$$\int_{\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx = \frac{\pi}{e}.$$

E11) We write $f(z) = \frac{(z+1)e^{iz}}{z^2+4z+5} = \frac{(z+1)e^{iz}}{(z+2+i)(z+2-i)}$.

The point $z=-2+i$, lying in the upper-helf plane, is a simple pole of f with residue

$$b = e^{-1}(1+i)(\cos 2 - i\sin 2).$$

For $r > \sqrt{5} = |-2+i|$ and C_r denoting the upper half of the positively oriented circle $|z| = r$ (see Fig. 10) the Cauchy residue theorem yields

$$\int_{-r}^{r} \frac{(x+1)e^{ix}\,dx}{x^2+4x+5} = 2\pi ib - \int_{C_r} f(z)\,dz.$$

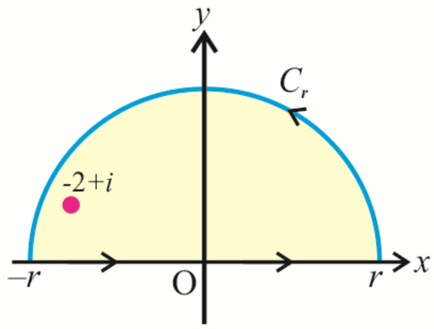

Fig. 10

We shall show that $\left|\int_{C_r} f(z)\,dz\right| \to 0$ as $r\to\infty$. When z is a point on C_r

$$|f(z)| = \left|\frac{(z+1)e^{iz}}{z^2+4z+5}\right| \le \frac{(r+1)}{(r-\sqrt{5})^2}|e^{iz}|$$

and $|e^{iz}| \le 1$ for such a point z. We cannot conclude that integral of $f(z)$ along $C_r \to 0$ as $r\to\infty$ (because $\frac{\pi r(r+1)}{(r-\sqrt{5})^2} \not\to 0$ as $r\to\infty$).

If we put $z = re^{i\theta}$ where $(0\le\theta\le\pi, r>\sqrt{5})$, we get for z on C_r (using Jordan Inequality)

$$\left|\int_{C_r} f(z)\right| \le r\int_0^{\pi} |f(re^{i\theta})|\,d\theta$$

$$\le \frac{r(r+1)}{(r-\sqrt{5})^2}\int_0^{\pi} e^{-r\sin\theta}\,d\theta$$

$$\le \frac{2r(r+1)}{(r-\sqrt{5})^2}\int_0^{\pi/2} e^{-r\sin\theta}\,d\theta \le \frac{\pi(r+1)}{(r-\sqrt{5})^2}$$

and, we have

$$\left|\int_{C_r} f(z)\,dz\right| \to 0 \text{ as } r\to\infty. \text{ Thus,}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi ib.$$

Hence, by equating the real parts

$$P.V.\int_{-\infty}^{\infty} \frac{(x+1)\cos x\,dx}{x^2+4x+5} = Re[2\pi ib] = \frac{\pi(\sin 2-\cos 2)}{e}.$$

E12) i) Let $f(z) = \frac{e^{aiz}}{(z^2+b^2)^2}dz$. The Singularities of this function are at the points $z=\pm bi$. Hence, $z=bi$ is a double pole lying in the upper-half plane. Residue at this point is given by

$$B = \lim_{z \to bi} \frac{d}{dz}(z - bi)\frac{e^{aiz}}{(z^2 + b^2)^2} = \frac{-i}{4b^3}(1 + ab)e^{-ab}$$

Using Theorem 2

$$P.V. \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + b^2)^2}\, dx = -2\pi\, Im\, B = \frac{\pi}{2b^3}(1 + ab)\, e^{-ab}$$

Since the integrand is an even function and $P.V.$ exists therefore

$$\int_0^{\infty} \frac{\cos a\, x}{(x^2 + b^2)^2}\, dx = \frac{\pi}{4b^3}(1 + ab)\, e^{-ab}\,.$$

ii) $$\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + 1)\,(x^2 + 9)}\, dx = \frac{\pi}{16}\left(\frac{9}{e^3} - \frac{1}{e}\right).$$

