UNIT 4  COMPLEX INTEGRATION-I

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4.1 INTRODUCTION

In the earlier units you have studied differentiation of complex analytic functions. In this unit we will introduce you to complex integration. We begin by discussing the integration of complex valued function of a real variable. We also discuss some basic results regarding differentiation of such functions. In Sec. 4.3 we introduce you to contour integration, a powerful tool in Complex Analysis. In Sec. 4.4, we discuss the existence of antiderivatives. In Sec. 4.5 we prove Cauchy-Goursat theorem an important result in complex analysis.

Objectives
After studying this unit, you should be able to:
• find the derivatives of complex valued functions of a real variable;
• state and apply the chain rule for differentiation of complex valued functions;
• define the concepts of arc, contour, rectifiable arc and the arc length of a rectifiable arc;
• define, state and apply the properties of complex valued functions of a real variable;
• define the integral of a function over a contour and state its basic properties; and
• state the Cauchy-Goursat theorem and apply it to evaluate contour integrals whenever possible.

4.2 COMPLEX VALUED FUNCTIONS OF A REAL VARIABLE

In your undergraduate courses in Calculus and Real Analysis, you have learnt how to differentiate and integrate real valued functions of a real variable. In this section we will see how to differentiate and integrate complex valued functions of a real variable. As we will see, this is similar to differentiation of real valued functions of a real variable. Most of the time, you can differentiate or integrate a complex valued function of a real variable like a real valued function, treating $i$ as a constant.
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However, some of the theorems that are true for the real valued functions are no longer true for complex valued functions of a real variable. The aim of this section is to sketch the proofs of some of the theorems for complex valued functions of a real variable that are analogous to the ones for real valued functions of real variables. Please go through the section, fill up the gaps in the proofs, if there are any, and give the missing proofs.

Given a complex valued function \( f \) defined on a subset of \( \mathbb{R} \), we can write
\[
f(t) = u(t) + iv(t)
\]
We call \( u(t) \) and \( v(t) \) the real and imaginary parts of the function \( f(t) \), respectively. Let us look at an example to understand this.

Example 1: Write the real and imaginary parts of the following complex valued function \( f \) of a variable \( t \):

i) \( f : \mathbb{R} \rightarrow \mathbb{C} \) defined by \( f(t) = (1+it)^2 - 3(1+it) + 4t + i \).

ii) \( f : \mathbb{R} \rightarrow \mathbb{C} \) defined by \( f(t) = (t+i)e^{2+3it} \).

Solution: i) We have
\[
(1+it)^2 - 3(1+it) + 4t + i = 1 - t^2 + 2it - 3 - 3it + 4t + i = (-t^2 + 4t - 2) + i(1-t).
\]
\[\therefore u(t) = -t^2 + 4t - 2 \quad \text{and} \quad v(t) = 1 - t.\]

ii) We have
\[
e^{2+3it} = e^2 e^{3it} = e^2 (\cos 3t + i \sin 3t)
\]
\[= e^2 \cos 3t + ie^2 \sin 3t
\]
\[\therefore (t+i)e^{2+3it} = (t+i)(e^2 \cos 3t + ie^2 \sin 3t)
\]
\[= te^2 \cos 3t - e^2 \sin 3t + i(e^2 \cos 3t + te^2 \sin 3t)
\]
\[\therefore u(t) = te^2 \cos 3t - e^2 \sin 3t \quad \text{and} \quad v(t) = e^2 \cos 3t + te^2 \sin 3t
\]

Try the following exercises to check your understanding of Example 1.

E1) Write the real and imaginary parts of the function \( f : \mathbb{R} \rightarrow \mathbb{C} \) defined by \( f(t) = \frac{1}{i + \sin^2 t} \).

Next, we define the derivative of a complex valued function of a real variable. You will see that this is quite similar to the real variable case.

Definition 1: Let \( ]a,b[ \subseteq \mathbb{R} \) and \( t_0 \in ]a,b[ \). We say that \( f : ]a,b[ \rightarrow \mathbb{C} \) is differentiable at \( t_0 \) if
\[
\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} (2)
\]
exists. If the limit in Eqn. (2) exists, we call the limit the derivative of \( f \) at \( t_0 \) and denote it by \( f'(t_0) \).
In other words, \( f \colon [a,b] \to \mathbb{C} \) is differentiable at \( t_0 \in ]a,b[ \) with derivative \( \alpha \) if, given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that
\[
\left| \frac{f(t) - f(t_0)}{t - t_0} - \alpha \right| < \varepsilon \quad \text{if} \quad |t - t_0| < \delta
\]  
(3)

A function \( f : [a,b] \to \mathbb{C} \) is differentiable in \([a,b]\) if it is differentiable in \([a,b]\) and the limits
\[
f'(a^+) = \lim_{t \to a^+} \frac{f(t) - f(a)}{t - a}
\]
and
\[
f'(b^-) = \lim_{t \to b^-} \frac{f(t) - f(b)}{t - b}
\]
exists.

Before we discuss some examples, we prove a proposition which relates the derivative of a complex valued function of a real variable to the derivatives of its real and imaginary parts. This proposition will help us to prove quickly some of the basic results we need about complex valued functions by appealing to results from one variable calculus.

**Proposition 1:** Let \( f : [a,b] \to \mathbb{C} \) be a function and suppose \( f(t) = u(t) + iv(t) \), where \( u(t) : [a,b] \to \mathbb{R} \) and \( v(t) : [a,b] \to \mathbb{R} \). Then, \( f \) is differentiable at \( t_0 \) if and only if \( u(t) \) and \( v(t) \) are differentiable at \( t_0 \). Further, if \( f(t) \) is differentiable at \( t_0 \),
\[
f'(t_0) = u'(t_0) + iv'(t_0)
\]  
(4)

**Proof:** Suppose \( u(t) \) and \( v(t) \) are differentiable at \( t_0 \) and suppose \( u'(t_0) = \alpha_1 \) and \( v'(t_0) = \alpha_2 \). Then, by \( \varepsilon - \delta \) definition of derivative, given \( \varepsilon > 0 \), there are \( \delta_1, \delta_2 \) such that
\[
\left| \frac{u(t) - u(t_0)}{t - t_0} - \alpha_1 \right| < \frac{\varepsilon}{2} \quad \text{if} \quad |t - t_0| < \delta_1
\]  
(5)

and
\[
\left| \frac{v(t) - v(t_0)}{t - t_0} - \alpha_2 \right| < \frac{\varepsilon}{2} \quad \text{if} \quad |t - t_0| < \delta_2
\]  
(6)

Suppose we are given an \( \varepsilon > 0 \). To show that \( f'(t_0) = \alpha = \alpha_1 + i\alpha_2 \) we have to show that we can find a \( \delta \) such that
\[
\left| \frac{f(t) - f(t_0)}{t - t_0} - \alpha \right| < \varepsilon \quad \text{if} \quad |t - t_0| < \delta
\]
We have, using triangle inequality,
\[
\left| \frac{f(t) - f(t_0)}{t - t_0} - \alpha \right| = \left| \frac{u(t) - u(t_0)}{t - t_0} - \alpha_1 \right| + \left| \frac{v(t) - v(t_0)}{t - t_0} - \alpha_2 \right|
\]
\[
\leq \left| \frac{u(t) - u(t_0)}{t - t_0} - \alpha_1 \right| + \left| \frac{v(t) - v(t_0)}{t - t_0} - \alpha_2 \right| \leq \delta_1 + \delta_2
\]  
(7)

If we choose \( \delta = \min \{ \delta_1, \delta_2 \} \), where \( \delta_1 \) and \( \delta_2 \) are as in Eqn. (5) and Eqn. (6), then if \( |t - t_0| < \delta \), then \( |t - t_0| < \delta_1 \) and \( |t - t_0| < \delta_2 \) also. By Eqn. (5), Eqn. (6) and Eqn. (7) it follows that
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\[ \frac{f(t) - f(t_0)}{t - t_0} - \alpha < \epsilon \text{ if } |t - t_0| < \delta \]

So, if the real and imaginary parts \( u(t) \) and \( v(t) \) of \( f(t) \) are differentiable at \( t_0 \), \( f(t) \) is also differentiable at \( t_0 \) and Eqn. (4) holds good.

To prove the other implication, we start with a simple observation. If \( \gamma = u + iv \), we have

\[ |\text{Re}(\gamma)| = |u| \leq \sqrt{u^2 + v^2} = |\gamma| \]

(8)

\[ |\text{Im}(\gamma)| = |v| \leq \sqrt{u^2 + v^2} = |\gamma| \]

(9)

Suppose \( f \) is differentiable at \( t_0 \) with derivative \( f'(t_0) = \alpha + i\alpha_2 \). Then, given \( \epsilon > 0 \), we can find a \( \delta > 0 \) such that

\[ \frac{|f(t) - f(t_0)|}{t - t_0} - \alpha < \epsilon \text{ if } |t - t_0| < \delta \]

(10)

Taking

\[ \gamma = \frac{f(t) - f(t_0)}{t - t_0} - \alpha = \frac{u(t) - u(t_0)}{t - t_0} - \alpha_1 + i\left( \frac{v(t) - v(t_0)}{t - t_0} - \alpha_2 \right) \]

and applying Eqn. (8) and Eqn. (9), we get

\[ \left| \frac{u(t) - u(t_0)}{t - t_0} - \alpha_1 \right| = |\text{Re}(\gamma)| \leq |\gamma| < \epsilon \]

and

\[ \left| \frac{v(t) - v(t_0)}{t - t_0} - \alpha_2 \right| = |\text{Im}(\gamma)| \leq |\gamma| < \epsilon \]

if \( |t - t_0| < \delta \). By the definition of derivative, it follows that \( u'(t_0) = \alpha_1 \) and \( v'(t_0) = \alpha_2 \) and Eqn. (4) holds.

We now derive some basic properties of derivatives that will be helpful in calculating derivatives.

