$$= \frac{1}{z-a}\sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(z-a)^n} = \sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(z-a)^{n+1}}$$

Similarly, if $|z+a| > |a|$, then

$$f(z) = \sum_{n=0}^{\infty}\frac{a^n}{(z+a)^{n+1}}.$$

E16) For $0 < |z+1| < 2$, it is easy to see that

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{(z+1)(z+1+2)}$$

$$= \frac{1}{2(z+1)}\frac{1}{1+\frac{z+1}{2}}$$

$$= \frac{1}{2(z+1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots.$$

E17) The function f is analytic in all the three domains:
$D_1 = \{z \in \mathbb{C} : |z| < 1\}$, $D_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$,
$D_3 = \{z \in \mathbb{C} : 2 < |z| < \infty\}$.
For $z \in D_1, |z| < 1$ and $|z/2| < 1$ so that

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty}(2^{-n-1}-1)z^n.$$

For $z \in D_2$ $|1/z| < 1$ and $|z/2| < 1$, which gives

$$f(z) = \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

Finally, for $z \in D_3, |1/z| < 1$ and $|2/z| < 1$, that yields

$$f(z) = \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{z}\frac{1}{1-\frac{2}{z}}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}.$$

E18) We have $(w = e^{i\phi}, dw = ie^{i\phi} d\phi)$ for $n = 0, \pm 1, \pm 2, \ldots$

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(w)\ dw}{w^{n+1}} \qquad \text{[ref. Eqn. (28)]}$$

$$= \frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] dw}{w^{n+1}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp\left[\frac{z}{2}(e^{i\phi} - e^{-i\phi})\right]}{e^{i(n+1)\phi}}\ ie^{i\phi} d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp[i\ z\sin\phi]}{e^{in\phi}}\ ie^{i\phi} d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\ [i\ z\sin\phi] e^{-in\phi}\ ie^{i\phi} d\phi$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \exp\ [-i(n\phi - z\sin\phi)] d\phi$$

$$= J_n(z).$$

Therefore the Laurent series is given by

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)\ w^n \qquad (0 < |w| < \infty).$$

E19) i) $a_n = 1/e^n$. By the Cauchy-Hadamard formula,

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = \frac{1}{1/e} = e.$$

ii) $a_n = n!/n^n$. By D' Alembert's ratio test,

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_n + 1}\right| = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

iii) $a_n = (2 + (-1)^n)^n$ and

$$|a_n|^{a/n} = 2 + (-1)^n = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

$$\Rightarrow \limsup_{n\to\infty} |a_n|^{1/n} = 3.$$

By the Cauchy-Hadamard formula, the radius of convergence is $1/3$.

E20) $a_n = n + 2^n$ and $b_n = n/2^n$. Then

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} < 1.$$

By recalling the result of real sequences, we have

$$\lim_{n\to\infty} b_n = 0.$$

Now, by D'Alembert's ratio test

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n + 2^n}{n + 1 + 2^{n+1}} = \lim_{n\to\infty}\frac{\frac{n}{2^n} + 1}{\frac{n}{2^n} + \frac{1}{2^n} + 2}.$$

By making use of the above analysis and Example 2, the radius of convergence is $\frac{1}{2}$.

E21) By D'Alembert's ratio test,

$$R = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = \infty$$

so that the series converges for all values of z.

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