UNIT 7 RESIDUES AND POLES

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7.1 INTRODUCTION

You have seen in Unit 2 that a point \( z_0 \) is a singular point or a singularity of a function \( f \) if \( f \) is not analytic at \( z_0 \) but every neighbourhood of \( z_0 \) contains at least one point at which \( f \) is analytic. In this unit our emphasis will be on developing tools to evaluate contour integrals of complex functions with singularities. We shall start with discussing in Sec. 7.2, the zeros and singularities of complex valued functions and use the Laurent series representation to classify these singularities as poles, isolated essential or removable singularities. We shall also establish here a result called identity theorem which gives the condition for two analytic functions to be identical. When a complex valued function \( f \) is analytic at all points interior to and on a simple closed contour \( C \) then you know from Cauchy-Goursat theorem that the value of the complex integral of the function around the contour \( C \) is zero. However, if \( f \) has one or more isolated singularities inside \( C \) then Cauchy’s theorem cannot be used and the value of the integral of the function \( f \) around \( C \) may not be zero. Each of these singular points inside \( C \) contributes to the value of the complex integral. These contributions are called **residues**. In Sec. 7.3 of the unit we shall develop the theory of residues. Finally, in Sec. 7.4 Cauchy’s residue theorem which gives the evaluation of the integrals in terms of the sums of the residues is discussed and used for evaluating certain integrals.

Objectives

After studying this unit, you should be able to:

- define and obtain the zeros of analytic functions;
- use the identity theorem to establish equality of two analytic functions when they agree on a set having a limit point;
- classify the singularities of complex valued functions as poles, essential or removable singularities;
- obtain the residues of a complex valued function at its singular points; and
- use the Cauchy’s residue theorem for evaluating contour integrals.
7.2 SINGULARITIES AND ZEROS

You know that a number \( z_0 \) is a zero of a function \( f \) if \( f(z_0) = 0 \). You have already seen in Unit 3 on elementary functions of complex variables that their zeros are the same as that of their real counterparts. That is, their zeros lie on the real axis. They are isolated and countable. We shall now see whether this be the case, in general, for the complex valued functions.

7.2.1 Zeros of Analytic Functions

Let the function \( f \) be analytic in a domain \( D \) and \( p \in D \). Then the function \( f \) has a zero of order \( k \) at \( z = p \) if

\[
f(p) = 0, \quad f'(p) = 0, \quad f''(p) = 0, \ldots, \quad f^{(k-1)}(p) = 0 \quad \text{but} \quad f^{(k)}(p) \neq 0.
\]

A zero of order \( k \) is also referred to as a zero of multiplicity \( k \). For example, for \( f(z) = (z-2)^3 \), we have \( f(2) = 0, \quad f'(2) = 0, \quad f''(2) = 0 \quad \text{but} \quad f'''(2) = 6 \neq 0 \). Thus \( f \) has a zero of order 3 at \( z = 2 \).

Formally, consider the following definition:

**Definition 1:** Let \( f \neq 0 \) be an analytic function in a domain \( D \). Let \( p \in D \), then the smallest integer \( k \geq 0 \) such that \( f^{(k)}(p) \neq 0 \) is called the order (multiplicity) of the zero of \( f \) at \( p \).

A zero of order 1 of a function is called a simple zero of the function.

Consider the following example.

**Example 1:** Find the zeros and their order of the entire function \( f(z) = z \sin z \).

**Solution:** The zeros of \( f(z) = z \sin z \) are given by \( 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \). We observe that \( f'(z) = \sin z + z \cos z \neq 0 \) at \( \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \), but it is zero at \( z = 0 \). Further, \( f''(z) = 2 \cos z - z \sin z \neq 0 \) at \( z = 0 \). Therefore, \( z = 0 \) is a zero of \( f \) of the order 2 and all other zeros are of order 1.

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We now prove a result which gives the zeros of a function and their order explicitly using the algebraic method, without finding the derivatives of the function.

**Theorem 1:** Let a function \( f \neq 0 \) (it means there is at least a point \( z_0 \in D \)) such that \( f(z_0) \neq 0 \) be holomorphic (analytic) in a domain \( D \). Let \( p \in D \) be a zero of \( f \) of order \( k \). Then there is a unique holomorphic function \( \phi \) in a neighborhood of \( p \) such that \( \phi(p) \neq 0 \) and

\[
f(z) = (z - p)^k \phi(z),
\]

for all \( z \in D \).

**Proof:** Since \( f \) is analytic in \( D \) then for \( p \in D \), \( f \) has a Taylor series representation in powers of \( z - p \) throughout some neighbourhood \( B(p, r) \) for some \( r \). We thus have
\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n. \]  
\[ (2) \]

Since \( p \) is a zero of \( f \) of order \( k \), we have

\[ f(p) = f'(p) = f''(p) = \cdots = f^{(k-1)}(p) = 0 \]
\[ (3) \]

In view of Eqn. (3), Eqn. (2) reduces to

\[ f(z) = (z-p)^k \left[ \frac{f^{(k)}(p)}{k!} + \frac{f^{(k+1)}(p)}{(k+1)!}(z-p) + \frac{f^{(k+2)}(p)}{(k+2)!}(z-p)^2 + \cdots \right]. \]

Thus, there exists a function \( \phi \) such that

\[ f(z) = (z-p)^k \phi(z), \]

where

\[ \phi(z) = \frac{f^{(k)}(p)}{k!} + \frac{f^{(k+1)}(p)}{(k+1)!}(z-p) + \frac{f^{(k+2)}(p)}{(k+2)!}(z-p)^2 + \cdots \quad \text{for } |z-p| < r. \]

The convergence of Series (4) when \( |z-p| < r \) ensures that \( \phi \) is analytic in the neighbourhood \( B(p, r) \) and

\[ \phi(p) = \frac{f^{(k)}(p)}{k!} \neq 0. \]

For all points other than \( p \), Eqn. (1) defines \( \phi \) and hence \( \phi \) is unique. This completes the proof of the theorem.

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**Example 2:** Show that \( z = 0 \) is a zero of order 3 of the analytic function \( f(z) = z\sin z^2 \).

**Solution:** The analytic function \( f(z) = z\sin z^2 \) has a zero at \( z = 0 \). We can write the Maclaurin expansion

\[ \sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots, \quad \text{[replace } z \text{ by } z^2 \text{ in Eqn. (23) of Unit 6]} \]
\[ (5) \]

Using Eqn. (5) we can then write \( f(z) = z\sin z^2 = z^3 \phi(z) \), where

\[ \phi(z) = 1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots, \]
\[ (6) \]

and \( \phi(0) = 1 \neq 0 \). Comparing Eqns. (1) and (6) it follows that \( z = 0 \) is a zero of order 3 of \( f \).

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**Example 3:** Show that all the zeros of the polynomial \( p(z) = z^3 - 27 \) are of order one.

**Solution:** Zeros of \( p(z) \) are the cube roots of 27 and these root are clearly \( 3, 3\omega, 3\omega^2 \) where \( \omega \) is a cube root of unity. We can thus write

\[ p(z) = (z-3) (z-3\omega) (z-3\omega^2). \]

Let us put \( \phi_1(z) = (z-3\omega) (z-3\omega^2), \phi_2(z) = (z-3) (z-3\omega^2) \) and

\[ \phi_3(z) = (z-3) (z-3\omega). \]

We observe that \( \phi_1, \phi_2 \) and \( \phi_3 \) are analytic and
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i) \( p(z) = (z-3)\phi_1(z) \) implies that 3 is the zero of order one because \( \phi_1(3) \neq 0 \).

ii) \( p(z) = (z-3\omega)\phi_2(z) \) implies that \( 3\omega \) is the zero of order one because \( \phi_2(3\omega) \neq 0 \).

iii) \( p(z) = (z-3\omega^2)\phi_3(z) \) implies that \( 3\omega^2 \) is the zero of order one because \( \phi_3(3\omega^2) \neq 0 \).

Thus all the three zeros of \( p(z) \) are the simple zeros (zeros of order one). You may also observe that \( p'(3) = 27 \neq 0 \), \( p'(3\omega) = 27\omega^2 \neq 0 \) and \( p'(3\omega^2) = 27\omega \neq 0 \).

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If you look at the zeros of analytic functions in Examples 1-3, you would notice that all these zeros are isolated. That is, if \( z_0 \) is a zero of an analytic function \( f \) (which is not identically zero) then in some neighbourhood of this point \( z_0 \), \( f \) is non-zero at every point of that neighbourhood other than \( z_0 \). That is, there is no other zero of \( f \) in this neighbourhood other than \( z_0 \). We give the following theorem as a consequence to this result.

**Theorem 2:** Let \( f \neq 0 \) be analytic in a domain \( D \). Then the zero set \( Z_f \) of \( f \) is an isolated subset of \( D \).

**Proof:** Clearly, \( Z_f \) is a closed subset of \( D \). Let \( p \) be a zero of \( f \) then \( f(z) = (z-p)^k \phi(z) \), for some \( k \), such that \( \phi(p) \neq 0 \). By the continuity of \( \phi \) there exists a neighborhood \( U \) of \( p \) such that \( \phi(z) \) does not vanish in \( U \) and the only zero of \( (z-p)^k \) is \( p \). Therefore, \( p \) is the only zero of \( f \) in \( U \). Thus \( p \in Z_f \) is an isolated zero of \( f \) and hence \( Z_f \) is an isolated subset of \( D \).

Before moving on further, you may try the following exercises.

