UNIT 9 CONFORMAL MAPPING

9.1 INTRODUCTION

In Unit 3, we introduced you to various elementary transformations like translation, contraction/expansion, rotation and inversion, which map a given region in the \( z \)-plane onto a region in the \( w \)-plane. You saw there that how a specific composition of these transformations lead us to the notion of linear fractional transformations. Linear transformations which preserve the angle, both in magnitude and direction, between two curves are called conformal transformations, or conformal mappings. In this unit we shall discuss such transformations.

Sec 9.2 of the unit introduces you to the concept of conformal mappings which rely on the property of functions in terms of angle between the smooth arcs and their images. After defining conformal mapping formally, we have obtained here the necessary and sufficient conditions to check the conformality of an analytic function at a point. The existence of local inverse of a transformation at a point at which it is conformal is also established in this section.

You may recall that in Unit 2, we introduced you to harmonic functions which are real-valued functions of two real variables satisfying the Laplace’s equation. Because the harmonic functions satisfy the Laplace’s equation in two dimensions, they occur widely in applied mathematics. In Sec. 9.3 of the unit we shall see how they are used in finding the solutions of the boundary value problems. We shall discuss here the conjugate property of harmonic functions on a simply connected domain. The Dirichlet’s boundary value problem of finding a function that is harmonic in a specified domain and satisfies the prescribed conditions on the boundary of the domain is also discussed here. Some more applications of harmonic functions and applications of conformal mappings to physical problems are discussed, in detail, in the next unit.

Objectives

After studying this unit, you should be able to:

- check if a given transformation is conformal at a given point;
- determine the angle of rotation and scale factor at a point where the function is conformal;
- check if a given transformation is isogonal at a given point;
- obtain the critical points of a transformations;
• obtain the local inverse at a point of a transformation which is conformal at that point;
• obtain the harmonic conjugates of a harmonic function on a simple connected domain; and
• determine the solution of a boundary value problem by identifying it as real and imaginary component of an analytic function.

9.2 CONFORMALITY

Let us start by defining the notion of a tangent vector in the complex plane. Consider a smooth arc $C$ parameterized by the equation

$$z = z(t), \quad (a \leq t \leq b)$$

and passing through the point $z_0 = z(t_0)$ $(a < t_0 < b)$. The quantity $z'(t_0)$ is then the tangent vector to $C$ at $z_0$. If $\theta_0$ denotes a value of $\arg z'(t_0)$, then the number $\theta_0$ is the angle of inclination of the directed line tangent to $C$ at $z_0$ (see Fig. 1(a)). In other words, $\theta_0$ is the angle which the directed tangent to $C$ at $z_0$ makes with the positive real axis.

To give its geometrical interpretation, consider any point $z$ on the arc $C$ other than $z_0$. Then $\arg(z - z_0)$ is the angle which the straight line passing through the points $z$ and $z_0$ makes with the positive real axis. As $z$ approaches $z_0$ along the arc $C$, $\arg(z - z_0)$ approaches the value $\theta_0$. Thus, $\theta_0$ is the limiting value of $\arg(z - z_0)$ as $z$ approaches $z_0$ along $C$.

The angle $\alpha$ between two smooth arcs $C_1$ and $C_2$ intersecting at $z_0$ is defined to be the angle between their tangent vectors at $z_0$ (see Fig. 1(b)).

Let us now consider an example of the mapping of a complex valued function from $z$-plane to $w$-plane. You have studied such mappings in Sec. 1.3 of Unit 1.

**Example 1:** Let the angle between two smooth arcs:

$$C_1 : z(t) = t, \quad -1 \leq t \leq 1,$$

$$C_2 : z(t) = it, \quad -1 \leq t \leq 1$$

intersecting at $z_0 = 0$ in the $z$-plane is $\pi/2$. Let $\Gamma_1$ and $\Gamma_2$ be the images of $C_1$ and $C_2$ under the transformation $w = f(z)$. Determine the angle between $\Gamma_1$ and $\Gamma_2$ at their point of intersection for the following functions:

i) $f(z) = e^z$  ii) $g(z) = z^2$  and  iii) $h(z) = \overline{z}$.

**Solution:** i) The function $f(z) = e^z$ maps $C_1$ onto the closed interval $[1/e, e]$ on the real axis and $C_2$ onto the arc of unit circle $|w| = 1$ in the right-half plane. In this case, the angle between the images $\Gamma_1$ and $\Gamma_2$ intersecting at $f(z_0) = 1$ in the $w$-plane is also $\pi/2$ (see Fig. 2).
ii) Consider the function \( g(z) = z^2 \). It maps \( C_1 \) onto the closed interval \([0, 1]\) and \( C_2 \) onto the closed interval \([-1, 0]\) on the real axis. Clearly, the angle between the images \( \Gamma_1 \) and \( \Gamma_2 \) intersecting at \( g(z_0) = 0 \) in the \( w \)-plane is \( \pi \) (see Fig. 3).

iii) The function \( h(z) = \bar{z} \) maps \( C_1 \) onto itself in the same direction while \( C_2 \) onto itself in the reverse direction. Consequently, the angle between the image \( \Gamma_1 \) and \( \Gamma_2 \) intersecting at \( h(z_0) = 0 \) in the \( w \)-plane is \( \pi/2 \) (see Fig. 4).

In Example 1, the following observations are important:
Applications of Analytic Functions

- In i), the angle between $C_1$ and $C_2$ is the same in magnitude and sense as the angle between the images $\Gamma_1$ and $\Gamma_2$ under the exponential function.

- In ii), the angle between the images $\Gamma_1$ and $\Gamma_2$ is twice the angle between $C_1$ and $C_2$ under the transformation $z^2$.

- In iii), under the transformation $\bar{z}$, a counterclockwise angle is mapped onto a clockwise angle. As a result, the angle between $C_1$ and $C_2$ is same in magnitude but opposite in sense to the angle between the images $\Gamma_1$ and $\Gamma_2$.

The property of functions in terms of angles between the smooth arcs and their images is described in terms of conformality. Consider the following definition.

**Definition 1:** A transformation $w = f(z)$ defined on a domain $D$ is said to be **conformal** at a point $z_0$ in $D$ if it preserves angles, that is, the angle between any two smooth arcs $C_1, C_2$ in $D$ intersecting at $z_0$ is the same in **magnitude** and **sense** as the angle between their images $\Gamma_1, \Gamma_2$ intersecting at $f(z_0)$.

A transformation $w = f(z)$ defined in a domain $D$ is said be a **conformal mapping in** $D$ if $f$ is conformal at each point of $D$.

In Example 1, it is clear that the functions $g(z) = z^2$ and $h(z) = \bar{z}$ are not conformal at $z_0 = 0$. Is the function $f(z) = e^z$ conformal at $z_0 = 0$? The answer is YES, but we cannot deduce it just by considering two particular smooth arcs intersecting at $z_0 = 0$. In view of Definition 1, we need to show that the property of preservation of angles is satisfied for all the smooth arcs intersecting at $z_0 = 0$. It is quite a cumbersome task to prove it for all the smooth arcs if we proceed the way we did in Example 1. It is therefore necessary to obtain a condition under which a transformation is conformal at a point. We shall now consider a theorem which gives such a condition.

**Theorem 1:** If a function $f$ defined in a domain $D$ is analytic at a point $z_0 \in D$ and $f'(z_0) \neq 0$, then $f$ is conformal at $z_0$.

**Proof Step 1:** Let $C$ be a smooth arc parameterized by the equation $z = z(t), (a \leq t \leq b)$ and passing through the point $z_0 = z(t_0), a < t_0 < b$. Let $\Gamma$ be the image of $C$ under the function $f$ with the parametric equation $w = f(z(t)), a \leq t \leq b$. The tangent vector to $C$ at $z_0$ is $z'(t_0)$ and the tangent vector to $\Gamma$ at $w_0 = f(z_0)$ is

$$w'(t_0) = f'(z(t_0))z'(t_0) = f'(z_0)z'(t_0)$$

$$\Rightarrow \arg (w'(t_0)) = \arg (f'(z_0)) + \arg (z'(t_0))$$

$$\Rightarrow \arg (w'(t_0)) - \arg (z'(t_0)) = \arg (f'(z_0)).$$

Therefore, it follows that if $\theta_0$ is the angle of inclination of the directed tangent line to $C$ at $z_0$ and $\phi_0$ is the angle of inclination of the directed
tangent line to $\Gamma$ at $w_0$, then $\phi_0 - \theta_0$ is a fixed angle of rotation = value of $\arg (f'(z_0))$ (see Fig. 5). You may note here that $\arg (f'(z_0))$ is meaningful since $f'(z_0) \neq 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Fig. 5}
\end{figure}

**Step 2:** Let $C_1$, $C_2$ be the two smooth arcs intersecting at $z_0$ and $\Gamma_1$, $\Gamma_2$ be the image of $C_1$ and $C_2$ under $f$. Let $\theta_1$, $\theta_2$ be the angles of inclination of the directed tangent lines to $C_1$ and $C_2$, respectively at $z_0$. By Step 1, it is easy to deduce that $\theta_1 + \arg (f'(z_0))$ and $\theta_2 + \arg (f'(z_0))$ are the angles of inclination of the directed tangent lines to $\Gamma_1$ and $\Gamma_2$, respectively at $w_0 = f(z_0)$. Hence the angle from $C_1$ to $C_2$ is the same in magnitude and sense as the angle from $\Gamma_1$ to $\Gamma_2$, namely, $\theta_2 - \theta_1$ (see Fig. 6). Hence $f$ is conformal at $z_0$ by Definition 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Fig. 6}
\end{figure}

Application of Theorem 1 gives large number of examples of conformal mappings. Since the entire function $e^z \neq 0$ for all $z \in \mathbb{C}$, it is conformal in the entire complex plane. Also, you can easily see that the entire function $z^2$ is conformal at $z \neq 0$. Since $g'(z) = 2z \neq 0$ when $z \neq 0$. Let us consider an application of Theorem 1 in the context of level curves.