**Proposition 2:** Let \( f \) and \( g \) be functions defined on \( [a, b] \) with values in \( \mathbb{C} \). Suppose \( f \) and \( g \) are differentiable at \( t_0 \in ]a, b[ \). Then:

i) For all \( \alpha, \beta \in \mathbb{C} \), \( \alpha f + \beta g \) is differentiable at \( t_0 \) and

\[ (\alpha f + \beta g)'(t_0) = \alpha f'(t_0) + \beta g'(t_0). \]

(11)

ii) The function \( f(t)g(t) \) is differentiable at \( t_0 \) and

\[ (f g)'(t_0) = f'(t_0) g(t_0) + f(t_0) g'(t_0) \]

(12)

iii) If \( g(t_0) \neq 0 \), \( \frac{f(t)}{g(t)} \) is differentiable at \( t_0 \) and

\[ \left( \frac{f}{g} \right)'(t_0) = \frac{g(t_0) f'(t_0) - f(t_0) g'(t_0)}{g(t_0)^2}. \]

(13)

**Proof:** We give a sketch of the proof of Eqn. (13). We leave Eqn. (11) and Eqn. (12) to you as exercises.

We have
\[
\lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{1}{g(t)} \left\{ f(t)g(t_0) - f(t_0)g(t) \right\}
\]

Adding and subtracting \( f(t_0)g(t_0) \) in the numerator, we get

\[
\lim_{t \to t_0} \frac{1}{g(t_0)} \left\{ g(t) \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \right\} = \frac{f(t_0) - f(t)}{g(t) - g(t_0)}
\]

Note that, all the limits in the last step exist. We have

\[
\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = f'(t_0) \quad \text{and} \quad \lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} = g'(t_0)
\]

We get the result by substituting these values in Eqn. (16).

In the next exercises, we ask you to prove Eqn. (11) and Eqn. (12).

E2) Prove Eqn. (11) and Eqn. (12).

Note that Eqn. (11) implies in particular that we can ‘pull out constants’ while differentiating a complex valued function, i.e. if \( f : [a, b] \to \mathbb{C} \) is differentiable at \( t_0 \), \( \alpha f \)'(\( t_0 \)) = \( \alpha f \)'(\( t_0 \)) for all \( \alpha \in \mathbb{C} \).

Note also that, all the results we have proved in Proposition 2 hold for the derivatives at the points \( a \) and \( b \) of a closed interval \( [a, b] \) if we replace the derivatives in the statement by suitable one sided derivatives. This is because all we have used in the proofs are the properties of limits and these hold for one sided limits also.

The next thing we discuss is the ‘Chain rule’ or ‘function of function rule’. Actually, there are two situations in the case of complex valued functions of a real variable. We state both of them.

**Proposition 3:** i) Let \( f : [a, b] \to [c, d] \) and \( g : [c, d] \to \mathbb{C} \) be such that \( f \) is differentiable at \( t_0 \in [a, b] \) and \( g \) is differentiable at \( f(t_0) \) in \( [c, d] \).

Then, \( g \circ f \) is differentiable at \( t_0 \) and

\[
(g \circ f)'(t_0) = g'(f(t_0)) f'(t_0)
\]

ii) Let \( \Omega \) be a region in \( \mathbb{C} \) and \( f : [a, b] \to \Omega \) and \( g : \Omega \to \mathbb{C} \) be such that \( f \) is differentiable at \( t_0 \in [a, b] \) and \( g \) is analytic at \( t_0 \). Then we have

\[
(g \circ f)'(t_0) = g'(f(t_0)) f'(t_0)
\]
Proof:

i) Write \( g(t) = u(t) + iv(t) \). We have
\[
(g \circ f)(t) = (u \circ f)(t) + i(v \circ f)(t)
\]
Check that, by Eqn. (4),
\[
(g \circ f)'(t_0) = (u \circ f)'(t_0) + i(v \circ f)'(t_0)
\]
(20)
Notice that, in Eqn. (21), \( u, v \) and \( f \) are all real valued functions of one variable and we can apply the rule for real functions. So,
\[
(u \circ f)'(t_0) = u'(f(t_0)) f'(t_0)
\]
(21)
\[
(v \circ f)'(t_0) = v'(f(t_0)) f'(t_0)
\]
(22)
We leave it to you to deduce Eqn. (18) from Eqn. (20), Eqn. (21) and Eqn. (22).

ii) For convenience, we will write a complex number as an ordered pair. We will write
\[
(g \circ f)(t) = u(t) + iv(t)
\]
when \( z = x + iy \). We also write
\[
(g \circ f)'(t_0) = u'(t_0) + iv'(t_0)
\]
Remember that, the chain rule for real valued functions is usually proved using the mean value theorem. However, mean value theorem is not valid for complex valued functions of a real variable. It doesn’t hold for \( f : [0, 2\pi] \rightarrow \mathbb{C} \), \( f(t) = e^{it} \). This is because \( f(2\pi) - f(0) = 0 \), but \( f'(t) \neq 0 \) for all \( t \in [0, 2\pi] \). So, the proof of the real variable case doesn’t carry over to the proof of Eqn. (19).

Let us look at an example to understand the applications of the results that we have proved so far.

Example 2: Find the derivative of each of the following expressions when treated as a function of a real variable, wherever they are defined and whenever the derivatives exist.

i) \( \frac{1}{i + \sin^2 t} \)

ii) \( \sin (t + it^2) \).
Solution: i) Let \( f : \mathbb{R} \to [-1,1] \) be defined by \( f(t) = \sin t \) and \( g : [-1,1] \to \mathbb{C} \) be defined by
\[
g(t) = \frac{1}{i + t^2}.
\]
From Calculus we know that \( f(t) \) is differentiable on \( \mathbb{R} \). Since \( t \) is real, \( t^2 \neq i \) for all \( t \in \mathbb{R} \). So, \( g(t) \) is defined and differentiable on \( \mathbb{R} \) and on \([-1,1]\) in particular.

We have,
\[
(g \circ f)(t) = \frac{1}{i + \sin^2 t}
\]
Here, we are in the first situation discussed in Eqn. (18).

Using Eqn. (18), we have
\[
(g \circ f)(t) = g'(f(t)) f'(t)
\]
We have \( f'(t) = \cos t \).
Using Eqn. (13)
\[
g'(t) = \frac{(i + t^2)0 - 1 \cdot 2t}{(i + t^2)^2} = \frac{2t}{(i + t^2)^2}
\]
\[
\therefore g'(f(t)) = \frac{-2\sin t}{(i + \sin^2 t)^2}
\]
\[
\therefore g'(f(t))f'(t) = \frac{-2\sin t}{(i + \sin^2 t)^2} \cos t.
\]

ii) Let \( f : \mathbb{R} \to \mathbb{C} \) be defined by \( f(t) = t + it^2 \) and \( g : \mathbb{C} \to \mathbb{C} \) be defined by
\( g(z) = \sin z \).

We have
\[
(g \circ f)'(t) = g'(f(t)) f'(t)
\]
By using proposition 3(ii), we have
\( f'(t) = 1 + 2it \) and \( g'(z) = \cos z \)
\[
\therefore (g \circ f)'(t) = \cos(t + it^2)(1 + 2it)
\]

We end this section with an exercise for you.

E3) Find the derivative of each of the following expressions when treated as a function of a real variable \( t \), wherever it is defined and whenever the derivatives exist
i) \( 1 + i \tan^2 t \) \quad ii) \( \log(i + t^2) \)

We now begin our discussion of integration along a curve in the complex plane with the definition of an arc.

**Definition 2:** A parametrised arc is a continuous function \( z : [a, b] \to \mathbb{C} \). We call \( z(a) \) and \( z(b) \) the beginning and end point of the arc, respectively. An arc is called a closed arc if \( z(a) = z(b) \). A simple arc is a continuous 1–1 function \( z : [a, b] \to \mathbb{C} \). In other words, a simple arc is an arc that doesn’t
intersect itself, i.e. \( z(t_1) \neq z(t_2) \) if \( t_1 \neq t_2 \). We also say that a closed arc is simple if it doesn’t intersect itself at any point other than the end points. In other words, a simple closed arc is an arc \( z(t) : [a,b] \to \mathbb{C} \) such that \( z(t_1) = z(t_2) \) for \( t_1 \neq t_2 \) if and only if \( t_1 = a, t_2 = b \) or \( t_1 = b, t_2 = a \).

Fig. 1

Let us look at some examples to understand the concept of an arc.

**Example 3:** Consider \( z_1 : [0, 2\pi] \to \mathbb{C} \), defined by \( z_1(t) = e^{it} \). If we consider the point \( P \) on the circle given by \( P = (\cos t, \sin t) \), the point \( P \) traverses the circle in the anti-clockwise direction as \( t \) varies from 0 to \( 2\pi \). See Fig. 2. If you imagine a person walking along the arc as it is traced, the interior of the circle will be to her left. By convention, we call this a **positively oriented** arc.

![Fig. 2](image2.png)

Consider the function \( z_2(t) : [0, 2\pi] \to S \), given by \( z_2(t) = e^{-it} \). In this case, the point \( P = (\cos t, -\sin t) \) traverses the circle in the **clockwise direction**. See Fig. 3. If you imagine a person walking along the arc as it is traced, the area of the circle will be to her right. By convention, this is called a **negatively oriented** arc.
Consider the function \( z_1 : [0,1] \rightarrow S \), given by \( z_1(t) = e^{2\pi it} \). In this case also, the point \( P = (\cos 2\pi t, \sin 2\pi t) \) traverses the unit circle in the anti-clockwise direction. However, the domain is different in this case.

Consider the function \( z_2 : [0,2\pi] \rightarrow S \) given by \( z_2(t) = e^{2\pi t} \). In this case, \( P = (\cos 2t, \sin 2t) \) traverses the circle in the anti-clockwise direction, but ‘twice as fast as’ \( z_1(t) \). As \( t \) varies from 0 to \( 2\pi \), \( P \) goes around the circle twice. This is not a simple closed arc because \( \pi 4 = \frac{5\pi}{4} \), but \( \frac{\pi}{4} \) and \( \frac{5\pi}{4} \) are not the end points of the arc.

**Definition 3:** We call an arc \( z : [a,b] \rightarrow \mathbb{C} \) is smooth if:

i) The derivative \( z'(t) \) is continuous in \([a,b]\). By this we mean that \( z'(t) \) is continuous in \([a,b]\) and \( \lim_{t \to a^+} z'(x) = z'(a^+) \) and \( \lim_{t \to b^-} z'(x) = z'(b^-) \).

ii) For all \( t \in [a,b], \ z'(t) \neq 0 \).

Notice that in Example 3 all the three functions \( z_1(t), z_2(t) \) and \( z_3(t) \) have the same image in \( \mathbb{C} \), in other words

\[
\{z_1([0,2\pi]) = z_2([0,2\pi]) = z_3([0,1]) = \{z \in \mathbb{C} | |z| = 1\}.
\]

So, can we consider all of them the same? We also saw that \( z_1(t) \) is in the anti-clockwise direction and \( z_2(t) \) is in the clockwise direction. Are these arcs the 'same' in some sense because they have the same image in \( \mathbb{C} \)? This is the subject of our next definition.