**E1)** Show that the polynomial \( p(z) = z^3 - 64 \) has a zero of order 1 at \( z_0 = 4 \).

**E2)** Find all the zeros of the entire function \( f(z) = z^2(e^z - 1) \) and show that it has a zero of order 3 at \( z_0 = 0 \).

**E3)** Locate the zeros of the following functions and determine their order

i) \( z^3e^{z-1} \)

ii) \( z^2 \cosh z \).

We now consider a theorem which gives the condition for two analytic functions to be identical.

**Theorem 3 (Identity Theorem):** Let \( f \) and \( g \) be analytic functions on a domain \( D \). Suppose \( K \subset D \) is such that for every \( z \in K \), \( f(z) = g(z) \) and \( K \) has a limit point in \( D \). Then \( f = g \) on \( D \).
**Proof:** Let us assume that \( h: D \to \mathbb{C} \) be a non-zero function such that \( h = f - g \). Then the set \( K \) is a subset of the set of all the zeros of \( h \). By Theorem 2, \( K \) is then isolated subset of \( D \). But since \( K \) has a limit point, it follows that the zero set of \( h \) is not isolated, which is a contradiction. Thus \( h = 0 \) and hence \( f = g \) on \( D \).

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Consider the following example.

**Example 4:** Does there exist a function \( f: D \to \mathbb{C} \) such that \( f\left(\frac{1}{n}\right) = \frac{n}{n+1} \) for all \( n \in \mathbb{N} \), where \( D \) is the unit disk in \( \mathbb{C} \)?

**Solution:** We have

\[
f\left(\frac{1}{n}\right) = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} = g\left(\frac{1}{n}\right),
\]

where \( g \) is a function defined on \( D \) as \( g(z) = \frac{1}{1+z} \). Since the set \( K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \) has a limit point in \( D \), it follows from the identity theorem that \( f = g \) on \( D \).

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You may now try the following exercises.

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**E4)** Does there exist a function \( f: D \to \mathbb{C} \) such that \( f\left(\frac{1}{n}\right) = \frac{1}{n} \) for \( n \) even and \( f\left(\frac{1}{n}\right) = -\frac{1}{n} \) for \( n \) odd?

**E5)** Does there exist a non-constant analytic function \( f \) on unit disk \( D \) such that \( f\left(\frac{1}{n}\right) = 0 \) for all \( n \in \mathbb{N} \)?

**E6)** Does there exist a non-constant analytic function which is zero on an open interval on the real axis?

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You know that, a point \( z = z_0 \) is called a singular point or **singularity** of the complex function \( f \) if \( f \) is not analytic at \( z = z_0 \), but every neighbourhood of \( z_0 \) contains a point where \( f \) is analytic. Function \( f(z) = 1/z \) is an example of a function analytic everywhere except at \( z = 0 \). Another example is the function \( f(z) = \text{Log } z \), which is analytic everywhere except on the negative real axis including the point \( z = 0 \) (see Fig. 1).
Further, a point \( z = z_0 \) is called an **isolated singularity** of the complex function \( f \) if \( f \) is not analytic at \( z = z_0 \) and \( \exists \) a real number \( R > 0 \) so that \( f \) is analytic everywhere in the deleted neighbourhood \( 0 < |z - z_0| < R \) (see Fig. 2).

If you look at the above examples \( f(z) = 1/\sin z \) and \( g(z) = \log z \), then you will notice that \( z = 0 \) is an isolated singularity of \( 1/\sin z \), but \( z = 0 \) (in fact any point of the negative real axis) is not an isolated singularity of \( \log z \).

In the next sub-section, we shall discuss isolated singularities of the complex valued functions and classify them into various types.

### 7.2.2 Singularities–Poles, Essential, Removable

We now discuss different types of **isolated singularities** associated to a complex valued function.

Consider the following definitions.

**Definition 2 (Removable Singularity):** An isolated singularity \( z = z_0 \) of a function \( f \) is called a removable singularity if

\[
\lim_{z \to z_0} (z - z_0) f(z) = 0.
\]

**Remark:** If \( z = z_0 \) is a removable singularity of \( f \) in a domain \( D \) then the function \( f \) need not be defined at \( z_0 \), but \( \lim_{z \to z_0} f(z) \) exists. If \( \lim_{z \to z_0} f(z) = a_0 \) (say), then we can make \( f(z) \) an analytic function by redefining \( f(z) \) at \( z = z_0 \) as \( f(z_0) = a_0 \).

As an illustration, consider the following example.

**Example 5:** Consider the function \( f(z) = \frac{\sin z}{z} \), \( z \neq 0 \). Clearly, \( z = 0 \) is an isolated singularity of \( f \). Also we see that \( \lim_{z \to 0} z f(z) = 0 \). Hence \( z = 0 \) is a removable singularity. Since \( \lim_{z \to 0} f(z) = 1 \), we can define \( f(0) = 1 \) and make \( f \) analytic at \( z = 0 \).

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**Definition 3 (Pole):** An isolated singularity \( z = z_0 \) of a function \( f \) is called a pole if

\[
\lim_{z \to z_0} f(z) = \infty.
\]

Since \( f(z) \to \infty \) as \( z \to z_0 \), there exists a \( \delta > 0 \) such that \( B(z_0, \delta) \subset D \) and \( f(z) \neq 0 \) in \( B(z_0, \delta) \). Consider \( g = 1/f \) on \( B(z_0, \delta) \setminus \{z_0\} = D' \). Then \( g \) is analytic in \( D' \) and is bounded. Hence \( z = z_0 \) is a removable singularity of \( g \).

**Notice** that \( g(z) \to 0 \) as \( z \to z_0 \) and therefore we define \( g(z_0) = 0 \) in order to make \( g \) analytic in \( B(z_0, \delta) \). Further, if \( z_0 \) is a zero of \( g \) of order \( k \), then \( z_0 \) is a **pole of order** \( k \) of \( f \). In this case, we have

\[
f(z) = (z - z_0)^{-k} \phi(z)
\]
for all \( z \) in a neighborhood of \( z_0 \) where \( \phi \) is analytic and \( \phi(z_0) \neq 0 \).

For instance, \( z = 0 \) is a pole of function \( f(z) = 1/z^k \) of order \( k \). The function 
\[
f(z) = \frac{e^z}{(z-1)^3}
\]
has a pole of order 4 at \( z = 1 \). More precisely, we say that if \( f \) is a function such that for some positive integer \( k \), \( \lim_{z \to z_0} (z - z_0)^k f(z) \) is a non-zero finite complex number, then \( f \) has pole of order \( k \) at \( z = z_0 \). Pole of order one is called a simple pole.

Let us consider the following example.

**Example 6:** Determine the order of the poles of the function 
\[
f(z) = \frac{2z + 5}{(z-1)(z+5)(z-2)^2}.
\]

**Solution:** The denominator of the given function has the zeros of order 1 at \( z = 1 \) and \( z = -5 \), and a zero of order 4 at \( z = 2 \). Also note that the numerator of the given function is not zero at any of these points \( z = 1, -5 \) and 2. Thus, \( f \) has the simple poles at \( z = 1 \) and \( z = -5 \), and a pole of order 4 at \( z = 2 \).

In the case of isolated singularity of \( f \) at \( z = z_0 \), the function \( f \) can be expanded as a Laurent series in the deleted neighbourhood \( 0 < |z - z_0| < R \) of the point \( z_0 \) where \( f \) is analytic. Thus, we have 
\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.
\]
Here the sum 
\[
\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1 (z-z_0) + \cdots + a_n (z-z_0)^n + \cdots
\]
is called the analytic part of the Laurent series, whereas the sum 
\[
\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots
\]
involving negative powers of \( z-z_0 \) is called the principal part of the Laurent series. We shall now use Laurent series expansion of the function \( f \) to classify the isolated singularities of \( f \).

Consider the following definition.

**Definition 4:** If the principal part of \( f \) at \( z = z_0 \) consists of no terms, then \( z_0 \) is called a removable singularity of \( f \). In the case of removable singularity at the point \( z_0 \) all the \( b_n \)'s in Eqn. (7) are zero and we have 
\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1 (z-z_0) + a_2 (z-z_0)^2 + \cdots (0 < |z-z_0| < R)
\]
For instance, \( z_0 = 0 \) is a removable singular point of the function.
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\[ f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} \left[ 1 - \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right) \right] \]
\[ = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \quad (0 < |z| < \infty) \]

and on assigning the value \( f(0) = \frac{1}{2} \), \( f \) becomes analytic.

If the principal part of the Laurent series Expansion (7) of the function \( f \) has only a finite number of terms, i.e., it is of the form
\[ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^3} + \cdots + \frac{b_k}{(z-z_0)^k} \]
where \( k \) is a finite integer and \( b_{k+1}, b_{k+2}, \ldots \) are all zero, then \( z = z_0 \) is a pole of order \( k \). If \( k = 1 \), then \( z = z_0 \) is a simple pole.

Thus, a point \( z = z_0 \) is a pole if
\[ \lim_{z \to z_0} f(z) = \infty \quad \text{and} \quad \lim_{z \to z_0} (z-z_0)^k f(z) \]
exists for some \( k \geq 1 \).

The smallest value of \( k \) for which the second limit above exists defines the order of the pole. For instance, the function
\[ f(z) = \frac{\sin z}{z^2} \]
has simple pole at \( z = 0 \). Similarly \( z = 0 \) is a pole of order 3 of the function
\[ f(z) = \frac{\sin z}{z^4}. \]

Formally, we have the following definition.

**Definition 5:** If the principal part of a function \( f \) at an isolated singular point \( z_0 \) consists of a finite number of terms, say \( k \), then an isolated singularity \( z_0 \) of \( f \) is called a pole of order \( k \).