Let $f(z) = u(x, y) + iv(x, y)$ be a function that is analytic at a point $z_0$ with $f'(z_0) \neq 0$. Then by Theorem 1, $f$ is conformal at $z_0$. Consider two smooth arcs in the $z$-plane intersecting at $z_0$ which are the **level curves**

\[ u(x, y) = c_1 \text{ and } v(x, y) = c_2 \]
of the functions $u$ and $v$, respectively. Since the two arcs pass through $z_0$, we have $c_1 = u(x_0, y_0)$ and $c_2 = v(x_0, y_0)$, where $z_0 = x_0 + iy_0$. What can be said about the angle between these two smooth arcs intersecting at $z_0$? Since $f$ is conformal at $z_0$, we are sure that the angle between these two smooth arcs intersecting at $z_0$ is the same as the angle between their images intersecting at $f(z_0)$. But $f$ maps these smooth arcs into the lines $u = c_1$ and $v = c_2$ in the $w$-plane which are orthogonal at $f(z_0)$. Therefore, it follows that the two smooth arcs representing the level curves of $u$ and $v$ must be orthogonal at $z_0$. Thus, we have proved the following corollary.

**Corollary 1:** If the function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z_0$ with $f'(z_0) \neq 0$ and the level curves $u(x, y) = c_1$, $v(x, y) = c_2$ of $u$ and $v$, respectively pass through $z_0$, then the level curves intersect orthogonally at $z_0$.

You may note here that without invoking the concept of conformality, Corollary 1 can be proved independently also by making use of the Cauchy Riemann equations. We are leaving it to you to try and do it yourself (see E1)). However, we illustrate the result through an example.

**Example 2:** Verify Corollary 1 for the analytic function $f(z) = z^2$.

**Solution:** The function $f(z) = z^2$ is conformal at every non-zero point. It is easily seen that $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The level curves of $u$ and $v$ are given by $x^2 - y^2 = c_1$ and $2xy = c_2$. As the level curves pass through $z_0 = x_0 + iy_0 \neq 0$, we have $x_0^2 - y_0^2 = c_1$ and $2x_0y_0 = c_2$. Since $z_0 \neq 0$, we must have $c_1 \neq 0$. To prove that they intersect orthogonally, consider the following two cases: i) $c_2 \neq 0$ and ii) $c_2 = 0$.

**i)** If $c_2 \neq 0$ then both $x_0$ and $y_0$ are non-zero. Differentiate the equations $x^2 - y^2 = c_1$ and $2xy = c_2$ with respect to $x$ and obtain the slopes $m_1$ and $m_2$ of the tangent lines at $z_0$ as follows:

$$2x - 2y \frac{dy}{dx} = 0 \Rightarrow m_1 = \frac{dy}{dx} \bigg|_{z=z_0} = \frac{x_0}{y_0}$$

$$2y + 2x \frac{dy}{dx} = 0 \Rightarrow m_2 = \frac{dy}{dx} \bigg|_{z=z_0} = -\frac{y_0}{x_0}$$

Thus, $m_1m_2 = -1$ and hence the curves intersect orthogonally at $z_0$.

**ii)** If $c_2 = 0$, the point $z_0$ lies either on the real or on the imaginary axis. As a result, either $x_0 = 0$ or $y_0 = 0$. If $x_0 = 0$, then $c_1 = u(x_0, y_0) = -y_0^2 < 0$ and the equation of the tangent line to the curve $x^2 - y^2 = c_1$ at $z_0$ is $y = y_0$ which is perpendicular to the line $x = 0$. If $y_0 = 0$, then $c_1 = x_0^2 > 0$ and the equation of the tangent line to the curve $x^2 - y^2 = c_1$
at \( z_0 \) is \( x = x_0 \) which is perpendicular to the line \( y = 0 \) (see Fig. 7). Thus in both the cases the level curves intersect orthogonally.

You may now try the following exercise:

\[ f(z) = z^2 \]

**E1)** Making use of the Cauchy-Riemann equations show that the level curves of the real and the imaginary parts of an analytic function 
\[ f(z) = u(x, y) + iv(x, y) \]
are orthogonal at a point \( z_0 \) where \( f'(z_0) \neq 0 \).

In Example 1 you have seen that under the transformation \( h(z) = \overline{z} \), the angle between the arcs \( C_1 \) and \( C_2 \) is same in magnitude but opposite in sense to the angle between the images \( \Gamma_1 \) and \( \Gamma_2 \). Such transformations which preserve the magnitude of the angle between two smooth arcs but not necessarily the sense are called **isogonal**. The transformation \( w = h(z) = \overline{z} \), which is a reflection in the real axis, is isogonal but not conformal. Further, if a function \( f \) is analytic at \( z_0 \) and \( f'(z_0) = 0 \), then it fails to be conformal at \( z_0 \). Such a point \( z_0 \) is called a **critical point** of \( f \). The behavior of an analytic function in a neighbourhood of a critical point is given by the following theorem:

**Theorem 2:** Suppose that a function \( f \) defined in a domain \( D \) is analytic at a point \( z_0 \in D \) and 
\[ f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0, \]
for some positive integer \( m \geq 2 \). If the angle between two smooth arcs passing through \( z_0 \) is \( \theta \), then the corresponding angle between their image curves under \( f \) is \( m\theta \).

**Proof:** Since \( f \) is analytic at \( z_0 \), \( f \) has the Taylor’s series expansion
\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \]
Applications of Analytic Functions

about \( z_0 \), valid in some neighborhood of \( z_0 \). Using the hypothesis, it follows that

\[
f(z) = f(z_0) + \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\]

\[
\Rightarrow f(z) - f(z_0) = (z - z_0)^m g(z) \quad (1)
\]

where

\[
g(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-m}
\]

is analytic at \( z_0 \) and \( g(z_0) = f^{m}(z_0)/m! \neq 0 \).

Now, consider the following two cases:

**Case 1:** Let \( C \) be a smooth arc passing through the point \( z_0 \) and \( \Gamma \) be the image of \( C \) under the function \( f \). Let \( \theta \) be the angle of inclination of the directed tangent line to \( C \) at \( z_0 \) and \( \phi \) be the angle of inclination of the directed tangent line to \( \Gamma \) at \( f(z_0) \). Then as \( z \) approaches \( z_0 \) along \( C \), \( \arg(z - z_0) \) approaches the angle \( \theta \) and \( \arg(f(z) - f(z_0)) \) approaches the angle \( \phi \). Consequently, in view of Eqn. (1), we have

\[
\phi = m \theta + \arg(g(z_0)).
\]

**Note** that \( \arg(g(z_0)) \) makes sense since \( g(z_0) \neq 0 \).

**Case 2:** Let \( C_1, C_2 \) be two smooth arcs intersecting at \( z_0 \) and \( \Gamma_1, \Gamma_2 \) be the images of \( C_1 \) and \( C_2 \) under \( f \). Let \( \theta_1, \theta_2 \) be the angles of inclination of the directed tangent lines to \( C_1 \) and \( C_2 \), respectively at \( z_0 \). By Case 1, it follows that \( m \theta_1 + \arg(g(z_0)) \) and \( m \theta_2 + \arg(g(z_0)) \) are the angles of inclination of the directed tangent lines to \( \Gamma_1 \) and \( \Gamma_2 \), respectively at \( w_0 = f(z_0) \). Hence the angle between \( C_1 \) and \( C_2 \) is \( \theta = \theta_2 - \theta_1 \) and the angle between \( \Gamma_1 \) and \( \Gamma_2 \) is \( (m \theta_2 + \arg(g(z_0))) - (m \theta_1 + \arg(g(z_0))) = m(\theta_2 - \theta_1) = m \theta \).

This completes the proof of the theorem.

Consider for example, the function \( f(z) = z^2 \). The function \( f \) is analytic at the origin with \( f'(0) = 0 \) and \( f''(0) \neq 0 \). Then by Theorem 2, for any two smooth arcs intersecting at the origin, the angle between their images under \( f \) is twice the angle between the corresponding smooth arcs. However, you have already seen this in Example 1 ii). In general, the function \( z^n (n \geq 2) \) multiplies the angle at the origin by \( n \).