E13) i) Consider the following contour γ (see Fig. 11) and the function

$$f(z) = \frac{e^{inz}}{z^4 + 1}\,.$$

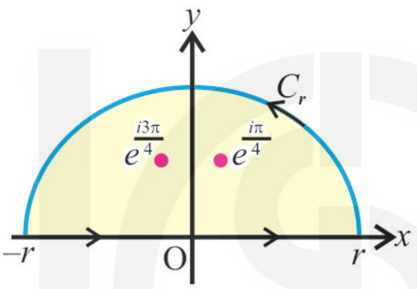

Fig. 11

We have $\left| \int_{C_r} \frac{e^{inz}}{z^4 + 1}\, dz \right| \le \int_{C_r} \frac{|e^{inz}|}{r^4 - 1}\, dz \le \frac{\pi r}{r^4 - 1} \to 0$ as $r \to \infty$

(note that for $z \in C_r, |e^{inz}| = e^{-Im\, nz} \le 1$).

Considering the real part and using the fact that for these roots $\frac{1}{z_1^3} = -z_1$, we have

$$\int_{-\infty}^{\infty} \frac{\cos nx}{x^4 + 1}\, dx = \lim_{r \to \infty} dx = \lim_{r \to \infty} Re\left(\int_{\gamma} \frac{e^{inz}}{z^4 + 1}\, dz \right)$$

$$= Re\left[2\pi i \left(\underset{z = e^{\frac{i\pi}{4}}}{Res} \frac{e^{inz}}{z^4 + 1} \right) + \left(\underset{z = e^{\frac{3i\pi}{4}}}{Res} \frac{e^{inz}}{z^4 + 1} \right) \right]$$

$$= -2\pi\;\; Im\left(\left[\frac{e^{inz}}{z^4 + 1} \right]_{z = e^{\frac{i\pi}{4}}} + \left[\frac{e^{inz}}{z^4} + 1 \right]_{z = e^{\frac{3i\pi}{4}}} \right)$$

$$= \frac{\pi e^{-\frac{n}{\sqrt{2}}}}{\sqrt{2}} \left[\cos\left(\frac{n}{\sqrt{2}}\right) + \sin\left(\frac{n}{\sqrt{2}}\right) \right].$$

ii) Consider the function $f(z) = \frac{e^{iz}}{(z + a)^2 + b^2}$. The singularities of $f(z)$ are $z = -a \pm bi$. Only $z = -a + bi$ lies in the upper-half plane and is a simple pole. Compute the residue at this point and use Theorem 2 to get the answer

$$P.V. \int_{-\infty}^{\infty} \frac{\cos a\, x}{(x + a)^2 + b^2}\, dx = \frac{\cos a}{be^{-b}}\,.$$

E14) Let us consider $f(z)=\dfrac{p(z)}{q(z)}$. Hence we have $p(z)=1$ and

$q(z)=(z-i)^2\,(z-1)$.

Note that $\deg q=3\geq \deg p+2$ as well as $f(z)$ has simple pole namely, $x=1$ lying on the real axis and a double pole $z=i$ lying in the upper half plane. Now we compute the residue of $f(z)$ at these poles.

$$Res[1, f(z)]=\lim_{z\to 1}\,(z-1)\,\frac{1}{(z-i)^2\,(z-1)}=\frac{i}{2}.$$

$$Res[i\,f(z)]=\lim_{z\to 1}\,\frac{d}{dz}\,(z-i)^2\,\frac{1}{(z-i)^2\,(z-1)}=-\frac{i}{2}.$$

Using Theorem 3, we get

$$P.V.\int_{-\infty}^{\infty}\frac{dx}{(x-i)^2\,(x-1)}=2\pi i\left(-\frac{i}{2}\right)+\pi i\left(\frac{i}{2}\right)=\frac{\pi}{2}.$$