**Definition 6:** An isolated singularity \( z = z_0 \) of \( f \) is called an essential singularity if it is neither removable nor a pole.

In other words, if the principal part of \( f \) at \( z = z_0 \) contains infinite number of terms, then \( z = z_0 \) is an isolated essential singularity of \( f \).

We know that the Laurent series of \( f(z) = e^{1/z} \) about \( z = 0 \) is given by
\[ e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots. \]

This shows that \( z = 0 \) is an isolated essential singularity of \( e^{1/z} \).

Let us consider the following examples.

**Example 7:** Locate the singularities of the function \( f(z) = z^{-2}(z - \sin z) \).

**Solution:**
\[ f(z) = z^{-2}(z - \sin z) = z^{-2} \left( z - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right), \quad (0 < |z| < \infty) \]
\[ f(z) = \frac{1}{z^3} \left( z - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right) \right) \]

\[ = \frac{z}{3!} - \frac{z^3}{5!} + \ldots \]

\[ \Rightarrow f(z) \text{ has a removable singularity at } z = 0. \]

Alternatively, we see that

\[ \lim_{z \to 0} \frac{z - \sin z}{z^2} = \lim_{z \to 0} \frac{1 - \cos z}{2z} \]

\[ = \lim_{z \to 0} \frac{\sin z}{2z} = 0. \]

Therefore, \( \lim_{z \to 0} f(z) \) exists and is a finite complex number. Hence, \( f(z) \) has a removable singularity at \( z = 0 \).

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**Example 8:** Determine and classify the singularities of the function

\[ f(z) = \frac{z - 3 - 2i}{z^2 - (4 + 3i)z + (1 + 5i)}. \]

**Solution:** The values of \( z \) for which the denominator of the function \( f \) is zero are given by

\[ z^2 - (4 + 3i)z + (1 + 5i) = 0 \]

\[ \Rightarrow 2z = 4 + 3i \pm \alpha_j \text{ where } \alpha_j^2 = 3 + 4i \]

\[ \Rightarrow 2z = 4 + 3i \pm (2 + i) \]

\[ \Rightarrow z = 3 + 2i, 1 + i \]

Therefore, we have \( f(z) = \frac{z - a}{(z - a) (z - b)} \) where \( a = 3 + 2i, b = 1 + i \).

Thus, \( f \) has a simple pole at \( z = b \) and a removable singularity at \( z = a \).

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**Example 9:** Show that the function \( e^z \) has an isolated essential singularity at \( z = \infty \).

**Solution:** Let \( f(z) = e^z \). The singularity of \( f(z) \) at \( z = \infty \) is same as singularity of \( f(1/w) \) at \( w = 0 \). We have already seen above that \( e^{1/w} \) has an isolated essential singularity at \( w = 0 \). Hence \( z = \infty \) is an isolated essential singularity of \( e^z \).

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**Note:** In the case of non-isolated singularities any neighbourhood of a non-isolated singular point of \( f \) contains other singularities and hence a non-isolated singular point of \( f \) is a limit point of its singular points. For example, the function \( f(z) = \frac{1}{\sin(1/z)} \) is singular at \( z = \frac{1}{k\pi} (k = \pm 1, \pm 2, \ldots) \) because \( \sin(1/z) = 0 \) at these points. The limit point of these singularities is the point \( z = 0 \). Hence \( z = 0 \) is non-isolated singularity of \( f(z) \). Each of the singularity \( z = \frac{1}{k\pi} \) is isolated, but the singular point \( z = 0 \) is not isolated.
because however small \( \epsilon \) we may choose, every annulus \( 0 < |z| < \epsilon \), contains at least one singular point (in fact, an infinite number of them). Thus \( z = 0 \) is **non-isolated essential singularity** of \( f(z) = \frac{1}{\sin \left( \frac{1}{z} \right)} \).

The classification of the singularities, we have discussed so far, is shown pictorially in Fig. 3.

![Fig. 3](image)

**Remark:**
- The limit point of a sequence of poles of a function \( f \) is a non-isolated essential singularity of \( f \).
- The limit point of a sequence of zeros of a function \( f \) is an isolated essential singularity of \( f \).

You may now try the following exercises.

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**E7)** Find the nature of the singularities of the following functions.

i) \( f(z) = \frac{z - \sin z}{z^4} \)  
ii) \( f(z) = \sin \frac{1}{1-z} \)

iii) \( f(z) = \frac{1-e^z}{1+e^z} \)  
iv) \( f(z) = \tan z \).

**E8)** Locate the poles and determine their order for the following functions

i) \( z^{-1} \csc z \)  
ii) \( (z^2 \sin z)^{-1} \).

**E9)** Locate the singularities of the following functions and determine their types

i) \( z e^{1/z} \)  
ii) \( (\cos z - \cos 2z)/z^4 \).

**E10)** Write the function \( f(z) = \frac{8a^3 z^2}{(z^2 + a^2)^3} (a > 0) \) as \( f(z) = \frac{\phi(z)}{(z - ai)^2} \) where \( \phi(z) = \frac{8a^3 z^2}{(z + ai)^3} \).

Point out why \( \phi(z) \) has a Taylor series representation about \( z = ai \) and
then use it to show that the principal part of \( f \) at that point is
\[
\frac{\phi'(ai)/2}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = \frac{i/2 - a/2}{(z-ai)^2} - \frac{a^2i}{(z-ai)^3}.
\]

As we mentioned in the introduction, if a function \( f \) has one or more isolated singularities inside a contour \( C \) then each of these singularities contribute to the value of the integral of \( f \) around \( C \). In the next section, we shall compute these contributions called the residues of a complex valued function.

### 7.3 RESIDUES

We know that a function \( f \) having an isolated singularity at the point \( z_0 \) has the Laurent series representation as given by Eqn. (7), viz,
\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n},
\]
valid for \( 0 < |z - z_0| < r \), for some positive number \( r \). Further, the coefficients \( a_n \) and \( b_n \) have certain integral representations (ref. Sec. 6.4, Unit 6) and we can write
\[
b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad (n = 1, 2, \ldots) \tag{10}
\]
where \( C \) is a positively oriented simple closed contour enclosing the point \( z_0 \) and lying in the domain \( 0 < |z - z_0| < r \) (see Fig. 4). When \( n = 1 \), the expression for \( b_n \) as given by Eqn. (10) takes the form
\[
b_1 = \frac{1}{2\pi i} \oint_C f(z)dz.
\]
This number \( b_1 \), which is the coefficient of \( \frac{1}{(z-z_0)^1} \) in Expansion (7), is called the residue of \( f \) at \( z_0 \). It is usually represented as
\[
\text{Res}[f(z)] = b_1 \text{ or } \text{Res}[z_0, f] = b_1.
\]

Thus, we have
\[
\text{Res}[z_0, f] = b_1 = \frac{1}{2\pi i} \oint_C f(z)dz \tag{11}
\]
where \( C(t) = z_0 + \varepsilon e^{2\pi it}, \ 0 \leq t \leq 1 \), for any \( 0 < \varepsilon < r \).

There are methods available for finding the residues of a given function \( f \) having singularities of the types discussed above. We now discuss these methods.

**Residue at removable singularity:** If \( z = z_0 \) is a removable singularity of a function \( f \) then we know that its Laurent series expansion around \( z = z_0 \) contains no negative powers of \( (z-z_0) \). Thus the coefficient of \( (z-z_0)^{-1} \) in the expansion is equal to zero and consequently, \( \text{Res}[z_0, f] = 0 \).

We illustrate this situation through an example.
Example 10: Find the residues of the function \( \frac{\sin \pi z}{z^2 - 1} \) at its singular points.

Solution: Here \( z^2 - 1 = (z - 1)(z + 1) = 0 \) at \( z = \pm 1 \).

Moreover, \( \lim_{z \to 1} \frac{\sin \pi z}{z^2 - 1} = \frac{\sin \pi 1}{1^2 - 1} \) and \( \lim_{z \to 1} \frac{\sin \pi z}{z^2 - 1} = 0 \).

Therefore, \( z = \pm 1 \) are removable singularities of the function. Hence by the result above

\[
\text{Res} \left[ 1, \frac{\sin \pi z}{z^2 - 1} \right] = 0, \text{ and } \text{Res} \left[ -1, \frac{\sin \pi z}{z^2 - 1} \right] = 0.
\]

You may now try to do the following exercise.

E11) Find the singularities of \( f(z) = \frac{\sin z^2}{z^2(z - 1)} \) which lie interior to the contour \( |z| = 1 \). Compute the residues at those singularities.

Residue at poles: Here we are giving some results for computing the residues at the poles.

If \( f \) has a simple pole at \( z_0 \), then

\( \text{Res}[z_0, f] = \lim_{z \to z_0} (z - z_0) f(z). \) \hspace{1cm} (12)

If \( f \) has a double pole at \( z_0 \), then

\( \text{Res}[z_0, f] = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z). \) \hspace{1cm} (13)

If \( f \) has a pole of order \( m \) at \( z_0 \), where \( m \) is a finite integer, then

\( \text{Res}[z_0, f] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{(m-1)}}{dz^{(m-1)}} (z - z_0)^m f(z). \) \hspace{1cm} (14)

Let \( f \) and \( g \) be two functions analytic at the point \( z_0 \) satisfying

\( f(z_0) \neq 0, g(z_0) = 0, \text{ and } g'(z_0) \neq 0, \)

then \( z_0 \) is a simple pole of the quotient \( \frac{f(z)}{g(z)} \) and

\( \text{Res} \left[ z_0, \frac{f}{g} \right] = \frac{f(z_0)}{g'(z_0)}. \) \hspace{1cm} (15)

If \( z_0 \) is an isolated essential singular point of \( f \) then we need to expand \( f(z) \) in the Laurent series about the point \( z = z_0 \) and obtain the residue (coefficient \( b_1 \) of \( (z - z_0)^{-1} \)) directly. In this case, this is the only way of computing the residue at \( z_0 \).