Combining Theorems 1 and 2, we obtain the result which characterizes the conformality of an analytic function at a point.

**Theorem 3:** An analytic function \( f \) is conformal at a point \( z_0 \) if and only if \( f'(z_0) \neq 0 \).

**Proof:** Suppose that an analytic function \( f \) is conformal at a point \( z_0 \). Let \( f'(z_0) = 0 \). Then the following two cases arise:
Case 1: If \( f^{(m)}(z_0) = 0 \) for all \( m \in \mathbb{N} \), then \( f \) is constant in a neighborhood of \( z_0 \) which is obviously not conformal (because the angles are not preserved).

Case 2: If there exists a positive integer \( m \geq 2 \) such that \( f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \), and \( f^{(m)}(z_0) \neq 0 \), then by Theorem 2 \( f \) does not preserve the angles. As a result, \( f \) is not conformal at \( z_0 \).

In both the cases, we arrive at a contradiction. Hence we must have \( f'(z_0) \neq 0 \). The converse of the theorem is obviously true by Theorem 1.

---

Theorem 3 provides the necessary and sufficient condition for an analytic function to be conformal at a point.

Let us consider an example to illustrate the theorem.

**Example 3:** Show that the translation \( f(z) = z + \alpha (\alpha \in \mathbb{C}) \) and the complex multiplication \( g(z) = \beta z (\beta \neq 0) \) are conformal in \( \mathbb{C} \).

**Solution:** Functions \( f(z) \) and \( g(z) \), being polynomials, are both analytic functions. By Theorem 3, an analytic function \( f \) is conformal at a point \( z_0 \iff f'(z_0) \neq 0 \). For the analytic functions \( f'(z) = z + \alpha (\alpha \in \mathbb{C}) \) and \( g(z) = \beta z (\beta \neq 0) \), direct differentiation gives

\[
f'(z) = 1 \neq 0 \quad \text{and} \quad g'(z) = \beta \neq 0 \quad \text{for all} \quad z \in \mathbb{C}.
\]

Therefore, both \( f \) and \( g \) are conformal at each point of \( \mathbb{C} \).

***

You may now try the following exercises.

---

E2) Determine the points in the complex plane at which the following functions are conformal:

i) \( f(z) = \sin z \) \quad ii) \( f(z) = \cosh z \) \quad iii) \( f(z) = e^z + 100 \).

E3) Verify Corollary 1 for the following functions:

i) \( f(z) = \frac{1}{z} \) \quad ii) \( f(z) = e^z \).

E4) Let \( C \) be a smooth arc lying in a domain \( D \) throughout which a transformation \( w = f(z) \) is conformal, and let \( \Gamma \) denote the image of \( C \) under that transformation. Show that \( \Gamma \) is also a smooth arc.

---

Associated with a conformal mapping, there are two terms: angle of rotation and scale factor. We say that if an analytic function \( f \) is conformal at a point \( z_0 \), then its angle of rotation at \( z_0 \) is \( \arg(f'(z_0)) \) and its scale factor at \( z_0 \) is \( |f'(z_0)| \). We make the following two observations in this regard:

- From Theorem 1, it is evident that for a mapping conformal at a point, its angle of rotation at that point is a fixed value. This value represents the
difference between the angle of inclination of the directed tangent line to any smooth arc passing through that point and the angle of inclination of the directed tangent line to its image curve. Moreover, it is independent of the smooth arcs being chosen through that point for showing the conformality of the mapping.

- The scale factor for a mapping conformal at a point is the measure of the contraction or expansion of the image of a small region in a neighbourhood of that point. For its geometrical interpretation, you may note that if \( f \) is an analytic function at \( z_0 \), then

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.
\]

Using the properties of the modulus of continuous functions, you may write

\[
|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}.
\]

Since \( |z - z_0| \) is the line segment joining \( z \) and \( z_0 \) and \( |f(z) - f(z_0)| \) is the line segment joining \( f(z) \) and \( f(z_0) \), therefore the ratio \( |f(z) - f(z_0)|/|z - z_0| \) is approximately equal to \( |f'(z_0)| \) for the points \( z \) that are near to \( z_0 \). As a result, \( |f'(z_0)| > 1 \) represents an expansion and \( |f'(z_0)| < 1 \) represents a contraction.

The following examples will further help you in understanding the discussion above.

**Example 4:** Verify that the function \( f(z) = z^2 \) is conformal at the point \( z_0 = 1 + i \) by considering the half-lines \( C_1 : y = x (x \geq 0) \) and \( C_2 : x = 1 (y \geq 0) \) in the positive sense. Also, determine the angle of rotation and the scale factor of \( f \) at \( z_0 \).

**Solution:** In Fig. 8, you can see that the angle from \( C_1 \) to \( C_2 \) is \( \pi/4 \) at the point of intersection \( z_0 = 1 + i \). Also, \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = 2xy \). Let \( \Gamma_1, \Gamma_2 \) be the images of \( C_1 \) and \( C_2 \) under \( f \) at the point of intersection \( w_0 = f(z_0) = 2i \) in the \( w \)-plane. In order to prove the conformality of \( f \) at the point \( z_0 \), we need to show that the angle from \( \Gamma_1 \) to \( \Gamma_2 \) at the point \( w_0 \) is also \( \pi/4 \).

![Fig. 8](image-url)
The parametric representation of $\Gamma_1$ is
\[ u = 0 \quad \text{and} \quad v = 2x^2 \quad (x \geq 0) \]
\[ \Rightarrow \quad u = 0 \quad \text{and} \quad v \geq 0 \]
which represents the upper half of the $v$-axis in the $w$-plane. Similarly, the parametric representation of $\Gamma_2$ is
\[ u = 1 - y^2 \quad \text{and} \quad v = 2y \quad (y \geq 0) \]
\[ \Rightarrow \quad v^2 = -4 (u - 1) \quad (v \geq 0) \]
representing a parabola in the upper half plane with vertex $(1, 0)$ in the $w$-plane.

If $\phi_1$ and $\phi_2$ are the angles of inclination of the directed tangent lines to $\Gamma_1$ and $\Gamma_2$, respectively at $w_0$, then clearly $\phi_1 = \pi/2$. In order to determine $\phi_2$, let us calculate $dv/du$ in Eqn. (3) at the point $w_0$. We have
\[ 2v \frac{dv}{du} = -4 \Rightarrow \frac{dv}{du} = -\frac{2}{v} \Rightarrow \left. \frac{dv}{du} \right|_{w_0=2i} = -1. \]
This gives $\phi_2 = 3\pi/4$. Therefore, the angle from $\Gamma_1$ to $\Gamma_2$ at the point $w_0$ is $\phi_2 - \phi_1 = \pi/4$. This verifies that the function $f(z) = z^2$ is conformal at the point $z_0 = 1 + i$.

The angle of rotation of $f$ at the point $z_0 = 1 + i$ is $\pi/4$ which is one of the values of $\arg(f'(z_0)) = \arg(2(1+i))$ and the scale factor at the point $z_0 = 1 + i$ is $|f'(z_0)| = |2(1+i)| = 2\sqrt{2}$.

***

**Example 5:** Find the angle of rotation for the function $f(z) = 1/z$ at the point $z_0 = 1$ by considering the smooth arc $C: x^2 + y^2 = 1$ in the positive sense. Also, determine the scale factor at that point.

**Solution:** Let $\Gamma$ be the image of $C$ under $f$. Note that $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x/(x^2 + y^2)$ and $v(x, y) = -y/(x^2 + y^2)$. Using this, the parametric equation of $\Gamma$ becomes
\[ u = x \quad \text{and} \quad v = -y \]
\[ \Rightarrow \quad u^2 + v^2 = 1. \]
Eqn. (4) represents a unit circle centered at \((0, 0)\) in the reverse sense (Why?). This is because you can see in Fig. 9 that as the point \((x, y)\) traverses the circle in the counter clockwise direction, the corresponding point \((u, v)\) traverses the circle in the clockwise direction.

If \(\theta\) is the angle of inclination of the directed tangent line to \(C\) at \(z_0 = 1\) and \(\phi\) is the angle of inclination of the directed tangent line to \(\Gamma\) at \(f(z_0) = 1\), then obviously \(\theta = \pi / 2\) and \(\phi = 3\pi / 2\). Then the angle of rotation is \(\phi - \theta = \pi\) which is one of the values of \(\arg(f'(z_0)) = \arg(-1)\) and the scale factor at the point \(z_0 = 1\) is \(|f'(z_0)| = 1 - 1/|z_0^2| = 1\).

You may now try the following exercises.

E5) What angle of rotation is produced by the transformation \(w = \frac{1}{z}\) at the point \(z = i\)?

E6) Show that under the transformation \(w = \frac{1}{z}\), the images of the lines \(y = x - 1\) and \(y = 0\) are the circle \(u^2 + v^2 - u - v = 0\) and the line \(v = 0\), respectively. Sketch all the four curves. Determine the corresponding directions along them, and verify the conformality of the mapping at the point \(z = 1\).