E15) We choose a function $f(z)=\dfrac{e^{aiz}-e^{biz}}{z^2}$ whose real part is $\dfrac{\cos ax-\cos bx}{x^2}$. The singularity of $f(z)$ is $z=0$. The Laurent series expansion $f(z)$ about $z=0$ is given by

$$f(z)=\frac{e^{aiz}-e^{biz}}{z^2}=\frac{1}{z^2}\left[(a-b)iz-\frac{(a^2-b^2)}{2}z^2+\cdots\right]$$
$$=(a-b)i$$

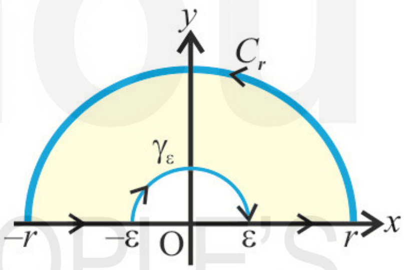

Fig. 12

So that the pole at $z=0$ is simple, with residue $(a-b)i$. Since the pole $z=0$ lies on the real axis, we make an indentation at 0 as shown in Fig.12. Our contour Γ consists of semicircle $C_r,(r>|a-b|)$ from r to $-r$, segment $[-r,-\varepsilon]$ $(0<\varepsilon<r)$, inner semicircle C_ε from $-\varepsilon$ to ε and segment $[\varepsilon, r]$. $f(z)$ is analytic inside and on Γ so, by the Cauchy-Goursat theorem

$$\int_{-r}^{-\varepsilon} f(x)dx-\int_{C_\varepsilon} f(z)\,dz+\int_{\varepsilon}^{r} f(x)\,dx+\int_{C_r} f(z)\,dz=0.$$

The first and the third integrals combine to give

$$\int_{\varepsilon}^{r}\frac{e^{-ax}-e^{-aix}}{x^2}dx+\int_{\varepsilon}^{r}\frac{e^{aix}-e^{bix}}{x^2}dx=2\int_{\varepsilon}^{r}\frac{\cos ax-\cos bx}{x^2}.$$

As we have seen that $z=0$ is a simple pole then applying indentation lemma, we get

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,dx=i(\pi-0)\,Res[0,\ f(z)]=\pi(b-a).$$

Finally, we have to estimate the integral $\int_{C_r} f(z)\,dz$. We see that on C_r

$$|\,f(z)\,|=\left|\frac{e^{aiz}-e^{biz}}{z^2}\right|\leq\frac{e^{-ay}+e^{-by}}{r^2}\leq\frac{2}{r^2},\quad (y\geq 0)$$

$$\Rightarrow\left|\int_{C_r} f(z)\,dz\right|\leq\frac{2\pi}{r}$$

$$\Rightarrow \left| \int_{C_r} f(z)\, dz \right| \to 0 \text{ as } r \to \infty$$

Letting $r \to \infty, \varepsilon \to 0$,

$$\int_0^\infty 2\frac{\cos ax - \cos bx}{x^2} dx = \pi(b-a) \Rightarrow \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a).$$

E16) Let us consider $f(z) = e^{iz} \dfrac{p(z)}{q(z)}$. Here we have $p(z) = 1$ and $q(z) = a^2 - z^2$. Note that $\deg q = 3 \geq \deg p + 1$. $f(z)$ has the simple poles, namely $z = \pm a$, lying on the real axis. Compute the residue of $f(z)$ at these poles

$$Res[a, f(z)] = \lim_{z \to a} (z-a) \frac{e^{iz}}{(a^2 - z^2)} = \frac{e^{ia}}{2a}.$$

$$Res[-a, f(z)] = \lim_{z \to -a} (z+a) \frac{e^{iz}}{(a^2 - z^2)} = \frac{e^{-ia}}{2a}.$$

$$\therefore P.V. \int_{-\infty}^{\infty} \frac{\cos x}{(a^2 - x^2)} dx = -\pi \left[\frac{-\sin a}{2a} + \frac{\sin(-a)}{2a} \right] = \frac{\pi \sin a}{a}.$$

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