Result (12)-(15) can be proved easily. We are leaving it for you to do it yourself.
We shall now illustrate these results through examples.

**Example 11:** Find the residue at the singular points of the function

\[ f(z) = \frac{z^2}{z^2 - 2z + 2}. \]

**Solution:** The singular points of \( f(z) \) are the roots of the equation \( z^2 - 2z + 2 = 0 \). We get the roots \( z_1 = 1 + i \) and \( z_2 = 1 - i \), which are simple poles. We write

\[ f(z) = \frac{z^2}{(z^2 - 2z + 2)} = \frac{z}{(z - z_1)(z - z_2)}. \]

Using the result given in Eqn. (12), we get

\[ \text{Res}[z_1, f] = \lim_{z \to z_1} (z - z_1)f(z) = \frac{z_1^2}{z_1 - z_2} = \frac{(1+i)^2}{2i} = 1. \]

\[ \text{Res}[z_2, f] = \lim_{z \to z_2} (z - z_2)f(z) = \frac{z_2^2}{z_2 - z_1} = \frac{(1-i)^2}{-2i} = 1. \]

Alternatively, \( z_1 \) and \( z_2 \) are simple poles of \( f(z) \) which is a rational function of the form \( p(z)/q(z) \), with \( p(z) = z^2 \) and \( q(z) = z^2 - 2z + 2 \) and \( q'(z) = 2z - 2 \). Using the result given by Eqn. (15), we get

\[ \text{Res}[z_1, f] = \frac{p(1+i)}{q(1+i)} = \frac{2i}{2i} = 1 \quad \text{and} \quad \text{Res}[z_2, f] = \frac{p(1-i)}{q(1-i)} = \frac{-2i}{-2i} = 1. \]

**Example 12:** Use the Laurent series representation to find the residue of the function \( f(z) = \frac{z^2 + 4z + 5}{z^2 + z} \) at \( z = 0 \). Also find \( \int_C f(z)dz \) where \( C_r = \{ z : |z| = r, \ 0 < r < 1 \} \) (see Fig. 5).

**Solution:** We have

\[ f(z) = \frac{z^2 + 4z + 5}{z(1+z)} = \frac{z^2 + 4z + 5}{z} (1+z)^{-1} \]

\[ = \frac{z^2 + 4z + 5}{z} [1 - z + z^2 - z^3 + \cdots] \quad (0 < |z| < 1) \]

\[ = (z^2 + 4z + 5) \left[ \frac{1}{z} - 1 + z - z^2 + \cdots \right] \]

\[ = \left[ \frac{5}{z} - 1 + 2z - \cdots \right]. \]

Thus

\[ \text{Res}[0, f] = \text{coefficient of } z^{-1} = 5. \]

Hence, from Eqn. (11), we get

\[ \int_C f(z) dz = 2\pi i b_i = 2\pi i \times 10\pi i = 100\pi^2. \]
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Example 13: Find the residue at \( z = 0 \) of the function \( f(z) = \frac{\sin z}{z^4} \).

Solution: You know that \( z = 0 \) is a pole of order 3 of the function \( \frac{\sin z}{z^4} \).

Applying the result given by Eqn. (14), we get

\[
\text{Res}[0, f] = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left( \frac{z^3 \sin z}{z^4} \right) = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left( \frac{\sin z}{z} \right).
\]

We have

\[
d^2 \left( \frac{\sin z}{z} \right) = -(z^2 - 2) \sin z + 2z \cos z (z \neq 0).
\]

You can see that the limit of the expression above as \( z \) tends to zero is an indeterminate form \( \left\{ \frac{0}{0} \right\} \). Therefore, applying the L'Hopital's rule, we get

\[
\lim_{z \to 0} \frac{(z^2 - 2) \sin z + 2z \cos z}{z^3} = -\lim_{z \to 0} \frac{\cos z}{3} = -\frac{1}{3}.
\]

Putting the above limiting value in Eqn. (16), we get the required residue as

\[
\text{Res}[0, f] = -\frac{1}{6}.
\]

Alternatively, we can use the following result which is more general than the one given by Eqn. (14) and helps in simplifying the computations involved.

If \( z_0 \) is a pole of order \( m \) of the function \( f \) then for any \( n \geq m \),

\[
\text{Res}[z_0, f] = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{(n-1)}}{dz^{(n-1)}} (z - z_0)^n f(z),
\]

where \( n \) and \( m \) are both integers.

If we apply the result given by Eqn. (17) to Example (13) with \( n = 4 > 3 = m \), we can easily compute the residue as

\[
\text{Res}[z_0, f] = \frac{1}{(4-1)!} \lim_{z \to 0} \frac{d^{(4-1)}}{dz^{(4-1)}} z^4 f(z) = \frac{1}{3!} \lim_{z \to 0} \frac{d^3}{dz^3} z^4 \frac{\sin z}{z^4} = -\frac{1}{6}.
\]

We now take up an example where the singularities of the functions involved are essential singularities.

Example 14: Find the residues at the singular points of the following functions:

i) \( f(z) = z \sin \frac{1}{z} \)

ii) \( f(z) = z \cos \frac{1}{z} \).

Solution: The singularities of both the given functions is the point \( z = 0 \) which is an isolated essential singular point of \( f(z) \) in both the cases. We write the Laurent series expansion of \( f(z) \) about \( z = 0 \) to compute the residue in each case

i) \( f(z) = z \sin \frac{1}{z} = z \left[ \frac{1}{z} - \frac{1}{3! z^3} + \cdots \right] = 1 - \frac{1}{3! z^3} + \cdots \)
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\[ \text{Res}\left[0, \frac{z \sin \frac{1}{z}}{z}\right] = \text{coefficient of } \frac{1}{z} \text{ in the Laurent series} = 0 \]

ii) \[ f(z) = z \cos \frac{1}{z} = z \left[1 - \frac{1}{2!z^2} - \frac{1}{4!z^4} - \ldots\right] = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \ldots \]

\[ \text{Res}\left[0, \frac{z \cos \frac{1}{2}}{z}\right] = \text{coefficient of } \frac{1}{z} \text{ in the Laurent series} = -\frac{1}{2}. \]

Before we ask you to try a few exercises to check your understanding of finding the residues, we give you a result as a theorem, which you may find useful while solving the problems.

**Theorem 4:** Suppose that 0 is an isolated singularity of an even function \( f \) analytic on the punctured plane \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). Then \( \text{Res}[0, f] = 0 \).

**Proof:** Let \( f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \) be the Laurent series expansion of \( f \) around 0 in \( \mathbb{C}^* \). We then need to show that the coefficient \( a_{-1} \) in the expansion is 0. If we replace \( z \) by \( -z \) in this expansion and use the fact that \( f \) is an even function i.e., \( f(-z) = f(z) \), we get \( \sum_{n=-\infty}^{\infty} (-1)^n a_n z^n = \sum_{n=-\infty}^{\infty} a_n z^n \). Now using the uniqueness of Laurent expansion, we have \( (-1)^n a_n = a_n \). If \( n \) is odd then \( a_n = 0 \). In particular, \( a_{-1} = 0 \). Hence \( \text{Res}[0, f] = a_{-1} = 0 \).

Consider the following example.

**Example 15:** Compute the \( \text{Res}\left[0, e^{-\frac{1}{z}} \cos \frac{1}{z}\right] \).

**Solution:** Let \( f(z) = e^{-\frac{1}{z}} \cos \frac{1}{z} \) then \( f(-z) = f(z) \) for all \( z \in \mathbb{C}^* \). Also \( z = 0 \) is an isolated essential singularity of \( f(z) \). Then by Theorem (4)

\[ \text{Res}\left[0, e^{-\frac{1}{z}} \cos \frac{1}{z}\right] = 0. \]

You may now try the following exercises.

**E13)** Consider the function in Example 12. Find its residue at \( z = -1 \) by using the Laurent series expansion. Also find

\[ \int_{C_r} f(z)dz \] where \( C_r = \{z : |z + 1| = r, 0 < r < 1\} \) (see Fig. 6).

**E14)** Compute each of the following residues:
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i) \( \text{Res} \left[ -3, \frac{z^2 + 1}{z(z + 3)^2} \right] \) 

ii) \( \text{Res} \left[ i + 1, \frac{e^z}{(z - i - 1)^3} \right] \).

E15) Find the residues of the given functions at the given points:

i) \( \text{Res} \left[ 0, \frac{\sin z}{z^3(z - 2)(z + 1)} \right] \)

ii) \( \text{Res} \left[ 0, \frac{(e^{iz} - 1)}{\sin^2 z} \right]. \)

E16) Show that the singular point of each of the following functions is a pole.

Determine the order of the pole and the corresponding residue

i) \( \frac{1 - \cosh z}{z^3} \)

ii) \( \frac{\exp(2z)}{(z - 1)^2} \).

E17) Show that

i) \( \text{Res} \left[ i, \frac{\log z}{(z^2 + 1)^2} \right] = \frac{\pi + 2i}{8} \)

ii) \( \text{Res} \left[ i, \frac{z^{1/2}}{(z^2 + 1)^2} \right] = \frac{1 - i}{8\sqrt{2}} \) \((|z| > 0, 0 < \arg z < 2\pi)\).