E7) Verify that the function \(f(z) = e^z\) is conformal at the point \(z_0 = 1\) by considering the half lines \(C_1 : y = 0(x \geq 0)\) and \(C_2 : x = 1(y \geq 0)\) in the positive sense. Also, determine the angle of rotation and the scale factor of \(f\) at that point.

We say that a transformation \(w = f(z)\) that is conformal at a point \(z_0\) has a local inverse there. What do we mean by a local inverse of a transformation \(w = f(z)\)? We shall now try to find the answer to this question.

**Local Inverses**

By a local inverse of a transformation \(w = f(z)\), we mean the existence of a unique transformation \(z = g(w)\) defined in a neighbourhood of \(w_0 = f(z_0)\) with \(g(w_0) = z_0\) and \(f(g(w)) = w\) for all \(w\) in that neighbourhood. Assuming that an analytic function \(f\) is conformal at a point \(z_0\), let us prove the existence of such an inverse. The proof makes use of the result viz., the inverse function theorem, which you would have studied in your undergraduate course of advanced calculus (calculus of multi-variables).

Suppose that \(f\) is an analytic function at \(z_0\) where it is conformal. By Theorem 3, \(f'(z_0) \neq 0\). Since \(f\) is analytic at \(z_0\), therefore there is a neighbourhood of \(z_0\) in which \(f\) is analytic. Consequently, if we write
Conformal Mapping

\[ f(z) = u(x, y) + iv(x, y), \]
then the component functions \( u(x, y) \) and \( v(x, y) \) have partial derivatives of all orders and are continuous throughout a neighbourhood \( S \) of the point \((x_0, y_0)\), where \( z_0 = x_0 + iy_0 \) (ref. Sec. 2.3, Unit 2). Further, the Cauchy-Riemann equations

\[ u_x = v_y \text{ and } u_y = -v_x, \]

are satisfied in the neighbourhood. Using these Cauchy-Riemann equations, you can see that the Jacobian of the transformation \( u = u(x, y) \) and \( v = v(x, y) \) given by the determinant:

\[
J = \begin{vmatrix}
    u_x & u_y \\
    v_x & v_y \\
\end{vmatrix} = u_xv_y - u_yv_x = (u_x)^2 + (v_y)^2 = |f'(z)|^2, \quad z = x + iy
\]

is non-zero at \((x_0, y_0)\) because \( f'(z_0) \neq 0 \). Without the loss of generality, we assume that \( J \neq 0 \) throughout a neighbourhood \( S \) of the point \((x_0, y_0)\).

Existence of such a neighbourhood \( S \) is always possible because \( f' \) is continuous and \( f'(z_0) \neq 0 \).

Let \( u_0 = u(x_0, y_0) \) and \( v_0 = v(x_0, y_0) \). By the inverse function theorem in two-dimension, there is a unique transformation \( x = x(u, v) \) and \( y = y(u, v) \) defined in the neighbourhood \( N \) of the point \((u_0, v_0)\) such that for each \((u, v) \in N\), there is a unique \((x, y) \in S\) for which \( u = u(x, y) \) and \( v = v(x, y) \). Also, it satisfies the following two conditions:

- \( x(u_0, v_0) = x_0 \) and \( y(u_0, v_0) = y_0 \);
- \( u = u(x(u, v), y(u, v)) \) and \( v = v(x(u, v), y(u, v)) \).

Moreover, \( x(u, v) \) and \( v(x, y) \), along with their first-order partial derivatives are continuous and satisfy the equations

\[
x_u = \frac{1}{f_y}, \quad x_v = -\frac{1}{f_x}, \quad y_u = -\frac{1}{f_y}, \quad y_v = \frac{1}{f_x}
\]

(5)

throughout the neighbourhood \( N \).

Let \( w = u + iv \) and \( w_0 = u_0 + iv_0 \). Define

\[
g(w) = x(u, v) + iy(u, v).
\]

Then \( g(w_0) = x(u_0, v_0) + iy(u_0, v_0) = x_0 + iy_0 = z_0 \) and

\[
f(g(w)) = f(x(u, v) + iy(u, v))
\]

\[
= u(x(u, v), y(u, v)) + iv(x(u, v), y(u, v)) = u + iv = w,
\]

for all \( w \) in \( N \). The transformation \( z = g(w) \) is the desired local inverse of the transformation \( w = f(z) \) at \( z_0 \). A more stronger version of the result is given in the following theorem.

**Theorem 4:** Suppose that an analytic function \( f \) is conformal at a point \( z_0 \). Then the transformation \( w = f(z) \) has a local inverse at \( z_0 \), that is, there exists a unique transformation \( z = g(w) \), which is defined and analytic in a neighbourhood \( N \) of \( w_0 = f(z_0) \) such that \( g(w_0) = z_0 \) and \( f(g(w)) = w \) for all points \( w \) in \( N \). Moreover, \( g'(w) = 1/ f'(z) \) for all points \( w \) in \( N \). In particular, the transformation \( z = g(w) \) is itself conformal at \( w_0 \).
Applications of Analytic Functions

**Proof:** In view of the discussion above, it remains to show that the transformation \( g(w) \) defined by Eqn. (6) is unique, analytic and \( g'(w) = 1/ f'(z) \) for all points \( w \) in \( N \).

Let \( z = h(w) = x^*(u, v) + iv^*(u, v) \) is any other local inverse of the transformation \( w = f(z) \) at \( z_0 \), then

\[
h(w_0) = z_0 \Rightarrow x^*(u_0, v_0) + iy^*(u_0, v_0) = x_0 + iy_0
\]

and

\[
f(h(w)) = w \Rightarrow u(x^*(u, v), y^*(u, v)) + iv(x^*(u, v), y^*(u, v)) = u + iv
\]

But, since the inverse function theorem ensure the existence of unique transformation \( x = x(u, v) \) and \( y = y(u, v) \) satisfying the conditions:

\[
x(u_0, v_0) = x_0 \quad \text{and} \quad y(u_0, v_0) = y_0; \quad u = u(x(u, v), y(u, v)), \quad \text{and} \quad v = v(x(u, v), y(u, v)),
\]

therefore, we must have \( x^*(u, v) = x(u, v) \) and \( y^*(u, v) = y(u, v) \) which implies that \( h(w) = g(w) \) for all points \( w \) in \( N \).

Next, we will show that the function \( g(w) \) defined by Eqn. (6) is analytic in \( N \). By the sufficient condition of differentiability (see Theorem 5, Unit 2), since \( x(u, v) \) and \( v(x, y) \), along with their first-order partial derivatives are continuous in \( N \), it suffices to show that the Cauchy-Riemann equations

\[
x_u = y_v \quad \text{and} \quad x_v = -y_u
\]

hold in \( N \). But you can see that these are automatically satisfied if we use the Cauchy-Reimann equations \( u_x = v_y \) and \( u_y = -v_x \) in Eqn. (5).

Now we only need to show that \( g'(w) = 1/f'(z) \) for all points \( w \) in \( N \). Since \( f(g(w)) = w \) for all \( w \in N \), therefore its differentiation gives

\[
f'(g(w))g'(w) = 1 \quad \text{for all} \quad w \in N.
\]

As \( z = g(w) \), we get \( f'(z)g'(w) = 1 \), or \( g'(w) = 1/f'(z) \) as desired. This completes the proof of the theorem.

We now take up an example to illustrate Theorem 4.

**Example 6:** Find the local inverse of the transformation \( f(z) = w = z^2 \) at the point i) \( z_0 = i \), ii) \( z_0 = -2i \), iii) \( z_0 = 3 \) and iv) \( z_0 = -4 \).

**Solution:** Let \( f(z) = z^2 \). Then \( f \) is conformal at every non-zero point.

i) For \( z_0 = i \), \( f(z_0) = -1 = w_0 \) and let \( g(w) = w^{1/2} \) where \( w^{1/2} \) denotes the branch

\[
w^{1/2} = e^{\theta/2} \log |w| = \sqrt{|w|} e^{i\theta/2}, \quad 0 < \theta < 2\pi.
\]

Then \( g(w_0) = g(-1) = e^{\pi/2} = i = z_0 \) and \( f(g(w)) = w \). Here the branch is chosen such that the point \( w_0 = -1 \) lies in the domain of \( g \) and \( w_0 \) is mapped onto \( z_0 \) by \( g \).

ii) For \( z_0 = -2i \), \( f(z_0) = -4 = w_0 \) and \( g(w) = w^{1/2} \) where \( w^{1/2} \) is the branch

\[
w^{1/2} = \sqrt{|w|} e^{i\theta/2}, \quad 0 < \theta < 4\pi.
\]
Then \( g(w_0) = g(-4) = 2e^{3\pi i/2} = -2i = z_0 \) and \( f(g(w)) = w \).

iii) For \( z_0 = 3 \), \( f(z_0) = 9 = w_0 \) and let \( g(w) = w^{1/2} \) where \( w^{1/2} \) denotes the principal branch
\[ w^{1/2} = \sqrt{|w|} e^{i\theta/2}, \; |w| > 0, \; -\pi < \theta < \pi. \]
Then \( g(w_0) = g(9) = 3 = z_0 \) and \( f(g(w)) = w \).

iv) For \( z_0 = -4 \), \( f(z_0) = 16 = w_0 \) and let \( g(w) = w^{1/2} \) where \( w^{1/2} \) is the branch
\[ w^{1/2} = \sqrt{|w|} e^{i\theta/2}, \; |w| > 0, \; \pi < \theta < 3\pi. \]
Then \( g(w_0) = g(16) = 4e^{2\pi i} = -4 \) and \( f(g(w)) = w \).

