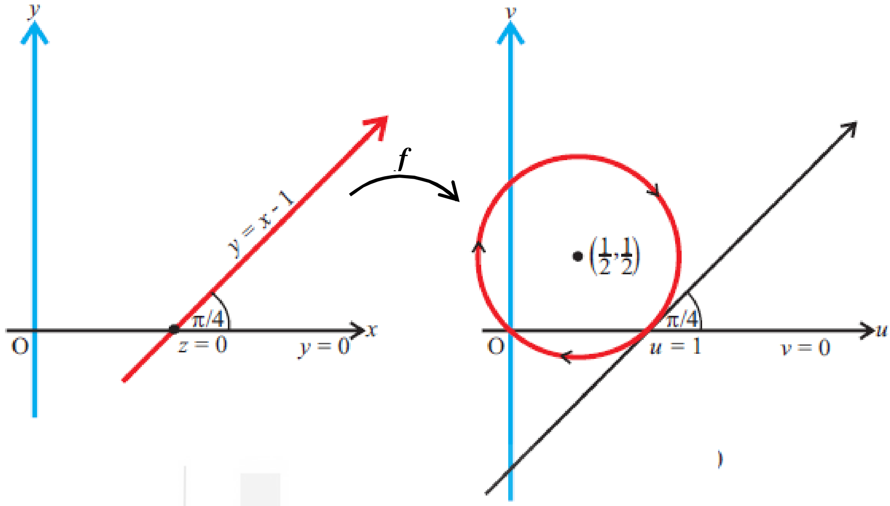


Fig. 14

It is a circle with centre $(1/2, 1/2)$ and radius $1/\sqrt{2}$ (see Fig. 14).

For $y = 0$, $\dfrac{-v}{u^2 - v^2} = 0 \Rightarrow v = 0$, a straight line.

Observe that $\arg f'(1) = \arg(-1) = \pi$.

E7) Clearly the angle between C_1 to C_2 is $\pi/2$ at the point of intersection $z_0 = 1$. Also, $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Let Γ_1, Γ_2 be the images of C_1 and C_2 under f at the point of intersection $w_0 = f(z_0) = e$ in the w-plane (see Fig. 15).

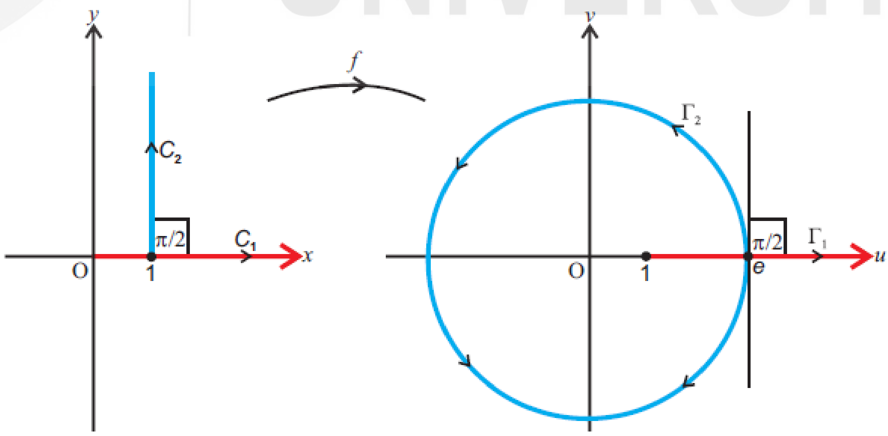


Fig. 15

It is easy to see that the parametric representation of Γ_1 is

$$u = e^x \text{ and } v = 0 \; (x \geq 0)$$

$$\Rightarrow \quad u \geq 1 \text{ and } v = 0$$

representing the strip $[1, \infty)$ on the u-axis in the w-plane. Similarity, the parametric representation of Γ_2 is

$$u = e\cos y \text{ and } v = e\sin y \ (y \geq 0)$$

$$\Rightarrow \quad u^2 + v^2 = e^2$$

which represents the circle centered at origin and radius e in the w-plane in the positive sense. If ϕ_1 and ϕ_2 are the angles of inclination of the directed tangent lines to Γ_1 and Γ_2, respectively at w_0, then $\phi_1 = 0$ and $\phi_2 = \pi/2$ (Think!). Hence the angle from Γ_1 to Γ_2 at the point w_0 is also $\phi_2 - \phi_1 = \pi/2$. This verifies the conformality. The angle of rotation of f at the point $z_0 = 1$ is 0. Which is one of the values of $\arg(f'(z_0))$ and the scale factor at the point $z_0 = 1$ is $|f'(z_0)| = e$.