**Definition 4:** Let \( z_1 : [a,b] \rightarrow \mathbb{C} \) and \( z_2 : [c,d] \rightarrow \mathbb{C} \) be two arcs. We say that \( z_1 \) is equivalent to \( z_2 \) if there is an onto function \( \phi : [c,d] \rightarrow [a,b] \) whose derivative is continuous on \([c,d]\) and \( \phi'(t) > 0 \) on \([c,d]\) and \( z_1(\phi(t)) = z_2(t) \) for all \( t \in [c,d] \). We write \( z_1 \sim z_2 \) in this case. (See Fig. 4.)
Let us now look at an example.

**Example 4:** Consider $z_1(t)$ and $z_2(t)$ in Example 3. Consider the function \( \phi(t):[0,1] \to [0,2\pi] \) defined by \( \phi(t) = 2\pi t \). Then, \( \phi'(t) = 2\pi \neq 0 \). So, $z_1(t)$ and $z_2(t)$ are equivalent.

However, note that $z_1(t)$ and $z_2(t)$ are not equivalent. Indeed, if there is a function \( \phi(t):[0,2\pi] \to [0,2\pi] \) such that \( z_1(t) = z_2(t) \), then \( e^{i\phi(t)} = e^{-i\pi} \) or \( e^{i(\phi(t)+\pi)} = 1 \). So, \( \phi(t) + t = 2k\pi \) for some \( k \in \mathbb{Z} \). The value of \( k \) *a priori* may depend on \( t \). Let us write \( \phi(t) + t = 2\pi f(t) \) where \( f(t) \in \mathbb{Z} \) for all \( t \in [0,2\pi] \).

Writing \( f(t) = \frac{\phi(t)}{2\pi} + t \), we see that \( f(t) \) has a continuous derivative on \( [0,2\pi] \) and hence continuous on \( [0,2\pi] \). So, \( f([0,2\pi]) \) is a connected subset of \( \mathbb{Z} \). Since the only connected subsets of \( \mathbb{Z} \) are singleton sets, \( f(t) \) is a constant, say \( k_0 \). Then \( \phi(t) = 2\pi k_0 = t \). So \( \phi'(t) = -1 \) and the condition \( \phi'(t) > 0 \) cannot be satisfied. Therefore $z_1(t)$ and $z_3(t)$ are not equivalent.

***

**Definition 5:** Let \([a,b]\) be a closed interval in \( \mathbb{R} \). By a *Partition* of \([a,b]\), we mean a subset \( \{t_0 < t_1 < \cdots < t_n\} \subset [a,b] \) such that \( t_0 = a \) and \( t_n = b \). We denote the set of all partitions of \([a,b]\) by \( \mathcal{P}([a,b]) \).

**Definition 6:** An arc \( z:[a,b] \to \mathbb{C} \) is *piecewise smooth* if there is a partition \( \{a = t_0, t_1, \ldots, b = t_n\} \) such that \( z(t) \) is smooth on each of the subintervals \([t_i, t_{i+1}]\). We also call a piecewise smooth arc a *contour*.

Here are some exercises for you.

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E4) Show that the relation ~ defined in Definition 4 is an equivalence relation.

E5) Let $z_1:[a,b] \to \mathbb{C}$ and $z_2:[c,d] \to \mathbb{C}$ be two arcs that are equivalent to each other. Show the following:
   
i) If $z_1$ is simple, so is $z_2$.
   
ii) If $z_1$ is closed, so is $z_2$.
   
iii) If $z_1$ is (piecewise) smooth, so is $z_2$.

---

We now introduce the concept of length of an arc in \( \mathbb{C} \). This involves some preliminaries. We begin with the concept of a function of bounded variation. A detailed discussion of functions of bounded variation is beyond the scope of this course. The aim of the discussion is to demystify some of the definitions that follow. If you are interested in more details, you may refer to Chapter 6, *Mathematical Analysis*, second edition, by T.M. Apostol published by Narosa publishers or Chapter 4, *Functions of Complex Variable*, second edition, by J.B. Conway, published by Springer. We begin by defining a function of bounded variation.
Definition 7: Let \( f \) be a real or complex valued function on \([a, b]\) and \( P = \{a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b\} \) is a partition of \([a, b]\). (See Fig. 5.)

We define

\[
S(P, f) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|
\]

We say that a function \( f : [a, b] \to \mathbb{R} \) is of **bounded variation**, if there is a positive number \( M \) such that \( S(P, f) < M \) for all partitions \( P \in \mathcal{P}([a, b]) \).

Intuitively speaking, functions of bounded variation are those that ‘don’t oscillate much’.

We now state some **sufficient conditions** for a function to be of bounded variation.

**Theorem 1:**

i) Let \( f : [a, b] \to \mathbb{R} \) be a function. If \( f \) is monotonic, then \( f \) is of bounded variation.

ii) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function that is differentiable on \([a, b]\). Further, assume that \( f' \) exists in \([a, b]\) and there is a constant \( A > 0 \) such that \( |f'(x)| \leq A \) for all \( x \in [a, b] \). Then, \( f \) is of bounded variation on \([a, b]\).

**Proof:**

i) Let us suppose \( f : [a, b] \to \mathbb{R} \) is an increasing function and let \( P = \{a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b\} \) be any partition of \([a, b]\).

Since \( f \) is increasing we have \( f(t_i) \geq f(t_{i-1}) \), so

\[
|f(t_i) - f(t_{i-1})| = f(t_i) - f(t_{i-1})
\]

So, we have

\[
S(f, P) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) = f(b) - f(a).
\]

\( \therefore S(f, P) \) is bounded and \( f \) is of bounded variation. The proof is along similar lines when \( f \) is decreasing. In this case we have

\[
|f(t_i) - f(t_{i-1})| = f(t_{i-1}) - f(t_i).
\]

ii) Applying mean value theorem we have \( f(t_i) - f(t_{i-1}) = f'(t'_i)(t_i - t_{i-1}) \)

where \( t'_i \in [t_{i-1}, t_i] \).

\( \therefore |f(t_i) - f(t_{i-1})| = |f'(t'_i)(t_i - t_{i-1})| \leq A(t_i - t_{i-1}) \). Summing from \( i = 1 \) to \( n \) we get \( S(P, f) \leq A \sum_{i=1}^{n} (t_i - t_{i-1}) = A(b-a) \). So, \( S(f, P) \) is bounded and \( f \) is of bounded variation.
**Definition 8:** Let \( z(t) : [a, b] \to \mathbb{C} \) be an arc. We say that \( z \) is **rectifiable** if there is a constant \( M > 0 \) such that \( S(P, z) < M \) for all partitions \( P \in \mathcal{P}([a, b]) \).

If \( z(t) \) is rectifiable, we define the **length of** \( z(t) \) to be
\[
\sup \{ S(P, f) | P \in \mathcal{P}([a, b]) \}
\]
and we denote it by \( L(z) \).

Let us now try to understand the definition of arc length. See Fig. 5. In the figure, we have the image of \( z(t) : [a, b] \to \mathbb{C} \). We also have a partition
\[
P_0 = \{ a = t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7 = b \}.
\]
We know that, in the Euclidean distance, the straight line joining two points give the shortest distance between two points. So the length of the line segment joining the two points \( z(t_i) \) and \( z(t_{i+1}) \), \( 0 \leq i \leq 5 \), is always less than the ‘length of the arc’ joining the two points. (We are using quotes around length of the arc because we are talking about intuitive notion of length of an arc.) Note that, the distance between \( z(t_i) \) and \( z(t_{i+1}) \) is \( |z(t_i) - z(t_{i+1})| \). So, \( S(P_0, z) \) is less than the ‘length of the arc’ joining \( z(a) \) and \( z(b) \).

Intuitively speaking for any partition \( P \in \mathcal{P}([a, b]) \) the sum in the RHS of Eqn. (23) is always less than the ‘length of the arc’ joining \( z(a) \) and \( z(b) \). Also, as we take partitions with more and more points, the sum gives closer and closer approximations of ‘arc length’. So, it makes sense to define the arc length to be supremum of all sums in Eqn. (23), taken over all the partitions of \([a, b]\).

We now give a criterion for an arc to be rectifiable.

**Proposition 4:** Let \( z(t) : [a, b] \to \mathbb{C} \) be an arc and suppose \( x(t), y(t) \) are the real and imaginary parts of \( z(t) \). Then \( z(t) \) is rectifiable if and only if \( x(t) : [a, b] \to \mathbb{R} \) and \( y(t) : [a, b] \to \mathbb{R} \) are of bounded variation.

**Proof:** We give a sketch of the proof. Suppose \( x(t) \) and \( y(t) \) are of bounded variation. There are constants \( M_1 \) and \( M_2 \) such that \( S(P, x) < M_1 \) and \( S(P, y) < M_2 \) for all partitions \( P \in \mathcal{P}([a, b]) \). Let us write \( M = M_1 + M_2 \).

From
\[
|z(t_i) - z(t_{i-1})| = |(x(t_i) - x(t_{i-1}) + i(y(t_i) - y(t_{i-1}))| \leq |x(t_i) - x(t_{i-1})| + |y(t_i) - y(t_{i-1})|,
\]
Summing up from \( i = 1 \) to \( n \) it follows that
\[
S(P, z) \leq S(P, x) + S(P, y) < M_1 + M_2 = M
\]
for all \( \mathcal{P}([a, b]) \). So, \( z(t) \) is rectifiable by definition.

Suppose \( z(t) \) is rectifiable. Then there is a positive constant \( M \) such that
\[
S(P, z) = \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| < M. \text{ From equations (8) and (9), we get}
\]
\[
|x(t_i) - x(t_{i-1})| \leq |z(t_i) - z(t_{i-1})| \quad \text{and} \quad |y(t_i) - y(t_{i-1})| \leq |z(t_i) - z(t_{i-1})|.
\]
Summing up both the inequalities from \( i = 1 \) to \( n \), we get
\[
\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \leq \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| < S(P, z) \quad \text{and}
\]
\[ \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})| \leq \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| \leq S(p, x) \leq M \text{ and } S(P, y) \leq M \] 
for all partitions \( P \) of \([a, b]\). It follows that \( x(t) \) and \( y(t) \) are of bounded variation.

The next result gives a formula for the length of the arc for ‘sufficiently nice’ arcs.

**Proposition 5:** Let \( z(t): [a, b] \to \mathbb{C} \) be a smooth arc. Then, \( z(t) \) is rectifiable and
\[
L(z) = \int_{a}^{b} |z'(t)| dt. \tag{24}
\]

**Proof:** We will sketch a proof the fact that \( z(t) \) is rectifiable. For the proof of Eqn. (24), you can refer to the books by either Apostol or Conway we referred earlier.