E18) Show that

i) \( \text{Res}[z_n, \sec z] = (-1)^{n+1}z_n \), where \( z_n = \frac{\pi}{2} + n\pi \), \( n = 0, \pm 1, \pm 2, \ldots \)

ii) \( \text{Res}[z_n, \tanh z] = 1 \) where \( z_n = \left( \frac{\pi}{2} + n\pi \right)i \), \( n = 0, \pm 1, \pm 2, \ldots \).

Now that you have learnt the methods of finding the residues of a complex function \( f \) at its isolated singularities which may be removable, pole or an essential singularity, we move on to discuss a theorem known as Cauchy’s residue theorem. The theorem states that under some circumstances we can evaluate complex integrals \( \int_C f(z)\,dz \) by summing the residues at the isolated singularities of \( f \) within the closed contour \( C \).

### 7.4 Cauchy’s Residue Theorem

The Cauchy’s residue theorem shows that if the function is analytic except for the finite number of points, inside and on a positively oriented, simple closed contour \( C \), then the value of the integral of \( f \) around \( C \) is \( 2\pi i \) times the sum of the residues of \( f \) at the singular points inside \( C \). We now state and prove the theorem.

**Theorem 5:** Let \( C \) be a simple closed contour described in the positive sense. If a function \( f \) is analytic inside and on \( C \) except for a finite number of singular points \( z_k (k = 1, 2, \ldots, n) \) inside \( C \), then

\[
\int_C f(z)\,dz = 2\pi i \sum_{k=1}^n \text{Res}[z_k, f(z)].
\]
Proof: It is given that $f$ is analytic inside and on $C$ except for finite number of singular points $z_k$'s which must be isolated. It means we can always construct positively oriented circles $C_k$, centered at $z_k$, such that they lie inside $C$ and are disjoint (see Fig. 7). The circles $C_k$, together with the simple closed contour $C$, form the boundary of a closed region throughout which $f$ is analytic and whose interior is a multiply connected domain. Let us now construct a polygonal path $L_1$ connecting the outer contour $C$ to the inner circle $C_1$. Construct another polygonal path $L_2$ connecting $C_1$ to $C_2$ and continue in this manner constructing polygonal path $L_{n+1}$ connecting inner circle $C_n$ back to contour $C$. Finally, we end up with two positively oriented simple closed contours $\Gamma_1$ and $\Gamma_2$ consisting of polygonal paths $L_k$ or $-L_k$ and parts of $C$ and $-C_k$ as shown in Fig. 8.

In order to distinguish between the two circles we have denoted the parts of $C_k$ by $C_k$ and $C_k$ and of $C$ by $C_1$ and $C_2$ in Fig. 8. Note that in this construction, the inner circles are clockwise (negatively oriented) and this is the reason for putting minus sign. The Cauchy-Goursat theorem can now be applied to the contours $\Gamma_1$ and $\Gamma_2$ to obtain

$$\int_{\Gamma_1} f(z)dz = 0$$

$$\int_{\Gamma_2} f(z)dz = 0$$

which can be combined to give

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0.$$  

We break the integrals on the left hand sides of the Eqns. (19) and (20) and write

$$\int_{\Gamma_1} f(z)dz = \int_{C_1} f(z)dz + \int_{L_1} f(z)dz + \int_{-C_{11}} f(z)dz + \int_{C_{11}} f(z)dz + \cdots + \int_{L_{n+1}} f(z)dz + \int_{C_{n+1}} f(z)dz$$

$$= \int_{C_1} f(z)dz + \int_{L_1} f(z)dz - \int_{-C_{11}} f(z)dz - \int_{C_{11}} f(z)dz - \cdots - \int_{L_{n+1}} f(z)dz + \int_{C_{n+1}} f(z)dz$$

$$= \int_{C_1} f(z)dz + \int_{-L_1} f(z)dz + \int_{-C_{11}} f(z)dz + \int_{C_{11}} f(z)dz + \cdots + \int_{-L_{n+1}} f(z)dz + \int_{C_{n+1}} f(z)dz$$

$$\int_{\Gamma_2} f(z)dz = \int_{C_2} f(z)dz + \int_{-L_1} f(z)dz - \int_{-C_{12}} f(z)dz - \int_{C_{12}} f(z)dz - \cdots - \int_{-L_{n+1}} f(z)dz - \int_{C_{n+1}} f(z)dz.$$
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\[ \int f(z)dz = \int f(z)dz - \int f(z)dz - \cdots - \int f(z)dz - \int f(z)dz. \quad (23) \]

Substituting from Eqns. (22) and (23) into Eqn. (21), we have

\[ \int f(z)dz + \int f(z)dz = \int f(z)dz - \int f(z)dz - \cdots - \int f(z)dz \]

\[ = \int f(z)dz - \sum_{k=1}^{n} \int f(z)dz = 0. \quad (24) \]

\[ \Rightarrow \int f(z)dz = \sum_{k=1}^{n} \int f(z)dz. \quad (25) \]

Observe that in writing Eqn. (24) the integrals along polygonal paths cancelled out and those along parts of inner circles and contours have been combined.

We have already seen from Laurent series representation in Eqn. (11) that

\[ \int_{C} f(z)dz = 2\pi i \text{Res}[z_{k}, f] \quad (k = 1, 2, \ldots, n). \]

Thus we get from Eqn. (25) the desired result

\[ \int_{C} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[z_{k}, f]. \]

We now evaluate some integrals using the Cauchy’s residue theorem.

**Example 16:** Use the Cauchy’s residue theorem to evaluate the integrals of the following functions around the circle \( C : |z| = 3 \), drawn in positive sense.

i) \( f(z) = \frac{e^{-z}}{(z-1)^{2}} \)

ii) \( f(z) = z^{2}e^{z} \).

**Solution:**

i) It can be seen easily that \( z = 1 \) is the singular point of \( f \) which lies inside the circle \( C : |z| = 3 \). It is a pole of order \( m = 2 \). Therefore

\[ \text{Res} \left[ 1, \frac{e^{-z}}{(z-1)^{2}} \right] = \frac{1}{(2-1)!} \left[ \frac{d}{dz} e^{-z} \right]_{z=1} = -e^{-1}. \]

Using the Cauchy’s residue theorem, we get

\[ \int_{C} \frac{e^{-z}dz}{(z-1)^{2}} = 2\pi i \left[ \frac{e^{-z}}{(z-1)^{2}} \right]_{z=1} = 2\pi i (-e^{-1}) = \frac{2\pi i}{e}. \]

ii) We know that \( z = 0 \) is an essential singularity of \( z^{1/2}e^{z} \) therefore it is an essential singularity of \( z^{2}e^{z} \). Also, \( z = 0 \) lies inside the contour \( C \). The Laurent series expansion of \( z^{2}e^{z} \) gives

\[ z^{2}e^{z} = z^{2} \left( 1 + \frac{1}{z} + \frac{1}{2!z^{2}} + \frac{1}{3!z^{3}} + \frac{1}{4!z^{4}} + \cdots \right) \]

\[ = z^{2} + \frac{1}{2!} + \frac{1}{3!z} + \frac{1}{4!z^{3}} + \cdots. \]

Thus \( \text{Res} \left[ 0, z^{2}e^{z} \right] = \frac{1}{3!} \).
The Cauchy’s residue theorem then gives
\[
\int_{C} z^{2} e^{z} dz = 2\pi i \text{Res} \left[ 0, z^{2} e^{2} \right] = 2\pi i \frac{1}{3!} = \frac{\pi i}{3}.
\]

**Example 17:** Use the residue theorem to evaluate
\[
\frac{1}{2\pi i} \oint_{C} f(z) dz
\]
where \( f(z) = \frac{e^{z}}{(\sin z)(\cos z)} \) and \( C \) is a positively oriented (counterclockwise) quadrilateral with vertices \( \pm 4 \pm 5i \).

**Solution:** The poles of \( f(z) \) are given by \( z = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pm 2\pi, \ldots \).

Only the points \( 0, \frac{\pi}{2}, \pm \pi \) lie in the interior of the quadrilateral (see Fig. 9).

They are all simple poles. The residues at these points are
\[
\text{Res}[0, f(z)] = \frac{e^{i\theta}}{\left( \frac{1}{2} \right)} = 1
\]
\[
\text{Res}[\pm \frac{\pi}{2}, f(z)] = \frac{e^{\pm i \frac{\pi}{2}}}{-1} = \pm i
\]
\[
\text{Res}[\pm \pi, f(z)] = \frac{e^{\pm i \pi}}{1} = -1.
\]

Now applying the residue’s theorem, we get
\[
\oint_{C} f(z) dz = 2\pi i [1 + i - i - 1] = -2\pi i.
\]

**Example 18:** Evaluate the integral
\[
\int_{\left[ 0, \frac{\pi}{4} \right]} \frac{dz}{z^{2} \tan z}.
\]

**Solution:** The integrand \( f(z) = \frac{1}{z^{2} \tan z} \) has singular points at \( z^{2} \tan z = 0 \), that is, \( z = 0, \pm \pi, \pm 2\pi, \ldots \). Here \( z = 0 \) is a pole of order three of \( f(z) \) as can be seen from the Laurent series representation of \( f(z) \), i.e.,
\[
\frac{1}{z^{2} \tan z} = \frac{1}{z^{2} \left( z + \frac{z^{3}}{3} + \cdots \right)}
\]
\[
= \frac{1}{z^{3}} \left( 1 + \frac{z^{2}}{3} + \cdots \right)^{-1}
\]
\[
= \frac{1}{z^{3}} \left( 1 - \frac{z^{2}}{3} + \cdots \right)
\]
\[
= \left( \frac{1}{z^{3}} - \frac{1}{3z} + \cdots \right).
\]
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Thus $z = 0$ is a pole of order three with $\text{Res}[0, f(z)] = -\frac{1}{3}$.

