Now $z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta$ where $-\pi \le \theta \le \pi$)

$$\therefore \int_C \frac{e^{az}\,dz}{z} = \int_{-\pi}^{\pi} \frac{e^{a e^{i\theta}}}{e^{i\theta}} ie^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + i\sin(a\sin\theta))\,d\theta$$

$$= \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta$$

$$= 0 + 2i\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta, \qquad \text{(ii)}$$

[Since $f(\theta) = e^{a\cos\theta}\sin(a\sin\theta)$ is odd function of θ whereas $g(\theta) = e^{a\cos\theta}\cos(a\sin\theta)$ is an even function.]

From (i) and (ii), we have, $2\pi i = 2i\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta$

$$\Rightarrow \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta = \pi.$$

E10) **Case-I:** $|a| > 4$ i.e., a lies outside contour C (see Fig. 11).

Clearly, $g(z) = \dfrac{z^2+3z-7}{(z-a)^2}$ is analytic in and on the contour C and by the Cauchy's theorem $\int_C g(z)\,dz = 0$. Thus, $f(a) = 0\ (|a| > 4)$.

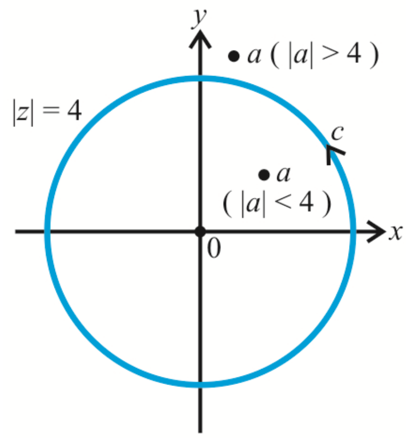

Fig.11

Case-II: $|a| < 4$ then

$$f(a) = \int_{|z|=4} \frac{z^2+3z-7}{(z-a)^2}dz = 2\pi i\left(\frac{d}{dz}(z^2+3z-7)\right)_{z=a}$$

$\Rightarrow f(a) = 2\pi i \times (2a+3)$ and $f'(a) = 4\pi i$.

Therefore, $f'(1+i) = f'(1-i) = 4\pi i$.

E11) Using Cauchy's inequality, we have

$$|f^{(3)}(1)| \le \frac{3!M_R}{R^3}.$$

Here $M_R = 10$ and $R = 3$, thus we get

$$|f^{(3)}(1)| \le \frac{3!\times 10}{3^3} = \frac{6\times 10}{27} = \frac{20}{9}.$$

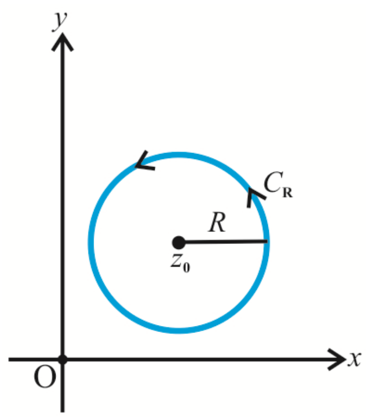

Fig. 12

E12) Let z_0 be any point of the plane. Consider a circle centered at z_0 and of radius R. Then for point $z = z_0 + Re^{i\theta}$ on C_R (see Fig. 12), we have

$$|f(z)| \le A|z_0 + Re^{i\theta}| \le A(|z_0| + R)$$

i.e. $M_R \le A(|z_0| + R)$ on C_R.

Using Cauchy's inequality, we get

$$|f''(z_0)| \le \frac{2!M_R}{R^2} \le \frac{2A(|z_0|+R)}{R^2} = \frac{2A|z_0|}{R^2} + \frac{1}{R}.$$

For R large enough $|f''(z_0)|$ is equal to zero $\Rightarrow f''(z_0) = 0$.

Since z_0 is any arbitrarily chosen point of complex plane, therefore $f''(z) = 0 \ \forall \ z$. $\Rightarrow f'(z) = a_1 \Rightarrow f(z) = a_1 z + b$ for complex constants a_1 and b. But $|f(z)| \le A|z| \Rightarrow f(0) = 0 \Rightarrow b = 0$. Thus $f(z)$ is of the form $f(z) = a_1 z$.

E13) No, if it were so, f will be bounded and hence a constant function by Liouville's theorem.

E14) Let $f(z)$ be entire function whose real part is bounded. So there exists $M > 0$ such that $Re(f(z))| \le M$. Consider the function $g(z) = e^{(f(z))}$. Then $|e^{(f(z))}| = Re^{(f(z))} \le e^M$.
This shows that the entire function g is bounded and hence is constant by Liouville's theorem and as a result, f is also constant.

E15) As $|f(z)| \ge \varepsilon$ for all z, so $f(z) \ne 0$ for all z. Consider the function $g(z) = \frac{1}{f(z)}$. Then g is analytic for all z, i.e. g is entire and $|g(z)| \le \frac{1}{\varepsilon}$. Hence g is entire and bounded and is constant by the Liouville's theorem. Thus $f(z)$ is also constant.

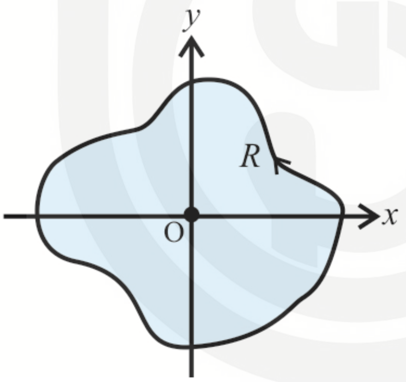

Fig. 13

E16) Using Theorem 8, the maximum value of $|f(z)|$ occurs on the boundary of R. As $|f(z)| = e^{Re(z)}$, and e^x being an increasing function, the maximum value occurs on the boundary line $1 + i\pi$.

E17) Consider $f(z) = z$. It is continuous and analytic everywhere and $f(z) = 0$ for $z = 0$ inside the region R. Clearly, $|f(z)|$ attains its minimum value 0 at an interior point without being a constant function (see Fig. 13).

E18) Consider $\phi(z) = e^{f(z)}$ which is continuous in R and analytic and non constant in the interior of R. Then by Example 13, $|\phi(z)| = e^{Re f(z)} = e^{u(x,y)}$, which is continuous in R must assumes its minimum value on the boundary of R. This implies that $u(x, y)$ assumes its minimum value on the boundary of R.

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