The proof of the first part is an application of Proposition 4 and second part of Theorem 1. By Proposition 4, it is enough to show that \( x(t) \) and \( y(t) \) are of bounded variation. Since \( z(t) \) is smooth, \( z'(t) \) is continuous in \((a, b)\). It follows that \( x'(t) \) and \( y'(t) \) are continuous. If we define a function \( g(t) \) by
\[
g(t) = \begin{cases} 
  x'(a^+) & \text{if } t = a \\
  x'(t) & \text{if } a < x < b \\
  x'(b^-) & \text{if } t = b
\end{cases}
\]
then \( g(t) \) is continuous and therefore bounded on the compact set \([a, b]\).

Hence, there is a constant \( A \) such that \( |g(t)| \leq A \). So, it follows that \( |x'(t)| < A \) for all \( t \in [a, b] \). By the second part of Theorem 1, it follows that \( x(t) \) is of bounded variation. We can use a similar argument to prove that \( y(t) \) is of bounded variation. It follows from Proposition 4 that \( z(t) \) is of rectifiable.

**Definition 9:** Suppose that \( z(t) \) is a piecewise smooth arc, i.e. there is a partition \( \{a = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_n = b\} \) and \( z \) is smooth on each of the subintervals \([t_i, t_{i+1}]\). We define the arc length of \( z(t) \) by
\[
L(z) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |z'(t)| dt. \tag{25}
\]

Note that, although it appears as if the arc length of a piecewise smooth arc depends on the partitions chosen, it is not so. We leave the checking to you as an exercise. Note that, all the arcs in Example 3 are smooth. Let us look at an example of a piecewise smooth contour.

**Example 5:** Consider the function \( z(t) = t + it \) defined on \([-1, 1]\) and the partition \( \{-1, 0, 1\} \) of \([-1, 1]\). See Fig. 6. Let us first check that \( z(t) \) is differentiable on \([-1, 0]\). We have
\[
z(t) = \begin{cases} 
  t - it & -1 \leq t \leq 0 \\
  t + it & 0 \leq t \leq 1
\end{cases}
\]
Therefore,
\[ z'(t) = \begin{cases} 
1 - i & -1 < t < 0 \\
1 + i & 0 < t < 1 
\end{cases} \]

So, \( z'(t) \) is continuous in \(-1 < t < 0\) and \( 0 < t < 1 \). You can check that
\[
\lim_{t \to -1^+} \frac{z(t) - z(-1)}{t + 1} = \lim_{t \to -1^+} \frac{t - it - (-1 + i)}{t + 1} = \frac{(t + 1) - i(t + 1)}{t + 1} = 1 - i
\]

\[
\lim_{t \to -1^-} z'(t) = 1 - i
\]

\[
\lim_{t \to 0^-} \frac{z(t) - z(0)}{t - 0} = \lim_{t \to 0^-} \frac{t - it}{t} = 1 - i
\]

\[
\lim_{t \to 0^+} z'(t) = 1 - i
\]

Also, \( z'(t) = 1 - i \neq 0 \) for \(-1 < t < 0\) and

\[
\lim_{t \to 0^-} z'(t) = 1 - i = \lim_{t \to 0^-} \frac{z(t) - z(0)}{t - 0}.
\]

So, \( z(t) \) is smooth on \([-1, 0]\).

\[
\lim_{t \to 0^+} \frac{z(t) - z(0)}{t} = \lim_{t \to 0^+} \frac{(1 + i)t}{t} = 1 + i
\]

\[
\lim_{t \to 0^-} z'(t) = 1 + i
\]

\[
\lim_{t \to 1^-} \frac{z(t) - z(1)}{t} = \lim_{t \to 1^-} \frac{t + it - (1 + i)}{t - 1} = 1 + i
\]

\[
\lim_{t \to 1^+} z'(t) = 1 + i
\]

Again, \( z'(t) = 1 + i \neq 0 \) in \((0, 1)\). So, \( z(t) \) is smooth on \([0, 1]\). Therefore, according to our definition, it is piecewise smooth on \([-1, 1]\). However, it is not smooth on \([-1, 1]\) because \( z'(t) \) is discontinuous at \( t = 0 \). (Why?)

Let us find \( L(z) \) in this case. We have
\[
L(z) = \int_{-1}^{0} |z'(t)| \, dt + \int_{0}^{1} |z'(t)| \, dt
\]

\[
= \int_{-1}^{0} |1 - i| \, dt + \int_{0}^{1} |1 + i| \, dt = 2\sqrt{2}
\]

Here is an exercise for you to try.

E6) Show that length of a piecewise smooth arc is independent of the partition provided the arc is smooth in each of the subintervals defined by the partition.
E7) Consider $z(t) : [0,1] \rightarrow \mathbb{C}$ defined by

\[
z(t) = \begin{cases} 
(9 - 3i)t - (1 + i) & 0 \leq t \leq \frac{1}{3} \\
(-6 + 15i)t + 4 - 7i & \frac{1}{3} \leq t \leq \frac{2}{3} \\
-(3+12i)t + 11i + 2 & \frac{2}{3} \leq t \leq 1
\end{cases}
\]

Check that $z(t)$ is a contour. Find its length.

E8) Consider $z(t) : [0,1] \rightarrow \mathbb{C}$ defined by

\[
z(t) = \begin{cases} 
e^{2\pi i t} & 0 \leq t \leq \frac{1}{2} \\4t - 3 & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Check whether $z(t)$ is a contour.

E9) Let $y(t)$ be the real valued function defined on the interval $0 \leq t \leq 1$ by

\[
y(t) = \begin{cases} 
t^3 \sin t & \text{when } 0 < t \leq 1, \\
0 & \text{when } t = 0.
\end{cases}
\]

i) Show that the equation $z(t) = t + iy(t)$ represents an arc $C$ that intersects the real axis at the points $z = 1/n$, $(n = 1,2,\ldots)$ and $z = 0$, as shown in Fig. 7.

\[C \]

ii) Verify that $z(t)$ is a smooth arc.

We close this section here. In the next section we will discuss the integration of complex valued functions of real variables.

### 4.3 CONTOUR INTEGRATION

In your undergraduate classes, you may have studied line integrals. They were integrals of functions along a curve in $\mathbb{R}^3$. In this section, we will introduce you to something similar, namely integration of complex functions defined on $\mathbb{C}$ along a curve defined on the plane. We begin our discussion with the concept of integral of a complex valued function.

#### 4.3.1 Integration of Complex Valued Functions

We begin our discussion of integration of complex valued functions. As you
might have guessed, it is the sum of the integrals of the real and imaginary parts as in the case of differentiation. We begin with the formal definition of the integral of a complex valued function of a real variable.

**Definition 10:** Let \( f : [a, b] \to \mathbb{C} \) be a function with real and imaginary parts \( u(t) \) and \( v(t) \). Then, \( f \) is **integrable** on \([a, b]\), if \( u : [a, b] \to \mathbb{R} \) and \( v : [a, b] \to \mathbb{R} \) are integrable. We also define

\[
\int_a^b f(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt
\]  

(26)

We now show that many of the results for integrals of real valued functions carry over for complex valued functions also. In many cases, they can be deduced from the corresponding statements for real valued functions. We now prove some results analogous to the results for real valued integrable functions on an interval.

In what follows, we will often write \( \int_a^b f \) instead of \( \int_a^b f(t) \, dt \) if the dummy variable \( t \) is not important for the discussion.

**Proposition 6:**

i) Let \( f : [a, b] \to \mathbb{C} \) be a complex valued, integrable function and suppose that \( \alpha \in \mathbb{C} \). Then, \( \alpha f \) is integrable on \([a, b]\) and we have

\[
\int_a^b \alpha f = \alpha \int_a^b f
\]  

(27)

ii) Let \( f_1 \) and \( f_2 \) be complex valued integrable functions on \([a, b]\). Then, \( f_1 + f_2 \) is integrable on \([a, b]\) and we have

\[
\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2
\]  

(28)

iii) If \( f \) is a complex valued function that is integrable on \([a, c]\) and \([c, b]\), then \( f \) is integrable on \([a, b]\) and

\[
\int_a^b f = \int_a^c f + \int_c^b f
\]  

(29)

**Proof:** We prove only the first statement and leave proof of the other statements to you as an exercise. Let \( \alpha = \alpha_1 + i \alpha_2, \alpha_1, \alpha_2 \in \mathbb{R} \). Then,

\[
\alpha f = (\alpha_1 + i \alpha_2)(u(t) + iv(t)) = (\alpha_1 u_1 - \alpha_2 u_2) + i (\alpha_1 u_2 + \alpha_2 u_1)
\]  

(30)

Note that, \( u_1 \) and \( u_2 \) are real valued integrable functions and \( \alpha_1 \) and \( \alpha_2 \) are real constants. So, by the properties of real valued, integrable functions, it follows that \( \alpha_1 u_1 - \alpha_2 u_2 \) is a real integrable function. (How?) Similarly, \( \alpha_1 u_2 + \alpha_2 u_1 \) is also a real valued integrable function.

Again, from our definition of complex valued integrable functions, it follows that \( \alpha f \) is integrable. By the definition of the complex integral and we have

\[
\int_a^b \alpha f = \int_a^b (\alpha_1 u_1 - \alpha_2 u_2) + i \int_a^b (\alpha_1 u_2 + \alpha_2 u_1)
\]

Using the properties for integrals of real valued functions, we get

\[
\begin{align*}
&= \alpha_1 \int_a^b u_1 - \alpha_2 \int_a^b u_2 + i \alpha_1 \int_a^b u_2 + i \alpha_2 \int_a^b u_1 \\
&= (\alpha_1 + i \alpha_2) \left( \int_a^b u_1 + i \int_a^b u_2 \right) = (\alpha_1 + i \alpha_2) \int_a^b f = \alpha \int_a^b f
\end{align*}
\]
Another important result that we will need is the analogue of the **Fundamental Theorem of Calculus** from the theory of integration of real variables. We now state this result.

**Theorem 2 (Fundamental Theorem of Calculus):** Let
\[ w(t) = u(t) + iv(t) \text{ and } W(t) = U(t) + iV(t) \]
and \( w(t) \) is integrable on \([a, b]\). Also, suppose that \( W(t) \) is differentiable in an open interval containing \([a, b]\) and \( W'(t) = w(t) \) in \([a, b]\). Then,
\[ \int_a^b w(t) \, dt = W(a) - W(b) \quad (31) \]

**Proof:** We have
\[ \int_a^b W(t) \, dt = \int_a^b U(t) \, dt + i \int_a^b V(t) \, dt \quad (32) \]
Further,
\[ U'(t) = u(t) \text{ and } V'(t) = v(t). \]

We can complete the proof by applying the Fundamental Theorem of Calculus for real valued functions to the two integrals on the RHS of Eqn. (32). We leave it to you to complete the details of the proof.