To check the nature of singularities at points $z = \pm n \pi, n \in \mathbb{Z} \setminus \{0\}$ you may observe that

$$\lim_{z \to \pm n \pi} f(z) = \lim_{z \to \pm n \pi} \frac{1}{z^2 \tan z} = \infty$$

and

$$\lim_{z \to \pm n \pi} (z \pm n \pi) f(z) = \lim_{w \to 0} w f(w \pm n \pi) = \lim_{w \to 0} \frac{w}{(w \pm n \pi)^2 \tan(w \pm n \pi)} = \frac{1}{n^2 \pi^2} \quad \text{(for } n \neq 0)\text{.}$$

Thus $z = \pm n \pi, n \in \mathbb{Z} \setminus \{0\}$ are the simple poles.

Only the singularity at the point $z = 0$ lies inside the circle $C \left(0, \frac{\pi}{4}\right)$ (see Fig. 10).

Using the Cauchy’s residue theorem, we get

$$\int_{C \left(0, \frac{\pi}{4}\right)} \frac{dz}{z^3 \tan z} = 2\pi i \text{Res}[0, f(z)] = 2\pi i \times -\frac{1}{3} = -\frac{2\pi i}{3} \text{.}$$

***

You may now try the following exercises.

E19) Evaluate the following integral using the Cauchy’s residue theorem:

$$\int_{C} \frac{dz}{z^2 + 1} \text{ where } C \text{ is the circle } x^2 + y^2 = 2x \text{.}$$

E20) Compute the following integral using the Cauchy’s residue theorem.

$$\frac{1}{2\pi i} \oint_{C} \frac{e^z}{(z + 1) \sin z} \frac{dz}{dz} \quad \left( C = \{z : |z| = \frac{1}{2}\} \right) \text{.}$$

E21) Evaluate the integral $\int_{C} \frac{\cosh \pi z}{z(z^2 + 1)} dz$.

E22) Let $C_N$ denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm \left(N + \frac{1}{2}\right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2}\right) \pi$$

where $N$ is positive integer. Further, taking the limit $N \to \infty$ in the above equation, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12} \text{.}$$

E23) Consider the function $f(z) = \frac{1}{|q(z)|^2}$, where $q$ is analytic at $z_0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$. Show that $z_0$ is a pole of order 2 of the function $f$, with residue $B_0 = -\frac{q''(z_0)}{|q'(z_0)|^3}$.
We now end the unit by giving a summary of what we have covered in it.

### 7.5 SUMMARY

In this unit we have covered the following:

1) **Zero set** of an analytic function is an isolated set.

2) **Identity theorem** provides condition under which two analytic functions defined on a domain are identical.

3) A point \( z_0 \) is called a singular point of a function \( f \) if \( f \) fails to be analytic at \( z_0 \) but is analytic at some point in every neighbourhood of \( z_0 \).

4) A point \( z_0 \in D \) is said to be an isolated singularity of \( f \) if \( f \) is defined and holomorphic in a neighbourhood of \( z_0 \) except possibly at \( z_0 \).

5) Laurent series can be used to classify the isolated singular point \( z = z_0 \) of the function as follows:
   - If the function is not defined at \( z = z_0 \) but \( \lim_{z \to z_0} f(z) \) exists, then the point is called a **removable singular point**. In this case the principal part of the Laurent series is zero.
   - If the principal part of the Laurent series expansion of the function \( f \) about a point has only a finite number of terms then the singularity is called a **pole**.
   - If the principal part of the Laurent series expansion of the function \( f \) about a point has infinite number of terms then the singularity is called an **essential singularity**.

6) If a function \( f \) has one or more isolated singularities inside a simple closed contour \( C \) then each of these singularities contributes to the value of the integral \( \int_C f(z) \, dz \). These contributions come from the **residues** evaluated at these isolated singularities.

7) If \( z = z_0 \) is a simple pole of \( f \), then
   \[
   \text{Res}[z_0, f(z)] = \lim_{z \to z_0} (z - z_0) f(z) .
   \]

8) If \( z = z_0 \) is a pole of order \( m \) of \( f \), then
   \[
   \text{Res}[z_0, f(z)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{(m-1)}}{dz^{(m-1)}}(z - z_0)^m f(z) .
   \]

9) If \( z = z_0 \) is an essential singularity of \( f \) then the coefficient of \( (z - z_0)^{-1} \) in the Laurent series expansion of \( f \) gives the residue of \( f \) at \( z = z_0 \).

10) **Cauchy’s Residue Theorem:** Let \( C \) be a simple closed contour, described in the positive sense. If a function \( f \) is analytic inside and on \( C \) except for a finite number of singular points \( z_k (k = 1, 2, \ldots, n) \) inside \( C \), then
   \[
   \int_C f(z) \, dz = 2\pi i \sum_{k=1}^n \text{Res}[z_k, f(z)].
   \]
7.6 SOLUTIONS/ANSWERS

E1) It can be seen that $f(z) = z^3 - 64 = (z - 4) \left( z^2 + 4z + 16 \right) = (z - 4)\phi(z)$ where $\phi(z) = z^2 + 4z + 16$. Now we find that $f(4) = 0$, $\phi(4) = 16 + 16 + 16 = 48 \neq 0$. Thus $z = 4$ is the zero of order 1 (simple zero).

E2) It is known that $e^i = 1$ if and only if $z = 2n\pi i, n \in \mathbb{Z}$. Therefore, all the zeros of the entire function $f$ are precisely $2n\pi i, n \in \mathbb{Z}$ (it should be noted here that $z^2 = 0$ if and only if $z = 0$). Note that $f(0) = 0$ and

$$f'(z) = 2ze^{-1} + z^2e^z \Rightarrow f'(0) = 0.$$  
$$f''(z) = 2(e^z - 1) + 4ze^z + z^2e^z \Rightarrow f''(0) = 0.$$  
$$f'''(z) = 6e^z + 6ze^z + z^2e^z \Rightarrow f'''(0) = 6 \neq 0.$$  

Therefore $z_0 = 0$ is the zero of order 3 of $f$. All other zeros are simple zeros.

E3) i) Let $f(z) = z^3 e^{z-1}$. Now $f(z) = 0 \Rightarrow z^3 = 0$, thus $z = 0$ is a zero of $f$. We write the Taylor series expansion of $f$ as

$$f(z) = z^3 e^{z-1} = \frac{z^3}{e^z} = \frac{z^3}{e} \left( 1 + \frac{z^2}{2} + \cdots \right) = z^3 \phi(z).$$  

Therefore, $z = 0$ is a zero of order 3.

ii) $f(z) = z^2 \cosh z$. Zeros of $f$ are given by $z = 0, \left( \frac{\pi}{2} + n\pi \right)i (n = 0, \pm 1, \pm 2, \ldots)$. Further, check that

$$f(0) = 0, f'(0) = 0, f''(0) \neq 0.$$  
$$f\left( \left( \frac{\pi}{2} + n\pi \right)i \right) = 0, f\left( \left( \frac{\pi}{2} + n\pi \right)i \right) \neq 0 (n = 0, \pm 1, \pm 2, \ldots).$$  

Thus, $z = 0$ is a zero of order 2 and the remaining zeros are simple zeroes (that is, of order one).

E4) Consider a function $g$ defined as $g(z) = z$ and a function $h$ defined as $h(z) = -z$. On similar lines as in Example 4 we can observe that $f = g$ on a set $A = \left\{ \frac{1}{2n} : n \in \mathbb{N} \right\}$ and $f = h$ on a set $B = \left\{ \frac{1}{2n+1} : n \in \mathbb{N} \right\}$. Both the sets have limit point 0 inside the domain. Thus we arrive at a contradiction by the identity theorem.

E5) Assume that such a function exists, then proceeding as in E4) prove that such a function will be a constant function.

E6) Since an open interval has a limit point therefore by the identity theorem an analytic function on an open interval will be identically zero. That is, it is a constant function. Thus no such non constant function exists.
E7) i) Since \( \lim_{z \to 0} z f(z) = z \frac{-\sin z}{z^4} = 1/6 \). Therefore \( z = 0 \) is a pole of order 1.

ii) Zeros of \( f \) are \( z = 1 - \frac{1}{n \pi} n = 0, \pm 1, \pm 2, \ldots \) and 1 is the limit point of the zero set of \( f \). Therefore \( z = 1 \) is an isolated essential singularity.

iii) Poles of \( f \) are given by the zeros of \( e^z + 1 \), i.e., \( z = \frac{1}{2} \pm \frac{1}{2} \pi i n \). Since the zeros of \( 1 + e^z \) are of order 1, therefore the poles of \( f \) are of order 1. Also, \( z = \infty \) is the limit point of poles, therefore \( z = \infty \) is a non-isolated essential singularity.

iv) **Hint:** Poles of \( f \) are the zeros of \( \cos z \).

