$$\int_{|z|=2} \frac{\cosh(\pi z)}{z(z^2+1)} dz = 2\pi i \times \left[1+\frac{1}{2}+\frac{1}{2}\right] = 4\pi i.$$

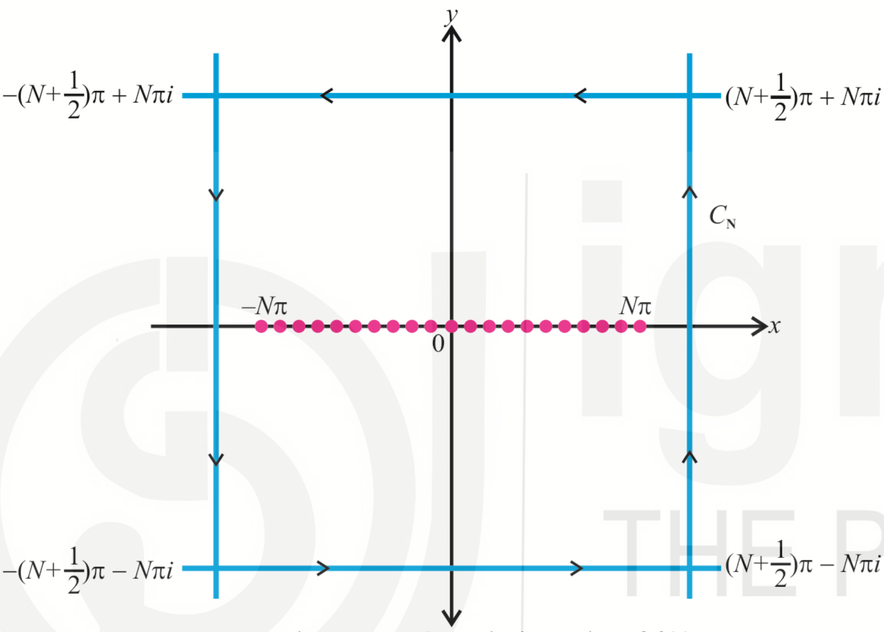
E22) We have $f(z) = \frac{1}{z^2 \sin z}$. Let $p(z) = 1,\ q(z) = z^2 \sin z$.

Clearly, $q(z) = 0 \Leftrightarrow z^2 \sin z = 0 \Leftrightarrow z = n\pi,\ (n = 0, \pm 1, \pm 2, \ldots)$.

Now $p(n\pi) = 1 \neq 0$ and $q(n\pi) = 0,\ q'(z) = 2z \sin z + z^2 \cos z$.

Thus, $q'(n\pi) = n^2\pi^2 \cos n\pi \neq 0$. $(n \neq 0)$. Clearly, each singular point $z = n\pi$ is a simple pole $(n \neq 0)$

$$Res[n\pi, f] = \frac{1}{n^2\pi^2 \cos n\pi} = (-1)^n / n^2\pi^2 \text{ if } n \neq 0.$$



rectangular contour C_N enclosing poles of $f(z)$

Fig. 13

Further, $\frac{1}{z^2 \sin z} = \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \cdots$.

Thus $z = 0$ is a pole of f of order 3 and $Res[0, f] = 1/6$.

$$\therefore \int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \times \frac{1}{6} + \left(2\pi i \sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right) \times 2$$

No. of singular points (see Fig. 13) are symmetric with respecto to the origin.

$$= 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

We know that $\int_{C_N} \frac{dz}{z^2 \sin z} \to 0$ as $N \to \infty$.

$$\Rightarrow \quad 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right] \to 0 \text{ as } N \to \infty$$

$$\Rightarrow \quad \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{N} \frac{(-1)^n}{n^2} \to 0 \text{ as } N \to \infty \Rightarrow \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \to \frac{\pi^2}{12} \text{ as } N \to \infty.$$

E23) Since $q(z_0)=0$ and $q'(z_0)\neq 0 \Rightarrow z=z_0$ is a zero of order $m=1$ of the function and therefore $q(z)=(z-z_0)\ g(z)$, where $g(z)$ is analytic at z_0 and $g'(z_0)\neq 0$.

Now $f(z)=\dfrac{1}{(z-z_0)^2[g(z)]^2}$. Put $\phi(z)=\dfrac{1}{[g(z)]^2}$.

Therefore, $f(z)=\dfrac{\phi(z)}{(z-z_0)^2} \Rightarrow z=z_0$ is a pole of order $m=2$ for the function f. We have

$Res[z_0,\ f]=B_0=\dfrac{\phi'(z_0)}{1!}=\phi'(z_0)$.

Now $\phi'(z)=-2[g(z)]^{-3}g'(z)=\dfrac{-2g'(z)}{[g(z)]^3}$

$\Rightarrow\quad \phi'(z_0)=-\dfrac{2g'(z_0)}{[g(z_0)]^{-3}}$, where $q'(z_0)=g(z_0)$ and $q''(z_0)=2g'(z_0)$.

$\therefore\quad B_0=\phi'(z_0)=-\dfrac{2.q''(z_0)/2}{[q'(z_0)]^3}=-\dfrac{q''(z_0)}{[q'(z_0)]^3}$.

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