Let us now look at an example.

**Example 6:** Consider the function \( w(t) = (i + at)^2 \). Let us find \( \int_0^1 w(t) \, dt \). We can expand \((i + at)^2\) using binomial theorem and integrate the real and imaginary parts separately. However, we will use Theorem 2 instead.

We find the primitive of \((i + at)^2\) just the same way we do in real variable case:
\[ \int (i + at)^2 \, dt = \frac{1}{a} \left( \frac{(i + at)^3}{3} \right) = W(t) \text{ (say.)} \]
Here, we treat \( i \) as just another constant. You can verify directly that \( W'(t) = w(t) \). So, from Eqn. (32) it follows that
\[ \int_0^1 w(t) = W(1) - W(0) = \frac{1}{a} \left\{ \frac{(i + a)^3}{3} - \frac{-i^3}{3} \right\}. \]

Here are some exercises for you to test your understanding.

---

E10) Prove part ii) of Proposition 6 on the preceding page.

E11) Fill in the details of the proof of Theorem 2.

E12) Evaluate \( \int_0^{\frac{\pi}{2}} \sin i t \, dt \).

---

Next, we define **improper integrals**.

**Definition 11:** Suppose \( a \in \mathbb{R} \); and \( f \) is a complex valued function that is integrable on every sub-interval of \([a, \alpha]\). We say that the integral \( \int_a^\infty f \) exists
if the limit
\[ \lim_{r \to \infty} \int_a^r f \]
exists. We write
\[ \int_a^\infty f = \lim_{r \to \infty} \int_a^r f. \]
We can similarly define \( \int_\infty^a f \).

Suppose that \( f \) is integrable on every sub-interval of \( ]-\infty, \infty[ \). We say that
\( \int_{-\infty}^\infty f \) exists if \( \int_{-\infty}^0 f \) and \( \int_0^\infty f \) exist. We set
\[ \int_{-\infty}^\infty f = \int_{-\infty}^0 f + \int_0^\infty f \]
Suppose that \( f \) is integrable on every sub-interval of \( ]-\infty, \infty[ \). We say that the
Cauchy principal value \( \int_{-\infty}^\infty f \) exists if, the limit
\[ \lim_{r \to \infty} \int_{-r}^r f \]
eists. We write
\[ PV \int_{-\infty}^\infty f = \lim_{r \to \infty} \int_{-r}^r f \]
Note that Proposition 6 is true for improper integrals also because of the
properties of limits.

Let us look at some examples.

Example 7: Consider the function \( f(x) = e^{-x} \). Then \( f \) integrable on every
sub-interval of \([0, \infty]\). Further, we have
\[ \int_0^\infty e^{-x} \, dx = 1 - e^{-a} \]
We have
\[ \lim_{r \to \infty} \int_0^r e^{-x} \, dx = 1 - \lim_{r \to \infty} e^{-r} = 1 \]
Therefore, \( \int_0^\infty e^{-x} \, dx = 1 \)

Here are some exercises for you.

E13) Evaluate the following integrals: i) \( \int_1^\infty \frac{1}{x^r}, r > 1 \) ii) \( \int_0^\infty \frac{1}{1+x^2} \, dx \).

Some times we will be interested in a bound for the integral rather than its
exact value. We know that, if \( f \) is a real valued function which is integrable
on \( [a, b], |f| \) is also integrable on \( [a, b] \) and
\[ \left| \int_a^b f \right| \leq \int_a^b |f| \]
(34)
The next result tells us that the result holds for complex valued integrable
functions also.

Proposition 7: Let \( f \) be a complex valued function which is integrable on
\( [a, b] \). Then, \( |f| \) is integrable on \( [a, b] \) and Eqn. (34) holds.
**Proof:** Let us first show that $|f|$ is integrable. Suppose $f(t) = u(t) + iv(t)$.

Recall that, if $f$ is a real valued function that is integrable on $[a,b]$, $f^2$ is also integrable on $[a,b]$. Since $u_1$ and $u_2$ are real valued, integrable functions, $u^2$ and $v^2$ are also integrable on $[a,b]$. So, $u^2(t) + v^2(t)$ is also integrable on $[a,b]$. Further, if $g$ is a real valued integrable function on $[a,b]$ and $g \geq 0$ on $[a,b]$, then $\sqrt{g}$ is also integrable on $[a,b]$. Applying this with $g = u^2(t) + v^2(t)$, we have $|f| = \sqrt{u^2(t) + v^2(t)}$ is integrable on $[a,b]$.

Let us now prove Eqn. (34). Since $\alpha = \int_a^b f$ is a complex number, let us write it in the form

$$\alpha = r_0 e^{i\theta_0}, \quad (0 \leq \theta < 2\pi)$$

If $\alpha = 0$ is 0, the LHS of Eqn. (34) is 0. Since $|f| \geq 0$ on $[a,b]$, $\int_a^b |f| \geq 0$, so Eqn. (34) is true in this case.

Let now suppose that $\alpha \neq 0$. We have

$$r_0 e^{i\theta_0} = \int_a^b f$$

$$\therefore r_0 = e^{-i\theta_0} \int_a^b f = \int_a^b e^{-i\theta_0} f \quad (\because e^{-i\theta_0} \text{ is a constant.})$$

$$\therefore \left| \int_a^b f \right| = |\alpha| r_0 = \text{Re}(r_0) \quad (\because r_0 \text{ is real})$$

$$= \text{Re}\left( \int_a^b e^{-i\theta_0} f \right) = \int_a^b \text{Re}(e^{-i\theta_0} f)$$

by the definition of complex integral. (How?) Using Eqn. (8), we have

$$\int_a^b \text{Re}(e^{-i\theta_0} f) \leq \int_a^b |e^{-i\theta_0} f| = \int_a^b |f| \quad (\because |e^{-i\theta_0}| = 1)$$

It follows that

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Before we move on to the next topic, we ask you to fill in the details of the proofs we have skipped in the next exercise.

---

E14) Check that

$$\text{Re}\left( \int_a^b e^{-i\theta_0} f \right) = \int_a^b \text{Re}(e^{-i\theta_0} f)$$

E15) Show that, if the integrals $\int_a^b f \left| \int_a^b f \right.$ exist, then

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(35)

Next, we introduce the notion of contour integral. We begin with its definition.

**Definition 12:** Let $R$ be a region in $\mathbb{C}$ and suppose that $f : R \to \mathbb{C}$ be a continuous function. Further, suppose that $\gamma : [a,b] \to R$ be a contour and let
Introduction to Analytic Functions

Let \( P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\} \) be a partition of \([a, b]\) such that \( \gamma \) is smooth on \([t_i, t_{i+1}], 0 \leq i \leq n\). We define the integral of \( f \) along the contour \( \gamma \) by

\[
\int_{\gamma} f(z) \, dz = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(\gamma(t)) \gamma'(t) \, dt
\]

(36)

You can easily verify that Eqn. (36) is independent of the partition chosen. Often, we will omit the dummy variable \( z \) and will write \( \int_{\gamma} f \) instead of \( \int_{\gamma} f(z) \, dz \). Let us now look at some examples of contour integrals.

**Example 8:** Consider the contour \( z(t) \) we defined in E7. Let us integrate \( f(z) = \frac{1}{z^2} \) along this contour. Note that, \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) is continuous and \( z([0,1]) \subset \mathbb{C} \setminus \{0\} \) By definition,

\[
\int_{\gamma} f(w) \, dw = \int_{\gamma} f(z(t)) z'(t) \, dt + \int_{\gamma} f(z(t)) z'(t) \, dt
\]

\[
= \int_{\gamma} \frac{(9-3i)}{((9-3i)t-(1+i))^2} \, dt + \int_{\gamma} \frac{(-6+15i)}{((-6+15i)t+(4-7i))^2} \, dt + \int_{\gamma} \frac{(-3+12i)}{((-3+12i)t+2+11i)^2} \, dt.
\]

Note that

\[
\frac{d}{dt} \left( \frac{1}{((9-3i)t-(1+i))^2} \right) = \frac{(9-3i)}{((9-3i)t-(1+i))^3}
\]

\[
\frac{d}{dt} \left( \frac{-1}{((-6+15i)t+(4-7i))^2} \right) = \frac{(-6+15i)}{((-6+15i)t+(4-7i))^3}
\]

\[
\frac{d}{dt} \left( \frac{-1}{((-3+12i)t+2+11i))^2} \right) = \frac{(-3+12i)}{((-3+12i)t+2+11i)^3}
\]

By appealing to Theorem 2, we get

\[
\int_{\gamma} \frac{9-3i}{((9-3i)t-(1+i))^2} \, dt = \left. \frac{-1}{((9-3i)t-(1+i))^3} \right|_{0}^{1/3}
\]

\[
= \frac{-1}{2-2i} - \frac{-1}{-1-i} = \frac{-3+i}{4}
\]

\[
\int_{\gamma} \frac{-6+15i}{((-6+15i)t+(4-7i))^2} \, dt = \left. \frac{-1}{((-6+15i)t+(4-7i))^3} \right|_{0}^{1/3}
\]

\[
= \left\{ \frac{1}{3i} - \frac{1}{2-2i} \right\} = \frac{3+7i}{12}
\]

\[
\int_{\gamma} \frac{-3+12i}{((-3+12i)t+2+11i)^2} \, dt = \left. \frac{-1}{((-3+12i)t+(2+11i))^3} \right|_{0}^{1/3}
\]

\[
= \left\{ \frac{-1}{1+i} - \frac{1}{3i} \right\} = \frac{3-5i}{6}
\]
\[ f(w) dw = \left\{ \frac{-3+i}{4} + \frac{3+7i}{12} + \frac{3-5i}{6} \right\} = 0 \]

Here is an exercise for you to check your understanding of the earlier example.

E16) Let \( f : \mathbb{C} \to \mathbb{C} \) be defined by \( f(z) = z^2 + 2 \). Find \( \int_C f(w) dw \) where \( z(t) \) is the contour in E8).

E17) Let \( \gamma : [a, b] \to \mathbb{C} \) be a contour. Define \( \gamma^- : [a, b] \to \mathbb{C} \) by \( \gamma^-(t) = \gamma(a+b-t) \). Suppose \( f : [a, b] \to \mathbb{C} \) is a continuous function in a region containing \( \gamma([a,b]) \). Show that \[ \int_{\gamma^-} f = -\int_{\gamma} f . \]

The next result says that integrals of a function over different, but equivalent contours are the same.