***

You may further check your understanding of local inverses while doing the following exercises.

E8) Find the local inverse of the transformation \( w = 1/z \) at the point \( z_0 = 1 + i \).

E9) Find the local inverse of the transformation \( w = e^z \) at the origin.

In Unit 2 we introduced you to harmonic function. It is a real-valued function say, \( u(x, y) \) of two real variables \( x \) and \( y \) defined in a given domain \( D \) having continuous first and second order partial derivatives in \( D \) that satisfy the Laplace’s equation \( u_{xx} + u_{yy} = 0 \) in \( D \). You saw there that if a function \( f = u + iv \) is analytic in a domain \( D \) then \( u \) and \( v \) are harmonic in \( D \). Also, if two functions \( u \) and \( v \) are harmonic in a domain \( D \) and their first-order partial derivatives satisfy the Cauchy-Riemann equations throughout \( D \), then \( v \) is called the harmonic conjugate of \( u \). In the next section we shall discuss the conjugate property of harmonic functions and their role in solving boundary value problems in applied mathematics.

9.3 HARMONIC FUNCTIONS

In Sec. 2.5 of Unit 2, we discussed a method of finding the harmonic conjugate of a function harmonic in a given domain. But is this always possible to find, in a given domain, the conjugate of a harmonic function? The answer is- No. If \( u \) is a harmonic function in a domain \( D \), then it is not necessary that it has a harmonic conjugate there. Consider for example, the function
\[ u(x, y) = \log(x^2 + y^2)/2. \]
It is harmonic in \( \mathbb{C} \setminus \{0\} \) because
\[ u_x = \frac{x}{x^2 + y^2}, \; u_y = \frac{y}{x^2 + y^2}. \]
\[ \Rightarrow \quad u_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \; u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}. \]
\[ \Rightarrow \quad u_{xx} + u_{yy} = 0. \]
If \( v \) is the harmonic conjugate of \( u \), then the function \( f = u + iv \) is analytic in \( \mathbb{C} \setminus \{0\} \) (see Theorem 7, Sec. 2.5, Unit 2). The principal branch of logarithm

\[
\text{Log } z = \ln |z| + i\theta, \quad -\pi < \theta < \pi
\]

is analytic in the domain \( G = \{z \in \mathbb{C} \setminus \{0\} : -\pi < \text{Arg} \ z < \pi\} \). This gives

\[
\text{Re } f = \text{Re } (\text{Log } z) \quad \text{on } G
\]

\[
\Rightarrow \quad \text{Re } (f - \text{Log } z) = 0 \quad \text{in } G \quad \text{and } \quad f - \text{Log } z \quad \text{is analytic in } G
\]

\[
\Rightarrow \quad v = \text{Arg } z + a \quad \text{constant in } G
\]

which doesn’t extend continuously to \( \mathbb{C} \setminus \{0\} \). You know that \( \text{Arg } z \) is not continuous at the points on the negative real axis. Thus, no such function \( v \) exists.

We shall now determine the condition under which a harmonic function in a given domain has a harmonic conjugate.

You may recall from Unit 4 that a domain \( D \) with the property that every simple closed contour within \( D \) encloses only points of \( D \) is said to be a simply connected domain. Moreover, an analytic function \( f \) in a simply connected domain \( D \) must have an antiderivative in \( D \). That is, there exists an analytic function \( F \) defined in \( D \) such that \( F' (z) = f(z) \) for all \( z \in D \) (see Sec.4.5, Unit 4). Moreover, \( F \) has the form

\[
F(z) = \int_{z_0}^{z} f(s) \, ds
\]

where the value of the integral is independent of the contour lying in the domain \( D \) extending from a fixed point \( z_0 \) to \( z \).

Having recalled the above results, we are now ready to prove that if \( u \) is a harmonic function defined in a simply connected domain \( D \), then \( u \) has a harmonic conjugate \( v \) in \( D \).

Let us consider the function \( f(z) = u_x - iu_y = U + iV \), where \( U = u_x \) and \( V = -u_y \). Since \( u \) is harmonic, therefore \( U \) and \( V \) have continuous first-order partial derivatives in \( D \) and they satisfy \( U_x = V_y \) and \( U_y = -V_x \). By the sufficient condition of differentiability (Theorem 5, Sec. 2.3, Unit 2), \( f \) is analytic in \( D \). Since \( D \) is simply connected, \( f \) has an antiderivative. That is, there exists an analytic function \( F \) defined in \( D \) such that \( F'(z) = f(z) \) for all \( z \in D \). If we write \( F = A + iB \), then

\[
u_x - iu_y = f(z) = F'(z) = A_x + iB_x = A_x - iA_y \quad \text{(Why?, because } A \text{ and } B \text{ satisfy the Cauchy-Riemann equations)}.
\]

On comparing the real and imaginary parts of \( f(z) = F'(z) \), we get

\[
u_x = A_x \quad \text{and} \quad u_y = A_y.
\]  

(7)

Integrating the first equality of Eqn. (7) with respect to \( x \), treating \( y \) constant, we get

\[
A(x, y) = u(x, y) + C(y)
\]  

(8)

Differentiating Eqn. (8) with respect to \( y \) and using the second equality of Eqn. (7) in the resulting equation, we obtain

\[
C'(y) = 0 \Rightarrow C \quad \text{is a constant}.
\]
This implies that $A(x, y) = u(x, y) + C$. Hence $F(z) - C$ is an analytic function with $\text{Re}(F(z) - C) = u(x, y)$. It follows that $B$ is a harmonic conjugate of $u$. Thus, we have proved the following theorem.

Theorem 5: If $u$ is a harmonic function defined in a simply connected domain $D$, then $u$ must have a harmonic conjugate in $D$.

We now illustrate Theorem 5 through examples.

Example 7: Prove that the following functions are harmonic and hence find their harmonic conjugates:

i) $u(x, y) = x^3 - 3xy^2$, ii) $u(x, y) = e^x \cos y$ and iii) $u(x, y) = x^2 - y^2$.

**Solution:** The given three functions have continuous partial derivatives of all orders. Therefore, in order to prove that a function is harmonic, it suffices to show that it satisfy the Laplace’s equation. Further, to find the harmonic conjugates we follow Theorem 5. We define the function $f(z) = u_x - iu_y$ and determine the corresponding antiderivative function in each case.

i) $u(x, y) = x^3 - 3xy^2$ satisfies the Laplace’s equation since

$$u_{xx} + u_{yy} = 6x - 6x = 0.$$  

Here $f(z) = u_x - iu_y = (3x^2 - 3y^2) + 6xyi = 3z^2$. Function $f$ has an antiderivative $F$, given by

$$F(z) = \int_0^z f(s) ds = z^3 = u(x, y) + iv(x, y),$$

where $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$. Thus $v$ is the required harmonic conjugate of $u$.

ii) $u(x, y) = e^x \cos y$ satisfies the Laplace’s equation

$$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$$  

$f(z) = u_x - iu_y = e^x \cos y + ie^x \sin y = e^z$.

The antiderivative function $F$ is given by

$$F(z) = \int_0^z f(s) ds = e^z - 1.$$  

Thus the function $F(z) + 1 = e^z$ is analytic with $\text{Re}(F + 1) = u = e^x \cos y$ and the harmonic conjugate $u(x, y) = \text{Im}(F + 1) = e^x \sin y$.

iii) $u_{xx} + u_{yy} = 2 - 2 = 0$. Thus $u(x, y)$ satisfies the Laplace’s equation.

Here $f(z) = 2z$ and hence the anti-derivate function $F(z) = z^2$. The harmonic conjugate is $v(x, y) = 2xy$.

Example 8: Show that if $v$ is the harmonic conjugate of $u$ in a domain $D$, then $-u$ is the harmonic conjugate of $v$ in $D$. Give an example to show that if $v$ is the harmonic conjugate of $u$, then $u$ need not be the harmonic conjugate of $v$.

**Solution:** Since $v$ is the harmonic conjugate of $u$ in $D$, therefore $u$ and $v$ are
harmonic in $D$ and satisfy the Cauchy-Riemann equations $u_y = -v_x$. Let $U = v$ and $V = -u$ then we need to show that $V$ is the harmonic conjugate of $U$. Note that $U$ and $V$ are harmonic in $D$ and

$$U_x = v_x, \quad U_y = v_y, \quad V_x = -u_x, \quad V_y = -u_y$$

$$\Rightarrow \quad U_x = V_y \quad \text{and} \quad U_y = -V_x.$$  
Hence $V$ is the harmonic conjugate of $U$.