E8) i) Let \( f(z) = \frac{1}{\sin z} \). It can be seen that \( z = 0, \pm \pi, \pm 2\pi, \ldots \) are the isolated singularities of \( f \). Since
\[
\lim_{z \to n \pi} f(z) = \infty (n \in \mathbb{Z}) \quad \text{and} \quad \lim_{z \to 0} \frac{z^2}{\sin z} = 1
\]
\[
\lim_{z \to n \pi} \frac{z - n \pi}{\sin z} = \lim_{w \to 0} \frac{w}{(w + n \pi) \sin(w + n \pi)} \neq 0 \quad (n \neq 0),
\]
these points are poles. \( z = 0 \) is the pole of order 2 and \( z = n \pi, n \in \mathbb{Z} \setminus \{0\} \) are the simple poles.

ii) Let \( f(z) = (z^2 \sin z)^{-1} = \frac{1}{z^2 \sin z} \). Here \( f \) is analytic everywhere except at \( z = 0 \) and at the zeros of \( \sin z \), given by \( z = n \pi, n \in \mathbb{Z} \).

The Laurent’s expansion of \( f \) at \( z = 0 \) is given by
\[
\frac{1}{z^2 \sin z} = \frac{1}{z^2} \left( z^2 \frac{z^3}{3!} + z^5 \frac{z^5}{5!} + \cdots \right)^{-1}
\]
\[
= \frac{1}{z^3} \left( 1 - \frac{z^2}{3!} + \cdots \right)^{-1}
\]
\[
= \frac{1}{z^3} \left( 1 + \frac{z^2}{3!} - \frac{5z^4}{36} + \cdots \right) = \frac{1}{z^3} + \frac{1}{6z} - \frac{5z}{36} + \cdots.
\]
Thus \( z = 0 \) is a pole of order three. To check the nature of singularities at the points \( z = n \pi, n \in \mathbb{Z} \setminus \{0\} \), we observe that
\[
\lim_{z \to n \pi} f(z) = \infty \quad \text{and}
\]
\[
\lim_{z \to n \pi} (z - n \pi) f(z) = \lim_{w \to 0} w f(w + n \pi) = \begin{cases} \frac{1}{n^2 \pi^2}, & \text{if } n \text{ is even} \\ -\frac{1}{n^2 \pi^2}, & \text{if } n \text{ is odd} \end{cases}
\]
Therefore, \( z = n \pi, n \in \mathbb{Z} \setminus \{0\} \) are the simple poles.

E9) i) \( f(z) = z e^z \) is analytic everywhere except at the point \( z = 0 \). The
Laurent’s expansion about \( z = 0 \) is given by
\[
\frac{1}{ze^z} = z \left\{ 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots \right\} = z + 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \cdots.
\]
Thus at \( z = 0 \), \( f \) has an isolated essential singularity.

ii) Obviously, \( z = 0 \) is the singularity of \( f(z) = (\cos z - \cos 2z) / z^4 \). Further,
\[
\frac{\cos z - \cos 2z}{z^4} = \frac{1}{z^4} \left\{ \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right) - \left( 1 - \frac{4z^2}{2!} + \frac{16z^4}{4!} - \frac{64z^6}{6!} + \cdots \right) \right\} = \frac{1}{z^4} \left( \frac{3z^2}{2} - \frac{5z^4}{8} + \frac{7z^6}{80} + \cdots \right) = \frac{3}{2z^2} - \frac{5}{8} + \frac{7z^2}{80} - \cdots.
\]
Thus, \( z = 0 \) is the pole of order 2.

E10) Clearly, \( \phi(z) = \frac{8a^3z^2}{(z + ai)^3} \) is analytic at \( z = ai \) and its Taylor series is given by
\[
\phi(z) = \phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots.
\]

Now,
\[
f(z) = \frac{\phi(z)}{(z - ai)^3} = \frac{1}{(z - ai)^3} \left\{ \phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots \right\}
\Rightarrow f(z) = \frac{\phi(ai)}{(z - ai)^3} + \frac{\phi'(ai)}{(z - ai)^2} + \frac{\phi''(ai)}{2!(z - ai)} + \cdots.
\]

The principal part of \( f \) is given by:
\[
\frac{\phi(ai)}{(z - ai)^3} + \frac{\phi'(ai)}{(z - ai)^2} + \frac{\phi''(ai)}{2!(z - ai)} = \frac{8a^3(-a^2)}{-8a^3i} = \frac{a^2}{i} = -a^2i
\]
\[
\phi'(z) = 8a^3 \left\{ \frac{2z(z + ai)^3 - 3z^2(z + ai)^2}{(z + ai)^6} \right\} = 8a^3 \left( \frac{2(z + ai) = 3z}{(z + ai)^4} \right)
= \frac{8a^3z}{(z + ai)^4}(2ai - z)
\]

Now \( \phi'(ai) = \frac{8a^3(ai)}{(2ai)^4}(2ai - ai) = \frac{8a^3(ai)(ai)}{16a^4} = \frac{-8a^5}{16a^4} = -\frac{a^2}{2} \).

Similarly, \( \phi''(ai) = \frac{16a^3}{(z + ai)^5}\{-a^2 + z^2 - 4aiz\} \),
and \( \phi''(ai) = \frac{16a^3}{32a^5}\{-a^2 - a^2 + 4a^2\} = \frac{32a^5}{32a^5} = 1 = -i \).

Thus, the principal part of \( f \) is
\[
-\frac{a^2i}{(z - ai)^3} - \frac{a/2}{(z - ai)^2} - \frac{i/2}{(z - ai)}.
\]
E11) You can easily see that the singularities of the given function are at 0 and 1. Only 0 lies inside the given contour. Since \( \lim_{z \to 0} \frac{\sin z^2}{z^2(z-1)} = 0 \), Thus 0 is a removable singularity and

\[
\text{Res} \left[ 0, \frac{\sin z^2}{z^2(z-1)} \right] = 0.
\]

E12) Let us Prove Result (14).

Since \( f \) has a pole of order \( m \) at \( z = z_0 \), its Laurent series expansion convergent on the disk \( 0 < |z - z_0| < R \), for some positive number \( R \), is of the form

\[
f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}
\]

where \( b_m \neq 0 \). Multiplying Eqn. (26) by \( (z-z_0)^m \) we obtain

\[
(z-z_0)^m f(z) = a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \cdots + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \cdots + b_m
\]

Differentiating \( (m-1) \) times both the side of Eqn. (27), we get

\[
\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] = m!a_0(z-z_0) + \cdots + b_1(m-1)!.
\]

Since all the terms on the right hand side of Eqn. (28) except the last term involve positive integral powers of \( (z-z_0) \). The limit of (28) as \( z \to z_0 \) is

\[
\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] = b_1(m-1)!
\]

or

\[
\text{Res}[z_0, f] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]
\]

which proves Result (14). Result (12) and (13) can be proved on the similar lines.

In order to prove Result (15), you may note that since \( g(z_0) = 0 \) and \( g'(z_0) \neq 0 \), \( g \) has a zero of order 1 at \( z_0 \). From the definition of derivative (ref. Definition 1, Unit 2), we get

\[
g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{g(z)}{z-z_0}
\]

Using Result (12) in Eqn. (29), we obtain

\[
\text{Res}[z_0, f] = \lim_{z \to z_0} (z-z_0) \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}
\]

which proves Result (15).

E13) We have

\[
f(z) = \frac{z^2 + 4z + 5}{z(1+z)}
\]
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\[ z = 1 + \frac{5}{z} - \frac{2}{z+1} \]

Thus, \( \text{Res} [z = -1, f(z)] = \) the coefficient of \( (z+1)^{-1} = -2 \).

Hence

\[ \int f(z) dz = 2\pi i b_1 = 2\pi i \times -2 = -4\pi i . \]

E14) i) As \(-3\) is the double pole of the function, so

\[ \text{Res} \left[ -3, \frac{z^2+1}{z(z+3)^2} \right] = \frac{1}{1!} \lim_{z \to -3} \frac{d}{dz} \left[ (z+3)^2 \frac{z^2+1}{z(z+3)^2} \right] = \lim_{z \to -3} \frac{d}{dz} \left( \frac{z^2+1}{z} \right) = \frac{8}{9}. \]

ii) Here \( z = i+1 \) is a pole of order 3, we get

\[ \text{Res} \left[ i+1, \frac{e^z}{(z-i-1)^3} \right] = \frac{1}{(3-1)!} \lim_{z \to i+1} \frac{d^{(3-1)}}{dz^{(3-1)}} (z-(i+1))^{3} \frac{e^z}{(z-i-1)^3} = \frac{1}{2!} \lim_{z \to i+1} \frac{d^2}{dz^2} e^z = \frac{e^{i+1}}{2}. \]

E15) i) Observe that \( z = 0 \) is a pole of the given function of order 2. Hence

\[ \text{Res} \left[ 0, \frac{\sin z}{z^3 (z-2)(z+1)} \right] = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left[ \frac{\sin z}{z(z-2)(z+1)} \right] = \frac{-1}{2}. \]

ii) We see that \( z = 0 \) is a simple pole of the given function.

\[ \text{Res} \left[ 0, \frac{e^{iz} - 1}{\sin^2 z} \right] = \lim_{z \to 0} \frac{z (e^{iz} - 1)}{\sin^2 z}. \]

This is an indeterminate form \( \left( \frac{0}{0} \right) \) therefore applying L’Hospitals rule, we obtain the residue to be 4.

E16) i) \( \frac{1 - \cosh z}{z^3} = \frac{1 - \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}}{z^3} = \frac{-1}{2z} \cdot \frac{z}{24} \cdots \).

Thus, \( z = 0 \) is a pole of order 1 and the residue = \( -\frac{1}{2} \).

ii) \( \frac{e^{2z}}{(z-1)^2} = \left( \frac{e^{2z} - 2 - 2 + \frac{2}{(z-1)^2} - \cdots}{(z-1)^2} \right) = e^2 \left( \frac{1}{(z-1)^2} + \frac{2}{(z-1)} + 2 + \cdots \right) \).