**Proposition 8:** Let \( \gamma_1 : [a, b] \to \mathbb{C} \) and \( \gamma_2 : [c, d] \to \mathbb{C} \) be equivalent contours in a region \( R \) and suppose \( f : R \to \mathbb{C} \) is a continuous function. Then, \[ \int_{\gamma_1} f = \int_{\gamma_2} f. \]

We omit the proof of this proposition. We now derive an upper bound for the contour integrals in terms of an upper bound for the function and the length of the contour.

**Proposition 9:** [ML inequality] Let \( \gamma : [a, b] \to \mathbb{C} \) be a piecewise smooth contour and suppose \( f \) is a complex valued continuous function in a region containing \( \gamma([a,b]) \). Suppose \( |f(z)| \leq M \) for all \( z \in \gamma([a,b]) \) and write \( L = \int_a^b |\gamma'(t)| dt \). Then, \[ \left| \int_{\gamma} f(z) dz \right| \leq ML \] \[ (37) \]

**Proof:** We have \[ \left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))| \gamma'(t) dt = \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq M \int_a^b |\gamma'(t)| dt = ML \quad (\because |f(z)| \leq M \text{ for } z \in \gamma([a,b]).) \]

Let us look at some examples.

**Example 9:** Without evaluating the integral, show that \[ \left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3} \]
where \( C \) is the arc of the circle \( |z|=2 \) from \( z=2 \rightarrow z=2i \) lying in the first quadrant.

**Solution:** As \( z \) lies on the arc \( C \) we have \( |z|=2 \).

Now \( |z^2-11| \geq |z^2-1| = 3 \)

\[
\Rightarrow \frac{1}{|z^2-11|} \leq \frac{1}{3} \Rightarrow \frac{1}{|z^2-1|} \leq \frac{1}{3} \Rightarrow M = \frac{1}{3}
\]

\( L = \) length of arc \( C = \int_0^\pi 2i e^{it} \, dt = 2 \int_0^\pi e^{it} \, dt = \pi \) (see Fig. 8)

Using ML-inequality

\[
\left| \int_C \frac{dz}{z^2-1} \right| \leq ML = \frac{\pi}{3}.
\]

**Example 10:** Let \( C_R \) denote the upper half of the circle \( |z|=R \) \((R>2)\), taken in counterclockwise direction. Show that

\[
\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \, dz \leq \frac{\pi R (2R^2+1)}{(R^2-1)(R^2-4)}.
\]

Further, show that the value of the integral tends to zero as \( R \to \infty \).

**Solution:** We have \( |2z^2-1| \leq 2|z^4+1|=2R^2+1 \)

and \( |z^4+5z^2+4| = |(z^2+4)(z^2+1)| \)

\[
= |z^2+4||z^2+1| \geq ||z^2+1|-4|| = |R^2-4||R^2-1| = (R^2-4)(R^2-1)
\]

Therefore,

\[
\left| \frac{2z^2-1}{z^4+5z^2+4} \right| \leq \frac{2R^2+1}{(R^2-4)(R^2-1)} = M
\]

Length of \( C_R = \pi R \)

Then by ML-inequality (Note that \( f(z) \) is defined and continuous along \( C_R \), see Fig. 9)

\[
\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \, dz \leq \frac{\pi R (2R^2+1)}{(R^2-1)(R^2-4)}
\]

Now as \( R \to \infty \), we get

\[
\frac{\pi R (2R^2+1)}{(R^2-1)(R^2-4)} = \frac{\pi (2+1/R^2)}{R(1-1/R^2)(1-4/R^2)} \to 0.
\]

Thus,

\[
\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \, dz \to 0 \text{ as } R \to \infty.
\]

(Observe that using ML-inequality we have shown without actually integrating, the integral of \( f(z) = \frac{2z^2-1}{z^4+5z^2+4} \) along the semi circular path \( C_R \), tends to 0 as \( R \to \infty \).)

***

You may now try the following exercises:

E18) Let \( C \) denote the line segment from \( z=i \) to \( z=1 \). By observing that, of
all the points on that line segment, the mid point is the closest to the origin, show that

\[ \left| \oint_C \frac{dz}{z^4} \right| \leq 4\sqrt{2} \]

without evaluating the integral.

We shall now discuss in the next section the concept of an antiderivative of a continuous function \( f(z) \) in a domain \( D \).

### 4.4 ANTIDERIVATIVES

At this stage you may have a doubt. Suppose two contours \( C_1 \) and \( C_2 \) have the same starting and ending points and \( f \) is a function which is continuous on \( C_1 \) and \( C_2 \). Is it true that \( \oint_{C_1} f = \oint_{C_2} f \)? According to the next example, the answer is ‘NO’.

**Example 11:** Let \( C_1 \) and \( C_2 \) be the contours defined by

\[
\begin{align*}
  z_1(t) &= \begin{cases} 
    2i + 2t(1-2i) & 0 \leq t \leq \frac{1}{2} \\
    (3-4t) & \frac{1}{2} \leq t \leq 1,
  \end{cases} \\
  z_2(t) &= 2i - t(1+2i) & 0 \leq t \leq 1. 
\end{align*}
\]

See Fig. 10.

![Fig. 10](image)

The initial and endpoint are \( 2i \) and \( -1 \) for both of them. Let us integrate \( f(z) = \frac{1}{(z-i)} \). We have

\[
\int_{z_1} f(z) \, dz = \int_0^{\frac{1}{2}} f(z_1(t)) z_1'(t) \, dt + \int_{\frac{1}{2}}^1 f(z_1(t)) z_1'(t) \, dt
\]

\[
= \int_0^{\frac{1}{2}} \frac{1}{2i - 2t(1-2i) - i} (2-4i) \, dt + \int_{\frac{1}{2}}^1 \frac{1}{3-4t-i} (-4) \, dt
\]

Consider \( \int \frac{2-4i}{2i - 2t(1-2i)} \, dt. i - 2t(1-2i) = -2t + i(1+4t) \in \mathbb{R} \) only if \( t = -\frac{1}{4} \).

For \( t = -\frac{1}{4} \), \( -2t + i(1+4t) = \frac{1}{2} \notin \{ x \in \mathbb{R} | x \leq 0 \} \). So, \( \log (-2t + i(1+4t)) \) makes sense if we chose the principal branch of the logarithm.
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\[ \frac{d}{dt} \log(-2t+i(1+4t)) = \frac{1}{2t+i(1+4t)} - (2-4i). \] Using Theorem 2, we get

\[ \int_0^1 \frac{1}{3-4t-i} \, dt = \log((2t+i(1-4t))) \bigg|_0^1 = \log(1-i) - \log(i) \]

Since \( 3-4t-i \notin \mathbb{R} \) all, \( t \in \mathbb{R} \). So, we can choose the principal branch.

We have \( \int_0^1 \frac{1}{3-4t-i} \, dt = \log((3-4t-i)) \bigg|_0^1 = \log((-1+i)) - \log(1-i) \)

Using Theorem 2, we get \( \int f(z) \, dz = \int_{i-t(i+1+2i)} \frac{1}{i-t(i+1+2i)} \, dt. \) \( 1-t(1+2i) = -t(1+2t) i \in \mathbb{R} \) if \( 1-2t = 0 \) or \( t = \frac{1}{2} \). If \( t = \frac{1}{2} \), \( -t(1+2t) i = -\frac{1}{2} \). So, we cannot use the principal branch of logarithm here. Note that \( -t + (1 - 2t) i \notin \{ x \in \mathbb{R} \mid x \geq 0 \} \).

So, let us choose the branch of logarithm defined on \( \mathbb{C} \setminus \{ x \in \mathbb{R} \mid x \geq 0 \} \). So, we define \( \log_1 z = |z| + i \arg(z) \) where \( 0 < \arg(z) < 2\pi \). So,

\[ \int_0^1 \frac{1}{3-4t-i} \, dt = \log_1((1-t(i+1+2i)) \bigg|_0^1 = \log_1(-1) - \log_1(1) \]

\[ \log(1-i) = \sqrt{2} + i \left( -\frac{\pi}{4} \right), \log(i) = 1 + i \frac{\pi}{2}, \]

\[ \log((-1+i)) = \sqrt{2} + i \left( -\frac{\pi}{4} + \frac{3\pi}{4} \right) = \sqrt{2} - i \frac{3\pi}{4}, \]

\[ \log(1-i) = \sqrt{2} + i \left( -\frac{\pi}{4} \right), \log_1((-1+i)) = \sqrt{2} + \frac{5\pi i}{4}. \]

\[ \log_1(i) = 1 + i \frac{\pi}{2} \]

\[ \therefore \int f(z) \, dz = \left( \sqrt{2} - i \frac{\pi}{4} \right) + \left( \sqrt{2} - i \frac{3\pi}{4} + i \frac{5\pi}{4} \right) = \sqrt{2} - 1 + i \frac{3\pi}{4} \]

\[ \int f(z) \, dz = \sqrt{2} - 1 + i \frac{3\pi}{4} \]

So, \( \int f(z) \, dz \neq \int f(z) \, dz \).

***

Note that, in the previous example, the real parts of \( \int_{z_1} f(z) \, dz \) and \( \int_{z_2} f(z) \, dz \) are the same. We have \( \int_{z_2} f(z) \, dz + \int_{z_1} f(z) \, dz = \int_{z_2} f(z) \, dz - \int_{z_1} f(z) \, dz = 2\pi i \).

As you will see later, this follows directly from Cauchy’s integral formula.

So, there are functions whose contour integrals over different contours with the same end points are not the same. However, all is not lost. The next theorem gives a characterisation of functions where contour integrals depend only on the end points and not the particular contour. This characterisation gives us a large supply of such functions.

**Theorem 3:** Suppose that a function \( f \) is continuous on a domain \( D \). The following are equivalent:

i) There is an analytic function \( F(z) \) such that \( F'(z) = f(z) \).
ii) The contour integral of \( f(z) \) depends only on the initial and end points of the contour, i.e., if \( z_1, z_2 \in D \) and \( C_1 \) and \( C_2 \) are any two contour with end points \( z_1 \) and \( z_2 \), \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \).

iii) If \( C \) is any closed contour contained in \( D \), \( \int_{C} f(z) \, dz = 0 \)

**Proof:** We prove the equivalence as follows: (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i)

(i) \( \Rightarrow \) (ii) we will assume that \( C \) is smooth with parametric representation 
\[ z(t) \quad (a \leq t \leq b). \]

Then, \( \frac{d}{dt} (F(z(t))) = F'(z(t)) \, z'(t) \) by chain rule. Using the fundamental theorem of calculus, it follows that
\[ \int_{C} f(z) \, dz = \int_{a}^{b} f(z(t)) \, z'(t) \, dt = F(z(b)) - F(z(a)) \]
and \( z(b) = z_2, \int_{C} f(z) \, dz = F(z_2) - F(z_1) \) which is independent of the contour \( C \).