Further, you know that if we have the functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy, \quad v$ is the harmonic conjugate of $u$ because $f = u + iv = z^2$ is an entire function. However, the function $v + iu$ is not analytic anywhere (the corresponding Cauchy-Riemann equations are not satisfied). Consequently, $u$ is not a harmonic conjugate of $v$.

The following exercises will further strengthen your concepts about the harmonic conjugates.

---

**E10)** Show that the function $u(x, y) = \tan^{-1}(y/x)$ is harmonic in the domain $D = \{(x, y) : x > 0\}$ and find its harmonic conjugate.

**E11)** Find the constant $b$ such that the function $u(x, y) = x^2 + 2xy + by^2$ is harmonic in the complex plane. Hence find an entire function $F$ such that $ReF = u$.

**E12)** Show that the function $u(x, y) = x - e^y \sin y$ is harmonic in the complex plane. Find its harmonic conjugate as well.

---

In applied mathematics, the problem of finding a function that is harmonic in a specified domain and satisfies the prescribed conditions on the boundary of the domain is of significant importance. One of the boundary value problems is a Dirichlet Problem in which the values of the function are prescribed along the boundary of the domain. Another similar problem is a Neumann Problem in which the values of the normal derivative of the function are prescribed on the boundary. Here, we shall restrict our attention to Dirichlet problem only.

**Dirichlet Problem**

Let us consider the problem of the steady-state temperature distribution $T(x, y)$ in a thin, homogeneous semi-infinite plate (solid) in the $xy$-plane having no heat sources or sinks and whose edges are insulated. Most of the times, the domain specified in such problems is simply connected and in view of Theorem 5, a function harmonic in a simply connected domain always has the harmonic conjugate. Solutions of the boundary value problems for such domains are the real or imaginary parts of analytic functions. For instance, the function $T(x, y) = e^{-y} \sin x$ is harmonic and satisfies a Dirichlet problem in the semi-infinite vertical strip $0 < x < \pi, \quad y > 0$. We have

$$T_x = e^{-y} \cos x, \quad T_y = -e^{-y} \sin x$$
\[ T_{xx} = -e^{-\gamma} \sin x, \quad T_{yy} = e^{-\gamma} \sin x \]

\[ T_{xx}(x, y) + T_{yy}(x, y) = 0 \]

Moreover, it satisfies the following boundary conditions (see Fig. 10):

\[ T(0, y) = 0, \quad T(\pi, y) = 0 \]

\[ T(x, 0) = \sin x, \quad \lim_{y \to \infty} T(x, y) = 0. \]

You may note here that the harmonic function \( T(x, y) \) is the imaginary part of the entire function \( e^{ic} \). We shall discuss such similar problems, in detail in the next unit.

Our concern is to find the solution of the boundary value problems by making use of the harmonic conjugates. Given an analytic function mapping a domain \( D_\zeta \) in the \( \zeta \)-plane onto a domain \( D_w \) in the \( w \)-plane and a harmonic function on \( D_w \), the problem of finding the harmonic function defined on \( D_\zeta \) is taken care by the following theorem.

**Theorem 6:** Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function that maps a domain \( D_\zeta \) in the \( \zeta \)-plane onto a domain \( D_w \) in the \( w \)-plane. If \( h(u, v) \) is a harmonic function defined in the domain \( D_w \), then the function \( H \) defined by

\[ H(x, y) = h(u(x, y), v(x, y)) \]

is harmonic in the domain \( D_\zeta \).

**Proof:** **Case 1:** Let us assume that \( D_w \) is a simply connected domain. By Theorem 5, the function \( h(u, v) \) has a harmonic conjugate \( g(u, v) \) in \( D_w \). Consequently, the function \( \psi(w) = h(u, v) + ig(u, v) \) is analytic in \( D_w \). Since \( f(z) \) is analytic in \( D_\zeta \), therefore the composition \( \psi(f(z)) \) is also analytic in \( D_\zeta \). Thus

\[ \Re \psi(f(z)) = h(u(x, y), v(x, y)) = H(x, y) \]

is harmonic in \( D_\zeta \).

**Case 2:** Let us assume that \( D_w \) is not a simply connected domain. Let \( z_0 \in D_\zeta \). Then \( w_0 = f(z_0) \in D_w \). Since \( D_w \) is open, there exists a neighbourhood \( N_\varepsilon = \{w : |w - w_0| < \varepsilon\} \) of \( w_0 \) such that \( N_\varepsilon \subseteq D_w \). You may note that \( N_\varepsilon \) is
Applications of Analytic Functions

simply connected. By applying Case 1 above, to the simply connected domain \( N_\varepsilon \), the function \( \psi(w) = h(u, v) + ig(u, v) \) is analytic in \( N_\varepsilon \), where \( g(u, v) \) is the harmonic conjugate of \( h(u, v) \) in \( N_\varepsilon \). By the continuity of \( f \) at \( z_0 \), there exists a \( \delta > 0 \) such that \( |f(z) - f(z_0)| < \varepsilon \) whenever \( |z - z_0| < \delta \). In other words,

\[
f(N_\delta) \subseteq N_{\varepsilon}
\]

where \( N_\delta = \{ z : |z - z_0| < \delta \} \) (see Fig. 11).

![Fig. 11](image)

The composition \( \psi(f(z)) \) is analytic in \( N_\delta \) (because if \( z \in N_\delta \), then \( f(z) \in N_{\varepsilon} \)) so that \( \text{Re}\psi(f(z)) = h(u(x, y), v(x, y)) = H(x, y) \) is harmonic in \( N_\delta \). Since \( z_0 \in D_z \) is arbitrary, this argument is valid for each point of \( D_z \).

Therefore the function \( H(x, y) = h(u(x, y), v(x, y)) \) is harmonic in the domain \( D_z \).

We now consider examples to illustrate Theorem 6 in solving the boundary value problems. A better visualization of the real and imaginary components of analytic functions and their mapping properties will be needed in the following examples.

**Example 9:** Show that the function \( H(x, y) = e^{-2xy} \sin(x^2 - y^2) \) is harmonic in the first quadrant \( x > 0 \) and \( y > 0 \).

**Solution:** If \( f(z) = z^2 \), then \( f \) is analytic and \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = 2xy \). By making use of the polar coordinates you may check that \( f \) maps the domain \( D_z \) in the \( z \)-plane onto a domain \( D_w \) in the \( w \)-plane, where \( D_z = \{ z = x + iy : x > 0 \text{ and } y > 0 \} \) and \( D_w = \{ w = u + iv : v > 0 \} \) (see E13)). The real-valued function \( h(u, v) = e^{-v} \sin u \) is harmonic in \( D_w \) (for details, see the discussion preceding Theorem 6). By Theorem 6, it follows that the function \( H(x, y) = h(u(x, y), v(x, y)) = e^{-2xy} \sin(x^2 - y^2) \) is harmonic in the domain \( D_z \).

***
Example 10: Show that the function \( H(x, y) = e^{2x} \cos(2y) \) is harmonic in the horizontal strip \( 0 < y < \pi \).

Solution: Here we need to define the analytic function \( f \) and harmonic function \( h(u, v) \) such that \( H(x, y) = e^{2x} \cos(2y) = (e^x \cos y)^2 - (e^x \sin y)^2 \) is harmonic in the horizontal strip \( 0 < y < \pi \). We know that \( e^x \cos y \) and \( e^x \sin y \) are the components of the exponential function \( f(z) = e^z \). Thus we consider the analytic function \( f(z) = e^z \) satisfying \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \). You may check that the function \( f \) maps the domain \( D_z \) in the \( z \)-plane onto a domain \( D_w \) in the \( w \)-plane, where \( D_z = \{ z = x + iy : 0 < y < \pi \} \) and \( D_w = \{ w = u + iv : v > 0 \} \) (see E14)). The real-valued function \( h(u, v) = u^2 - v^2 \) is harmonic in \( D_w \) because

\[
h_u = 2u, \quad h_v = -2v \Rightarrow h_{uu} = 2, \quad h_{vv} = -2 \Rightarrow h_{uu} + h_{vv} = 0.
\]

By Theorem 6 the function \( H(x, y) = h(u(x, y), v(x, y)) = e^{2x} \cos^2 y - e^{2x} \sin^2 y = e^{2x} \cos(2y) \) is harmonic in the domain \( D_z \).

***

We now consider a theorem which shows that in a Dirichlet boundary value problem the values of the function prescribed along the boundary remain unaltered under a conformal transformation.

Theorem 7: Suppose that the transformation \( w = f(z) = u(x, y) + iv(x, y) \) is conformal on a smooth arc \( C \) and let \( \Gamma = f(C) \). If a function \( h(u, v) \) satisfies the condition:

\[
h = h_0 \text{ along } \Gamma
\]

then the function \( H(x, y) = h(u(x, y), v(x, y)) \) satisfies the corresponding condition

\[
H = h_0 \text{ along } C.
\]

We are leaving the proof of the theorem for you to do it yourself (see E15)). However, we take up an example to illustrate the theorem.