E8) Here $f(z) = 1/z$ and $z_0 = 1+i$. So $f(z_0) = (1-i)/2 = w_0$. Let $g(w) = 1/w$. Then $g(w_0) = z_0$ and $f(g(w)) = w$.

E9) Let $z_0 = 0$ and $f(z) = e^z$. Then $f(z_0) = 1 = w_0$. Let $g(w) = \text{Log}\, w$ where $\text{Log}\, w$ is the principal branch

$$\text{Log}\, w = \ln|w| + i\theta, |w| > 0 \quad -\pi < \theta < \pi.$$

This gives $g(w_0) = 0 = z_0$ and $f(g(w)) = w$.

E10) For the function $u(x, y) = \tan^{-1}(y/x)$, we have

$$u_x = \frac{-y}{x^2+y^2}, u_y = \frac{x}{x^2+y^2},$$

$$\Rightarrow \quad u_{xx} = \frac{2xy}{(x^2+y^2)^2}, u_{yy} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0.$$

Thus u is harmonic in D. Following Theorem 5, $f(z) = u_x - iu_y$ takes the form

$$f(z) = \frac{-y-ix}{x^2+y^2} = \frac{-i(x-iy)}{x^2+y^2} = -\frac{i\bar{z}}{|z|^2} = -\frac{i}{z}.$$

This function f has an antiderivative F given by

$$F(z) = \int_1^z f(s)\,ds = -i\log z = u(x, y) + iv(x, y)$$

where $u(x, y) = \tan^{-1}(y/x)$ and $v(x, y) = -\log(x^2+y^2)/2$. Thus v is the required harmonic conjugate of u.

E11) Let $u(x, y) = x^2 + 2xy + by^2$. Then $u_x = 2x + 2y, u_y = 2x + 2by \Rightarrow u_{xx} = 2$ and $u_{yy} = 2b$. Therefore Laplace equation is satisfied provided $b = -1$. Using the notations of Theorem 5, we obtain

$$f(z) = u_x - iu_y = 2(x+y) - 2i(x-y) = 2(1-i)z$$

and the required analytic function F with $Re\,F = u$ is

$$F(z) = \int_1^z f(s)\,ds = (1-i)z^2 .$$

E12) The function $u(x, y) = x - e^x \sin y$ satisfies

$$u_x = 1 - e^x \sin y,\ u_y = -e^x \cos y$$

$\Rightarrow \quad u_{xx} = -e^x \sin y,\ u_{yy} = e^x \sin y$

$\Rightarrow \quad u_{xx} + u_{yy} = 0.$

To find the harmonic conjugate, note that

$f(z) = u_x - iu_y = 1 - e^x \sin y + ie^x \cos y = 1 + ie^z$. So that the antiderivative function is

$$F(z) = \int_1^z f(s)\,ds = z + i(e^z - 1).$$

Clearly, $Re\,F = u$ and the conjugate of u is $Im\,F = y + e^x \cos y - 1$.

E13) If z is a point in the first quadrant, then its polar representation is $z = re^{i\theta}$ where $r > 0$ and $0 < \theta < \pi/2$. Hence

$$w = f(z) = z^2 = r^2 e^{2i\theta} = \rho\, e^{i\theta}$$

where $\rho = r^2$ and $\phi = 2\theta$ satisfies $\rho > 0$ and $0 < \phi < \pi$. Thus $w = f(z)$ lies in the upper-half of w-plane.

E14) If $z = x + iy$ where $0 < y < \pi$, then

$$w = f(z) = e^x e^{iy} = \rho\, e^{i\phi}$$

where $\rho = e^x$ and $\phi = y$ satisfies $\rho > 0$ and $0 < \phi < \pi$. Clearly, $w = f(z)$ lies in the upper-half of w-plane.

E15) Let $h = h_0$ along Γ. If $(x, y) \in C$, then $(u(x, y), v(x, y)) \in \Gamma$. By hypothesis,

$$h(u(x, y), v(x, y)) = h_0 \Rightarrow H(x, y) = h_0 \text{ along } C.$$

E16) The Function $f(z) = iz^2 = u(x, y) + iv(x, y) = -2xy + i(x^2 - y^2)$ is conformal on the smooth arc $C : y = x(x > 0)$. Let $\Gamma = f(C)$. For $z = x + iy,\ y = x,\ x > 0$

$$u = -2x^2,\ v = 0 \Rightarrow u < 0,\ v = 0$$

Thus Γ represents the negative u-axis. Along Γ, $h = 3$. Also, the function $H(x, y)$ takes the form

$$H(x, y) = h(u(x, y), v(x, y)) = 2(x^2 - y^2) + 3.$$

On C, it is easily seen that $H = 3$.

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