Suppose \( C \) is a piecewise smooth consisting of arcs \( C_1, C_2, \ldots, C_n \) and the end points of \( C_k \) are \( z_k \) and \( z_{k+1} \). Thus, we have
\[ \int_{C} f(z) \, dz = \sum_{k=1}^{n} \int_{C_k} f(z) \, dz. \]

Since \( C_k \) is smooth, we have
\[ \int_{C_k} f(z) \, dz = F(z_{k+1}) - F(z_k). \]
So,
\[ \sum_{k=1}^{n} \int_{C_k} f(z) \, dz = \sum_{i=1}^{n} (F(z_i) - F(z_{i+1})) = F(z_{n+1}) - F(z_1). \]

(ii) \( \Rightarrow \) (iii) Let \( z_1 \) and \( z_2 \) denote any two points on a closed contour and let \( C_1 \) and \( C_2 \) be two contours such that \( C = C_1 - C_2 \). See Fig. 11.

Since (ii) is true and \( C_1, C_2 \) have the same endpoints
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]
So,
\[ \int_{C} f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz \]
\[ = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = 0. \]

(iii) \( \Rightarrow \) (ii) Let \( C_1 \) and \( C_2 \) be two contours lying in \( D \) and join the point \( z_1 \) and \( z_2 \). Then \( C = C_1 - C_2 \) is a closed contour, so \( \int_{C} f(z) \, dz = 0 \) since we assume
(iii) to be true. But

\[ 0 = \int_C f(z) \, dz = \int_{C_1 - C_2} f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz \]

or

\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]

In other words, the contour integral of \( f(z) \) over any contour in \( D \) depends only on the initial and end points of the contour, proving (iii) \( \Rightarrow \) (ii). (ii) \( \Rightarrow \) (i). Using the fact that the contour integral of \( f(z) \) over path depends only on the end points, we construct a function \( F(z) \) such that \( F'(z) = f(z) \). We fix an element \( z_0 \in D \). Since we have assumed that \( D \) is a domain, there is a polygon in \( D \) connecting any two points in \( D \). In other words, any two points in \( D \) are connected by a piecewise smooth path. (See page 10, Unit 1) Let \( s(t) \) be any path with \( z_0 \) and \( z \) as the initial and final points. We define \( F(z) = \int_s f(z) \, dz \). This is well defined because of our assumption that the integral of \( f \) over any two contours with the same initial and end points are the same. Let \( \Delta z \) be small enough so that \( z + \Delta z \) lies in a small neighbourhood of \( z \) contained in \( D \). Then \( z \) and \( z + \Delta z \) can be connected by a line segment since any two points in a neighbourhood can be connected by a line segment. We have \( \int_z^{z+\Delta z} ds = \Delta z \). (See exercise 19). So,

\[ f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z) \, ds. \]

(Note here that \( f(z) \) is a constant since we are integrating with respect to \( s \).) So,

\[ \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(s) - f(z)) \, ds. \]

But, \( f \) is continuous at \( z \). So, given \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( |f(z) - f(s)| < \epsilon \) whenever \( |z - s| < \delta \). We can always assume \( |\Delta z| < \delta \). Then

\[ \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \leq \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon. \]

In other words,

\[ \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z), \]

or \( F'(z) = f(z) \)

Let us now look at some examples.

**Example 12:** Consider the function \( f(z) = z^3 \). The function \( f \) has antiderivative \( \frac{z^4}{4} \) throughout the complex plane. Hence \( \int_0^i z^3 \, dz = \frac{z^4}{4} \bigg|_0^i = 1 \) for every contour from 0 to \( i \).

---

E19) If \( f(z) = 1 \) and \( C \) is an arbitrary contour joining \( z_1 \) and \( z_2 \), check that \( \int_C f(z) \, dz = z_2 - z_1 \).

E20) Use antiderivatives to show that, for every contour \( C \) extending from a point \( z_1 \) to a point \( z_2 \).
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\[ \int_C z^n \, dz = \frac{1}{n+1} (z_2^{n+1} - z_1^{n+1}) \quad (n = 0, 1, 2, \ldots). \]

E21) By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

i) \( \int_{C_1} e^{z^2} \, dz \);  
ii) \( \int_{C_2} \cos \left( \frac{z}{2} \right) \, dz \);  
iii) \( \int_{C_3} (z-2)^3 \, dz \).

We close the section here. In the next section, we will discuss the Cauchy-Goursat Theorem, which is one of the important theorems in Complex Analysis.

4.5 CAUCHY GOursat THEOREM

Consider the function \( f(z) = \frac{1}{z-i} \) we discussed in Example 11. There we saw that the integral of the function over two different contours with same end points are not the same although \( f(z) \) is analytic in \( \mathbb{C} \setminus \{i\} \). The next theorem specifies additional conditions under which the contour integral of an analytic function is zero.

Theorem 4: If a function \( f \) is analytic at each point interior to and on a simple closed contour \( C \), described counterclockwise, and if the derivative \( f' \) of \( f \) is continuous in the closed region consisting of all points interior to and on \( C \) then

\[ \int_C f(z) \, dz = 0. \]

We shall not be proving the theorem here.

Theorem 3, proved by a French mathematician Cauchy (1789-1857) in the early part of the nineteenth century, was later improved by Goursat (1858-1936), who proved the result by omitting the condition of continuity of \( f' \) with other conditions remaining same. The revised result is known as the Cauchy-Goursat theorem which we shall prove now.

Theorem 5 (Cauchy-Goursat theorem): If a function \( f \) is analytic at all points interior to and on a simple closed contour \( C \), then

\[ \int_C f(z) \, dz = 0. \]  \hspace{1cm} (38)

Proof: We divide the region inside \( C \) into a large number of small parts by a network of lines parallel to the real and imaginary axes. Suppose that divides the insides of \( C \) into a number of complete squares \( C_1, C_2, \ldots, C_M \), coloured light green in Fig. 12, and a number of irregular regions, \( D_1, D_2, \ldots, D_N \) say, coloured light blue in Fig. 12, parts of whose boundaries are boundaries of \( C \). See Fig. 12.

Then \( \int_C f(z) \, dz = \sum_{i=1}^{M} \int_{C_i} f(z) \, dz + \sum_{j=1}^{N} \int_{D_j} f(z) \, dz \), \hspace{1cm} (39)

where we take all the integral in anticlockwise direction.
Let us now see why this is true. Suppose \( ABCD \) and \( DCEF \) are two squares with a common side \( CD \). The \( CD \) is described from \( C \) to \( D \) in the square \( ABCD \) and in the direction \( D \) to \( C \) in \( DCEF \). Hence the two integrals along \( CD \) cancel. So, all the integrals in Eqn. (39) cancel except those which form part of \( C \) itself. So Eqn. (39) is valid.

Let us now use the fact that \( f(z) \) is analytic at every point. If \( z_0 \) is any point inside \( C \), given any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that
\[
\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.
\]
In other words
\[
|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0| \quad \text{if} \quad 0 < |z - z_0| < \delta.
\]
(40)

If the number of subdivisions are high, the sizes of \( C_i \) and \( D_j \) will be so small that the distance between any two points in the same \( C_i \) or \( D_j \) will be less than \( \delta \). Our problem is that we may not be able to chose the same \( \delta \) so that, irrespective of the \( C_i \) or \( D_j \) that contains \( z \) and \( z_0 \), Eqn. (40) remain valid. This is like the difference between uniform continuity of a function on a set and pointwise continuity. We will complete the proof assuming this to be true and will prove this fact later.

Consider one of the squares \( C_m \) of side \( l_m \). By (40), if we write
\[
\phi(z) = f(z) - f(z_0) - (z - z_0)f'(z_0)
\]
we have \( |\phi(z)| \leq \varepsilon |z - z_0| \). So,
\[
\int_{C_m} f(z)dz = \int_{C_m} \{f(z_0) + (z - z_0)f'(z_0)\}dz + \int_{C_m} \phi(z)dz.
\]
Note that \( f(z_0) + (z - z_0)f'(z_0) = f'(z_0)z + (f(z_0) - z_0f'(z_0)) = az + b \) where \( a = f'(z_0) \) and \( b = (f(z_0) - z_0f'(z_0)) \). Note that \( a \) and \( b \) are constants. So,
\[
a \frac{z^2}{2} + bz \quad \text{is an antiderivative for this function over the entire complex plane.}
\]
Since \( C_i \) is a closed contour, \( \int_{C_i} \{f(z_0) + (z - z_0)f'(z_0)\}dz = 0 \). Note that, the distance between any two points in a square of side \( a \) is at most \( \sqrt{2}a \), the length of the diagonal of the square.

Since \( C_m \) has length \( l_m \), we have \( |z - z_0| \leq \sqrt{2}l_m \). Further, the perimeter of the square is \( 4l_m \). So,
\[
\left| \int_{C_m} \phi(z)dz \right| \leq \int_{C_m} |\phi(z)|dz \leq \varepsilon \int_{C_m} |z - z_0||dz| \leq \varepsilon \sqrt{2}l_m 4l_m.
\]
In the case of one the irregular regions $D_j$ the length of the boundary is at
most $4l_j + s_j$ where $s_j$ is the length of the curved portion of the boundary.

Hence $\left| \int_{D_j} \phi(z) \, dz \right| \leq \varepsilon \sqrt{2} l_j (4l_j + s_j)$.

Adding all the parts we get

$$\left| \int_{C} f(z) \, dz \right| < 4\sqrt{2} \varepsilon \left( \sum l_i^2 + \sum l_j^2 \right) + \varepsilon \sqrt{2} l \sum s_j. \quad (41)$$

Now $\left( \sum l_i^2 + \sum l_j^2 \right)$ is less than the area of the rectangle $PQRS$. The area of the
rectangle is bounded since $C$ is bounded. Also $\sum s_j$ is length of the curve $C$ which is bounded. So, we can make the RHS in Eqn. (41) as small as we please by choosing $\varepsilon$ small enough. Since the LHS is independent of $\varepsilon$ and is smaller than any positive real number, however small, the LHS has to be zero
and we are done if we can show that the assumption we made is correct.