Thus \( z = 1 \) is pole of order 2 and the residue = \( 2e^2 \).
E17) i) \[ \text{Res} \left[ i, \frac{\log(z)}{(z^2 + 1)^2} \right] = \text{Res} \left[ i, \frac{\log(z)}{(z + i)^3(z - i)^3} \right]. \]

Let \( \phi(z) = \frac{\log(z)}{(z + i)^2}. \) We know that \( \frac{d}{dz} \log(z) = \frac{1}{z}, \) so
\[ \phi'(z) = \frac{2(z + i)\log(z)}{(z + i)^4} \]
\[ \phi'(i) = \frac{4i - (\log(i))4i}{16} = \frac{i}{4}[1 - (\log 1) + i \arg i] = \frac{i}{4} \left[ 1 - \left( \frac{\pi}{2} \right) \right] = \frac{\pi + 2i}{8} \]
\[ \Rightarrow \text{Res} \left[ i, \frac{\log z}{(z^2 + 1)^2} \right] = \frac{\pi + 2i}{8}. \]

ii) Obviously \( z = i \) is a double pole. \( \phi(z) = \frac{\frac{1}{2}}{(z + i)^2} \) is analytic at \( z = i. \)
\[ \therefore \text{Res} \left[ i, \frac{\frac{1}{2}}{(z^2 + 1)^2} \right] = \text{Res} \left[ \frac{d}{dz} \phi(z) \right]_{z=i} = \frac{(z + i)^2 - 4z(z + i)}{2z^2(z + i)^4} \]
\[ = \frac{1 - i}{8\sqrt{2}} \left[ \frac{\sqrt{i}}{\sqrt{2}} = (1 + i) \right]. \]

E18) i) Put \( p(z) = z \) and \( q(z) = \cos z. \)
For \( n = 0, \pm 1, \pm 2, \ldots, p(z_n) = z_n = \frac{\pi}{2} + n\pi \neq 0, \)
\( q(z_n) = \cos(z_n) = -\sin(n\pi) = 0 \)
and \( q'(z_n) = -\sin(z_n) = -\cos(n\pi) = (-1)^{n+1}. \) Therefore
\[ \text{Res}[z_n, z \sec z] = \frac{p(z_n)}{q'(z_n)} = \frac{z_n}{-\cos(n\pi)} = (-1)^{n+1}z_n. \]

ii) Consider \( p(z) = \sinh z \) and \( q(z) = \cosh z \) and proceed as in i) above.

E19) It can be seen by factorising that \( z_1 = \frac{1+i}{\sqrt{2}}, z_2 = \frac{1-i}{\sqrt{2}}, z_3 = \frac{-1+i}{\sqrt{2}} \) and \( z_4 = \frac{-1-i}{\sqrt{2}} \) are the simple poles of the function \( f(z) = \frac{1}{z^4 + 1}. \) The poles \( z_1 \) and \( z_2 \) lie inside the contour \( C \) which is a circle \( (x-1)^2 + y^2 = 1 \) centred at \( (1,0) \) of radius \( 1 \) (see Fig. 11).
Now we compute the residues at \( z_1 \) and \( z_2: \)
\[ \text{Res} \left[ z_1, \frac{1}{z^4 + 1} \right] = \lim_{z \to z_1} \frac{(z-z_1)}{z^4 + 1} \]
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\[
\lim_{z \to z_1} \frac{(z - z_1)}{(z - z_3)(z - z_2)(z - z_4)} = \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{1}{2\sqrt{2}(i - 1)}
\]

\[
\text{Res} \left[ z_2, \frac{1}{z^4 + 1} \right] = \lim_{z \to z_2} \frac{(z - z_2)}{z^4 + 1}
\]

\[
= \lim_{z \to z_2} \frac{(z - z_2)}{(z - z_2)(z - z_1)(z - z_3)(z - z_4)} = \frac{1}{2\sqrt{2}(i + 1)}
\]

By the residue theorem

\[
\int_{C} \frac{dz}{z^4 + 1} = 2\pi i \left( \text{Res}_{z_1} \left[ \frac{1}{z^4 + 1} \right] + \text{Res}_{z_2} \left[ \frac{1}{z^4 + 1} \right] \right)
\]

\[
= 2\pi i \left[ \frac{1}{2\sqrt{2}(i - 1)} - \frac{1}{2\sqrt{2}(i + 1)} \right] = -\frac{\pi i}{\sqrt{2}}.
\]

Note that you can also solve this problem by using the partial fractions and the Cauchy integral formula.

E20) Given \( f(z) = \frac{e^z}{(z + 1) \sin z} \). The poles of \( f \) are 0 and \(-1\). Only the pole 0 lies inside the simple closed contour \( C \). Also it is a simple pole.

\[
\text{Res}_{z=0} \left[ \frac{e^z}{(z + 1) \sin z} \right] = \lim_{z \to 0} \frac{ze^z}{(z + 1) \sin z} = 1.
\]

Using the residue theorem, we get

\[
\frac{1}{2\pi i} \int_{C} \frac{e^z}{(z + 1) \sin z} \left( z : |z| = \frac{1}{2} \right) = \text{Res}_{z=0} \left[ \frac{e^z}{(z + 1) \sin z} \right] = 1.
\]

E21) Here \( f(z) = \frac{\cosh(\pi z)}{z(z^2 + 1)} \). Point 0 is a pole of order one and

\[
\phi(z) = \frac{\cosh(\pi z)}{z(z^2 + 1)} \text{ is analytic at } z = 0 \text{ (see Fig. 12). Thus}
\]

\[
\text{Res}[0, f] = \phi(0) = \frac{\cosh(0)}{1} = 1.
\]

Point \( z = i \) is a pole of order one and \( \phi(z) = \frac{\cosh(\pi z)}{z + i} \) is analytic at \( z = i \). Thus

\[
\text{Res}[i, f] = \phi(i) = \frac{\cosh(\pi i)}{i.2i} = \frac{1}{2}.
\]

Similarly, \( \text{Res}[-i, f] = \phi(-i) = \frac{\cosh(-\pi i)}{(-i)(-i - i)} = \frac{1}{2} \).

Then by the Cauchy’s residue theorem
\[
\int_{|z|=2} \frac{\cosh(\pi z)}{z(z^2+1)} \, dz = 2\pi i \times \left[ 1 + \frac{1}{2} + \frac{1}{2} \right] = 4\pi i.
\]

E22) We have \( f(z) = \frac{1}{z^2 \sin z} \). Let \( p(z) = 1 \), \( q(z) = z^2 \sin z \).

Clearly, \( q(z) = 0 \iff z^2 \sin z = 0 \iff z = n\pi, (n = 0, \pm 1, \pm 2, \ldots) \).

Now \( p(n\pi) = 1 \neq 0 \) and \( q(n\pi) = 0 \), \( q'(z) = 2z \sin z + z^2 \cos z \).

Thus, \( q(n\pi) = n^2\pi^2 \cos n\pi \neq 0 \). \( n \neq 0 \). Clearly, each singular point \( z = n\pi \) is a simple pole \( n \neq 0 \)

\[
Res[n\pi, f] = \frac{1}{n^2\pi^2 \cos n\pi} = (-1)^n / n^2\pi^2 \text{ if } n \neq 0.
\]

Further, \( \frac{1}{z^2 \sin z} = \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \cdots \).

Thus \( z = 0 \) is a pole of \( f \) of order 3 and \( Res[0, f] = 1/6 \).

\[
\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \times \left[ \frac{1}{6} + \frac{1}{2} \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right] \times 2 \quad \text{No. of singular points (see Fig. 13) are symmetric with respecto to the origin.}
\]

\[
= 2\pi i \left[ \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right].
\]

We know that \( \int_{C_N} \frac{dz}{z^2 \sin z} \to 0 \text{ as } N \to \infty \).

\[
\Rightarrow 2\pi i \left[ \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right] \to 0 \text{ as } N \to \infty
\]

\[
\Rightarrow \frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \to 0 \text{ as } N \to \infty \Rightarrow \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \to \frac{\pi^2}{12} \text{ as } N \to \infty .
\]

Fig. 13
E23) Since \( q(z_0) = 0 \) and \( q'(z_0) \neq 0 \) \( \Rightarrow \) \( z = z_0 \) is a zero of order \( m = 1 \) of the function and therefore \( q(z) = (z - z_0) \ g(z) \), where \( g(z) \) is analytic at \( z_0 \) and \( g'(z_0) \neq 0 \).

Now \( f(z) = \frac{1}{(z - z_0)^2 [g(z)]^2} \). Put \( \phi(z) = \frac{1}{[g(z)]^2} \).

Therefore, \( f(z) = \frac{\phi(z)}{(z - z_0)^2} \) \( \Rightarrow \) \( z = z_0 \) is a pole of order \( m = 2 \) for the function \( f \). We have

\[
\text{Res}[z_0, f] = B_0 = \frac{\phi'(z_0)}{1!} = \phi'(z_0).
\]

Now \( \phi'(z) = -2[g(z)]^{-3} \ g'(z) = \frac{-2g'(z)}{[g(z)]^3} \)

\[\Rightarrow \phi'(z_0) = -\frac{2g'(z_0)}{[g(z_0)]^3}, \text{ where } q'(z_0) = g(z_0) \text{ and } q''(z_0) = 2g'(z_0).\]

\[\therefore B_0 = \phi'(z_0) = -\frac{2q''(z_0)/2}{[q'(z_0)]^{3/2}} = -\frac{q''(z_0)}{[q'(z_0)]^{3/2}}.
\]