Example 11: Verify Theorem 7 for the transformation \( w = f(z) = e^z \), smooth arc \( C : x = 0, \ 0 \leq y \leq \pi \) and the function \( h(u, v) = 2 - u + u/(u^2 + v^2) \).

Solution: Since \( f'(z) \neq 0 \), therefore by Theorem 1, \( f \) is conformal on the smooth arc \( C \). Let \( \Gamma = f(C) \). For \( z = x + iy \in C \), \( x = 0 \) and

\[
0 \leq y \leq \pi, \ w = u + iv = f(z) = e^x(\cos y + i\sin y) \text{ takes the form } (\cos y + i\sin y).
\]

This gives

\[
u = \cos y, \ v = \sin y, \ 0 \leq y \leq \pi
\]

\[
\Rightarrow v \geq 0 \text{ and } u^2 + v^2 = 1.
\]

Thus \( \Gamma \) represents the semi-circle of radius 1 in the upper half-plane. Along \( \Gamma, \ h = 2 \). On simplification, the function \( H(x, y) \) takes the form

\[
H(x, y) = h(u(x, y), v(x, y)) = 2 - e^x \cos y + e^{-x} \cos y.
\]

On the arc \( C \), \( H(x, y) = 2 \) since \( x = 0 \). This verifies Theorem 7.

***
Applications of Analytic Functions

You may now try the following exercises.

E13) Show that the function \( w = f(z) = z^2 \) maps \( \{ z = x + iy : x > 0, y > 0 \} \) onto \( \{ w = u + iv : v > 0 \} \).

E14) Show that the function \( w = f(z) = e^z \) maps \( \{ z = x + iy : 0 < y < \pi \} \) onto \( \{ w = u + iv : v > 0 \} \).

E15) Prove Theorem 7.

E16) Verify Theorem 7 for the function \( f(z) = iz^2, h(u, v) = 2v + 3 \) and the smooth arc \( C : y = x, x > 0 \).

We now end this unit by giving a summary of what we have covered in it.

9.4 SUMMARY

In this unit, we have covered the following:

1) A transformation \( w = f(z) \) is **conformal at a point** \( z_0 \) if it preserves angles, that is, the angle between any two smooth arcs \( C_1, C_2 \) intersecting at \( z_0 \) is the same in **magnitude** and **sense** as the angle between their images \( \Gamma_1, \Gamma_2 \) intersecting at \( f(z_0) \).

2) An analytic function \( f \) is conformal at a point \( z_0 \) if and only if \( f'(z_0) \neq 0 \).

3) A function that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called **isogonal**.

4) If a function \( f \) is analytic at \( z_0 \) and \( f'(z_0) = 0 \), then the point \( z_0 \) is a **critical point** of \( f \).

5) If an analytic function \( f \) is conformal at a point \( z_0 \), then its angle of rotation at \( z_0 \) is \( \arg(f'(z_0)) \) and its **scale factor** at \( z_0 \) is \( |f'(z_0)| \).

6) If an analytic function \( f \) is conformal at a point \( z_0 \), then the transformation \( w = f(z) \) has a **local inverse** at \( z_0 \), that is, there exists a unique transformation \( z = g(w) \), which is defined and analytic in a neighbourhood \( N \) of \( w_0 = f(z_0) \) such that \( g(w_0) = z_0 \) and \( f(g(w)) = w \) for all points \( w \) in \( N \). Moreover, \( g'(w) = 1/f'(z) \) for all points \( w \) in \( N \).

7) If \( u \) is a **harmonic** function defined in a simply connected domain \( D \), then \( u \) must have a harmonic conjugate in \( D \).

8) The **Dirichlet’s boundary value problem** is the problem of finding a function that is harmonic in a specified domain and the values of the function are prescribed along the boundary of the domain.
9) Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function that maps a domain \( D_z \) in the \( z \)-plane onto a domain \( D_w \) in the \( w \)-plane. If \( h(u, v) \) is a harmonic function defined in the domain \( D_w \), then the function \( H(x, y) = h(u(x, y), v(x, y)) \) is harmonic in \( D_z \).

10) If a transformation \( w = f(z) = u(x, y) + iv(x, y) \) is conformal on a smooth arc \( C, \Gamma = f(C) \) and a function \( h(x, y) \) satisfies the condition \( h = h_0 \) along \( \Gamma \), then the function \( H(x, y) = h(u(x, y), v(x, y)) \) satisfies the corresponding condition \( H = h_0 \) along \( C \).

9.5 SOLUTIONS/ANSWERS

E1) Let the two level curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) intersect at the point \( z_0 = x_0 + iy_0 \) where \( f'(z_0) \neq 0 \). If \( L_1 \) be the tangent line to the level curve \( u(x_0, y_0) = c_1 \) at \( z_0 \) and \( L_2 \) be the tangent line to the level curve \( v(x_0, y_0) = c_2 \) at \( z_0 \), then we need to show that the lines \( L_1 \) and \( L_2 \) are perpendicular at \( z_0 \). Using the chain rule of partial differentiation, we have

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0
\]

\[
\therefore \text{The slope of the line } L_1 = \frac{dy}{dx} = \frac{-u_x}{u_y} \quad \text{and the slope of the line } L_2 = \frac{dy}{dx} = \frac{-v_x}{v_y}.
\]

Using the Cauchy-Riemann equations \( u_x = v_y, u_y = -v_x \) at \( (x_0, y_0) \), the product of the two slopes is

\[
\left( \frac{-u_x}{u_y} \right) \left( \frac{-v_x}{v_y} \right) = \left( \frac{v_y}{u_x} \right) \left( \frac{-v_x}{v_y} \right) = -1.
\]

Hence the lines \( L_1 \) and \( L_2 \) are perpendicular at \( z_0 \).

E2) i) For the function \( f(z) = \sin z, \ f'(z) = \cos z \) which equals zero \( \iff z = (2n+1)\pi/2 \ (n \in \mathbb{Z}) \). Therefore the analytic function \( f(z) = \sin z \) is conformal at all points except \( z = (2n+1)\pi/2 \ (n \in \mathbb{Z}) \) by Theorem 3.

ii) For the function \( f(z) = \cosh z, \ f'(z) = \sinh z = 0 \iff z = n\pi i \ (n \in \mathbb{Z}) \). Consequently, \( f \) is conformal at all points except \( z = n\pi i \ (n \in \mathbb{Z}) \).

iii) Function \( f(z) = e^z + 100 \) is conformal everywhere since \( f'(z) = e^z \neq 0 \) for all \( z \in \mathbb{C} \).

E3) i) The function \( f(z) = \frac{1}{z} \) is conformal at every non-zero point and \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = x/(x^2 + y^2) \) and
Applications of Analytic Functions

\[ v(x, y) = -y/(x^2 + y^2). \]

The level curves of \( u \) and \( v \) passing through a point \( z_0 = x_0 + iy_0 \neq 0 \) are given by \( x/(x^2 + y^2) = c_1 \) and \( -y/(x^2 + y^2) = c_2 \) (see Fig. 12).

![Level curves of \( f(z) = 1/z \)](image)

As \( z_0 \neq 0 \), \( x_0 \) and \( y_0 \) are not both zero simultaneously. As a result, the same is true for \( c_1 \) and \( c_2 \). To prove that they intersect orthogonally, consider the following three cases:

**Case 1:** If \( c_1 = 0 \), then \( x_0 = 0, c_2 \neq 0 \) and \( z_0 = -i/c_2 \) lies on the imaginary axis. Consequently, the level curves reduce to the curves \( x = 0 \) and \( -y/(x^2 + y^2) = c_2 \) which intersect orthogonally at \( z_0 \).

**Case 2:** If \( c_2 = 0 \), then \( y_0 = 0, c_1 \neq 0 \) and \( z_0 = 1/c_1 \) lies on the real axis. Consequently, the level curves simplify to the curves \( x/(x^2 + y^2) = c_1 \) and \( y = 0 \), which intersect orthogonally at \( z_0 \).

**Case 3:** If \( c_1 \) and \( c_2 \) are both non-zero, then the level curves represent the circles:

\[
\left(x - \frac{1}{2c_1}\right)^2 + y^2 = \frac{1}{4c_1^2}
\]

and

\[
x^2 + \left(y + \frac{1}{2c_2}\right)^2 = \frac{1}{4c_2^2},
\]

respectively. Differentiating these equations with respect to \( x \) and obtaining the slopes of the tangent lines at \( z_0 \), we get

\[
m_1 = -\left(\frac{x_0 - 1/2c_1}{y_0}\right) \quad \text{and} \quad m_2 = -\left(\frac{x_0}{y_0 + 1/2c_2}\right)
\]
\[
\Rightarrow \quad m_1 m_2 = \frac{x_0}{y_0} \left( \frac{x_0 - i/2c_1}{y_0 + i/2c_2} \right).
\]

Since \( x_0/(x_0^2 + y_0^2) = c_1 \) and \(-y_0/(x_0^2 + y_0^2) = c_2 \), it is a simple computation to show that \( m_1 m_2 = -1 \). Hence the curves intersect orthogonally at \( z_0 \).

ii) The function \( f(z) = e^z \) is conformal everywhere and \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \). The level curves of \( u \) and \( v \) passing through a point \( z_0 = x_0 + iy_0 \) are given by \( e^x \cos y = c_1 \) and \( e^x \sin y = c_2 \).

Directly differentiate these equations with respect to \( x \) and obtain the slopes of the tangent lines at \( z_0 \) as

\[
m_1 = \cot y \quad \text{and} \quad m_2 = -\tan y
\]

so that \( m_1 m_2 = -1 \).

E4) An arc \( z = z(t)(a \leq t \leq b) \) is said to be smooth if the derivative \( z'(t) \) is continuous on the closed interval \( a \leq t \leq b \) and non-zero on the open interval \( a < t < b \).

Let \( C \) be the smooth arc defined by \( z = z(t)(a \leq t \leq b) \) (see Fig. 13).

By definition \( z'(t) \) is continuous on the closed interval \( a \leq t \leq b \) and \( z'(t) \neq 0, a < t < b \). We have \( \omega(t) = f(z(t)) \) be the function defined at all points of \( C \). The image of \( C \) is \( \Gamma \) under this transformation and \( \omega'(t) = f'(z(t))z'(t)(a \leq t \leq b) \).

Since \( z'(t) \neq 0 \) and \( f \) is conformal \( f'(z(t)) \neq 0 \), therefore, \( \omega'(t) \neq 0, a < t < b \) and \( \omega'(t) \) is continuous. Hence, arc \( \Gamma \) is smooth.

E5) Given transformation is \( f(z) = w = 1/z \). This function is analytic everywhere except at \( z = 0 \). Now \( f'(z) = -1/z^2 \) and at \( z = i \), \( f'(i) = 1 \) and \( \arg (f'(i)) = 0 \).

E6) For the transformation \( w = 1/z \). \( u = \frac{x}{x^2 + y^2} \), \( v = \frac{-y}{x^2 + y^2} \) and
Applications of Analytic Functions

\[ x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}. \]

Now for \( y = x - 1 \)

\[ \frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1 \]

\[ \Rightarrow -v = u - u^2 - v^2 \Rightarrow u^2 + v^2 - u - v = 0 \]

\[ \text{Fig. 14} \]

It is a circle with centre \((1/2, 1/2)\) and radius \(1/\sqrt{2}\) (see Fig. 14).

For \( y = 0, \quad -\frac{v}{u^2 - v^2} = 0 \Rightarrow v = 0 \), a straight line.

Observe that \( \arg f'(1) = \arg(-1) = \pi \).

E7) Clearly the angle between \( C_1 \) to \( C_2 \) is \( \pi/2 \) at the point of intersection \( z_0 = 1 \). Also, \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \). Let \( \Gamma_1, \Gamma_2 \) be the images of \( C_1 \) and \( C_2 \) under \( f \) at the point of intersection \( w_0 = f(z_0) = e \) in the \( w \)-plane (see Fig. 15).

\[ \text{Fig. 15} \]

It is easy to see that the parametric representation of \( \Gamma_1 \) is

\[ u = e^x \text{ and } v = 0 \quad (x \geq 0) \]

\[ \Rightarrow u \geq 1 \text{ and } v = 0 \]
representing the strip $[1, \infty)$ on the $u$-axis in the $w$-plane. Similarity, the parametric representation of $\Gamma_2$ is

$$u = e \cos y \quad \text{and} \quad v = e \sin y \quad (y \geq 0)$$

$$\Rightarrow u^2 + v^2 = e^2$$

which represents the circle centered at origin and radius $e$ in the $w$-plane in the positive sense. If $\phi_1$ and $\phi_2$ are the angles of inclination of the directed tangent lines to $\Gamma_1$ and $\Gamma_2$, respectively at $w_0$, then $\phi_1 = 0$ and $\phi_2 = \pi / 2$ (Think!). Hence the angle from $\Gamma_1$ to $\Gamma_2$ at the point $w_0$ is also $\phi_2 - \phi_1 = \pi / 2$. This verifies the conformality. The angle of rotation of $f$ at the point $z_0 = 1$ is $0$. Which is one of the values of $\arg(f'(z_0))$ and the scale factor at the point $z_0 = 1$ is $|f'(z_0)| = e$.

E8) Here $f(z) = 1/z$ and $z_0 = 1 + i$. So $f(z_0) = (1 - i)/2 = w_0$. Let $g(w) = 1/w$. Then $g(w_0) = z_0$ and $f(g(w)) = w$.

E9) Let $z_0 = 0$ and $f(z) = e^z$. Then $f(z_0) = 1 = w_0$. Let $g(w) = \Log w$ where $\Log w$ is the principal branch

$$\Log w = \ln |w| + i \theta, \quad |w| > 0 \quad -\pi < \theta < \pi.$$

This gives $g(w_0) = 0 = z_0$ and $f(g(w)) = w$.

E10) For the function $u(x, y) = \tan^{-1}(y/x)$, we have

$$u_x = -\frac{y}{x^2 + y^2}, \quad u_y = \frac{x}{x^2 + y^2}$$

$$\Rightarrow u_{xx} = -\frac{2xy}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow u_{xx} + u_{yy} = 0.$$

Thus $u$ is harmonic in $D$. Following Theorem 5, $f(z) = u_x - iu_y$ takes the form

$$f(z) = -\frac{y - ix}{x^2 + y^2} = \frac{-i(x - iy)}{x^2 + y^2} = -\frac{i\pi}{1z^2} = -\frac{i}{z}.$$

This function $f$ has an antiderivative $F$ given by

$$F(z) = \int_1^z f(s)ds = -i \log z = u(x, y) + iv(x, y)$$

where $u(x, y) = \tan^{-1}(y/x)$ and $v(x, y) = -\log(x^2 + y^2)/2$. Thus $v$ is the required harmonic conjugate of $u$.

E11) Let $u(x, y) = x^2 + 2xy + by^2$. Then

$$u_x = 2x + 2y, \quad u_y = 2x + 2by \Rightarrow u_{xx} = 2 \quad \text{and} \quad u_{yy} = 2b.$$ Therefore Laplace equation is satisfied provided $b = -1$. Using the notations of Theorem 5, we obtain

$$f(z) = u_x - iu_y = 2(x + y) - 2i(x - y) = 2(1 - i)z$$

and the required analytic function $F$ with $Re F = u$ is
Applications of Analytic Functions

\[ F(z) = \oint_C f(s) ds = (1 - i)z^2. \]

E12) The function \( u(x, y) = x - e^x \sin y \) satisfies
\[ u_x = 1 - e^x \sin y, \quad u_y = -e^x \cos y \]
\[ \Rightarrow u_{xx} = -e^x \sin y, \quad u_{xy} = e^x \sin y \]
\[ \Rightarrow u_{xx} + u_{yy} = 0. \]

To find the harmonic conjugate, note that
\[ f(z) = u_x - iu_y = 1 - e^x \sin y + ie^x \cos y = 1 + ie^x. \]
So that the antiderivative function is
\[ F(z) = \oint_C f(s) ds = z + i(e^z - 1). \]
Clearly, \( \text{Re } F = u \) and the conjugate of \( u \) is \( \text{Im } F = y + e^x \cos y - 1. \)

E13) If \( z \) is a point in the first quadrant, then its polar representation is
\[ z = re^{i\theta} \text{ where } r > 0 \text{ and } 0 < \theta < \pi/2. \]
Hence
\[ w = f(z) = z^2 = r^2e^{2i\theta} = \rho e^{i\phi} \]
where \( \rho = r^2 \) and \( \phi = 2\theta \) satisfies \( \rho > 0 \) and \( 0 < \phi < \pi \). Thus \( w = f(z) \) lies in the upper-half of \( w \)-plane.

E14) If \( z = x + iy \) where \( 0 < y < \pi \), then
\[ w = f(z) = e^x e^{iy} = \rho e^{i\phi} \]
where \( \rho = e^x \) and \( \phi = y \) satisfies \( \rho > 0 \) and \( 0 < \phi < \pi \). Clearly, \( w = f(z) \) lies in the upper-half of \( w \)-plane.

E15) Let \( h = h_0 \) along \( \Gamma \). If \( (x, y) \in C \), then \( (u(x, y), v(x, y)) \in \Gamma \). By hypothesis,
\[ h(u(x, y), v(x, y)) = h_0 \Rightarrow H(x, y) = h_0 \text{ along } C. \]

E16) The Function \( f(z) = iz^2 = u(x, y) + iv(x, y) = -2xy + i(x^2 - y^2) \) is conformal on the smooth arc \( C : y = x(x > 0) \). Let \( \Gamma = f(C) \). For \( z = x + iy, \ y = x, \ x > 0 \)
\[ u = -2x^2, \ v = 0 \Rightarrow u < 0, \ v = 0 \]
Thus \( \Gamma \) represents the negative \( u \)-axis. Along \( \Gamma, \ h = 3 \). Also, the function \( H(x, y) \) takes the form
\[ H(x, y) = h(u(x, y), v(x, y)) = 2(x^2 - y^2) + 3. \]
On \( C \), it is easily seen that \( H = 3. \)