We need to prove the following assertion:

Given any $\varepsilon > 0$, there is a way of dividing the region inside $C$ in parts by lines parallel to real and imaginary axes such that we can choose a $\delta$ with true the following property. Each subdivision has a point $z_0$ such that Eqn. (41) holds.

Suppose we divide $C$ into parts by lines which are a constant distance $l$ apart. Some of the parts may contain an $z_0$ with the required property. We leave
these parts alone. We subdivide the remaining parts into two equal parts by a line midway. If there are still parts which do not have the required property, we subdivide these parts again. There are two possibilities. The process may terminate after a finite number of steps and we are done. The other possibility is this continues indefinitely. Let us now rule out this second possibility.

Suppose the process continues indefinitely. In this case there is at least one region $R_1$ which we can subdivide indefinitely without getting the required result. After subdividing $R_1$, we get a part $R_2 \subset R_1$ with the same property. So, we get a sequence of parts, $R_1 \supset R_2 \supset \ldots \supset R_i \supset \ldots$ for which (40) doesn’t hold. Note that, the $\{R_i\}$ is a family of closed sets with finite intersection property because they form a decreasing sequence since the rectangle $PQRS$ is a compact set $\bigcap_i R_i \neq \emptyset$. (See page 71, Unit 3, MMT-004). Let $z_0 \in \bigcap_i R_i$.

Since Area $(R_{i+1}) \leq \frac{1}{2}$ Area $(R_i)$, the area of $R_i$ are strictly decreasing. So, far
any $\delta > 0$ there is a $z_0$ such that $|z - z_0| < \delta$ if $z, z_0 \in R_n$ with $n \geq n_0$. Since $f(z)$ is analytic at $z_0$, we can chose a $\delta$ such that (40) will hold for $z_0 \in R_n$
with $n \geq n_0$. This is a contradiction.

Cauchy-Goursat theorem says that the integral of an analytic function over a closed contour is zero provided the function is analytic in a region containing the contour and its interior points. So, the question arises. For which kind of regions can we say that, if the region contains a contour it contains the interior points also? Intuitively speaking these are regions without any holes. For example, if we take the region $\mathbb{C} \setminus \{0\}$ and take any circle with centre as the origin, the origin, which is an interior point of the circle is not in $\mathbb{C} \setminus \{0\}$.

One concept that is useful in characterising a region without holes is that of homotopy.
**Definition 13:** Let $D$ be a domain and $\gamma_1 : [a, b] \to D$, $\gamma_2 : [a, b] \to D$ be two closed arcs in $\mathbb{C}$. We say that $\gamma_1$ and $\gamma_2$ are homotopic in $D$ if there is a continuous map $H(s, t) : [0, 1] \times [a, b] \to D$ such that $H(0, t) = \gamma_1(t), H(1, t) = \gamma_2(t)$ and $H(s, a) = H(s, b)$ for $0 \leq s \leq 1$.

Let us now look at an example.

**Example 13:** Consider the circles $r_1 : [0, 1] \to \mathbb{C}$ given by $r_1(t) = e^{2\pi it}$ and $r_2 : [0, 1] \to \mathbb{C}$ given by $r_2(t) = 2e^{2\pi it}$. Define $H(s, t) : [0, 1] \times [0, \pi] \to \mathbb{C}$ by $H(s, t) = (1 - s)e^{2\pi it} + 2se^{2\pi it}$. Then $H(s, t)$ is continuous. (Why?) Further, $H(0, t) = r_1(t)$ and $H(1, t) = r_2(t)$. Further, $H(s, 0) = H(s, 1) = 1 + s$.

The next theorem tells us why the concept of homotopy is useful. The contour integral of an analytic function over two different contours with the same endpoints are the same if the contours are homotopic.

**Theorem 4:** Let $f : D \to \mathbb{C}$ be an analytic function in a region $D$ and let $\gamma_1$ and $\gamma_2$ be arcs such that $\gamma_1$ is homotopic to $\gamma_2$. Then, $\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz$.

The proof of this theorem is beyond the scope of this course.
**Definition 14:** Let \( \gamma : [a, b] \to \mathbb{C} \) be a closed arc. Then, we say that \( \gamma \) is homotopic to 0 if \( \gamma \) is homotopic to a constant curve.

**Example 14:** Let \( \gamma : [0, 1] \to \mathbb{C} \) be defined by \( \gamma(t) = e^{2\pi it} \). Let \( \gamma_1 : [0, 1] \to \mathbb{C} \) be the constant arc given by \( \gamma_1(t) = 0 \). Let \( H(s, t) : [0, 1] \to \mathbb{C} \) be defined by \( H(s, t) = (1-s)\gamma(t) + s\gamma_1(t) \). Then, \( H(0, t) = \gamma(t) \) and \( H(1, t) = \gamma_1(t) \). Further \( H(s, 0) = (1-s)\gamma(0) = (1-s) \), \( H(s, 1) = (1-s)\gamma(1) = (1-s) \) because \( \gamma_1(t) = 0 \) for all the values of \( t \).

Since the integral over a constant arc is zero, the following is an easy consequence of Theorem (4).

**Corollary:** Let \( D \) be a domain and suppose \( \gamma : [a, b] \to D \) is homotopic to a constant arc in \( D \). then, \( \int_D f(z) \, dz = 0 \).

**Definition:** A domain \( D \) is **simply** connected if every simple closed arc in \( D \) is homotopic to zero. A domain is **multiply** connected if it is not simply connected.

The next theorem gives a nice characterisation of simply connected domains.

**Theorem 5:** A domain \( D \) is simply connected iff every (connected) component of \( \mathbb{C} \setminus D \) is unbounded.

The proof of this result is beyond the scope of this. In Fig. 15 you can see some example of simply connected and multiply connected regions.
Since every arc is homotopic to 0 in a simply connected region, it follows that if a function is analytic on a simply connected region, its integral over any simple closed arc is zero. The following general result is true.

**Theorem 6:** If a function \( f \) is analytic throughout a simply connected domain \( D \) and \( C \) is any contour in \( D \) then
\[
\int_C f(z) \, dz = 0
\]

**Corollary 1:** A function that is analytic throughout a simply connected domain \( D \) has an antiderivative everywhere in \( D \).

We can extend Cauchy-Goursat theorem in a way that involves integrals along the boundary of a multiply connected domain.

**Theorem 7:** Suppose that \( C \) is a simple closed contour described in counter clockwise direction. Further, suppose that \( C_1, C_2, \ldots, C_n \) are simple closed contours in \( D \) interior to described in the clockwise direction, they are disjoint and do not have interior points in common. (See Fig. 16(a).) If a function \( f \) is analytic on all of these contours and throughout the multiply connected region exterior to each \( C_k \) then
\[
\int_C f(z) \, dz + \sum_{k=1}^{n} \int_{C_k} f(z) \, dz = 0
\]

We will not prove this theorem in this course. As a consequence of Theorem 7 we have the following corollary:

**Corollary:** Let \( C_1 \) and \( C_2 \) denote positively oriented simple closed contours, where \( C_2 \) is interior to \( C_1 \). (See Fig. 16(b)). If a function is analytic in the closed region consisting of those contours and all the points between them, then
\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]

The above corollary is known as the principle of deformation of path. It tells us that passing through the points at which \( f \) is analytic, if \( C_1 \) is continiously deformed into \( C_2 \), then the value of the integral of \( f \) over \( C_1 \) does not change.

Try the following exercises to check your understanding of the discussion so far.

E22) Apply the Cauchy-Goursat Theorem to show that
\[
\int_C f(z) \, dz = 0
\]
where \( C \) is the circle \(|z| = 1\), in either direction, and when

i) \( f(z) = z e^{-z} \);

ii) \( f(z) = \frac{1}{z^2 + 2z + 2} \);

E23) Let \( C_1 \) denote the positively oriented circle \(|z| = 4\) and \( C_2 \) the positively oriented square whose lines lie along the lines \( x = \pm 1 \), \( y = \pm 1 \). (See Fig. 17). Use corollary to Theorem 7 to point out why

\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz
\]

when

i) \( f(z) = \frac{1}{3z^2 + 1} \)

ii) \( f(z) = \frac{z}{1 - e^z} \).

E24) According to E9), the path \( C_1 \) from the origin to the point \( z = 1 \) along the graph of the function defined by means of the equation

\[
y(x) = \begin{cases} 
  x^3 \sin \left( \frac{\pi}{x} \right) & \text{when } 0 < x \leq 1, \\
  0 & \text{when } x = 0
\end{cases}
\]

is a smooth arc that intersects the real axis an infinite number of times. Let \( C_2 \) denote the line segment along the real axis from \( z = 1 \) back to the origin, and let \( C_3 \) denote any smooth arc from the origin to \( z = 1 \) that does not intersect itself and has only its end points in common with the
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arcs $C_1$ and $C_2$ (Fig. 18). Consider Fig. 18(a) and 18(b) shown below. Apply the Cauchy-Goursat theorem to show that if a function $f$ is entire, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

and

$$\oint_{C_3} f(z)\,dz = -\oint_{C_1} f(z)\,dz.$$

Conclude that, even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\oint_C f(z)\,dz = 0.$$

We conclude this unit here. In the next section, we will summarise the contents of this units.

### 4.6 SUMMARY

In this unit we have discussed:

- Integration and differentiation of complex valued function of a real variable;
- The concept of an arc and the concept of a contour;
- The concept of a rectifiable arc, some conditions for an arc to be rectifiable and the concept of arc length of a rectifiable arc;
- Integration of a continuous function defined on a domain of the complex plane along a contour;
- The concept of antiderivative of a function and conditions for the existence of a function in a domain;
• The Cauchy-Goursat theorem which states that the contour integral of a function over a simple closed contour is zero provided that the function is analytic in a domain containing the contour as well as the interior of the contour.

4.7 SOLUTION/ANSWERS

E1) We have

\[
\frac{1}{i + \sin^2 t} = \frac{1}{i + \sin^2 t} \left( -i + \sin^2 t \right) = \frac{\sin^2 t - i}{1 + \sin^4 t} + i \left( \frac{-1}{1 + \sin^4 t} \right)
\]

\[
...u(t) = \frac{\sin^2 t}{\sin^4 t + 1}, v(t) = \left( \frac{-1}{1 + \sin^4 t} \right).
\]

E2) We give a sketch of the proof of Eqn. (11). It is enough to prove the following. (Why?):

i) Prove that, if \( f \) is differentiable at \( t_0 \) and \( \alpha \in \mathbb{C}, \alpha f \) is differentiable at \( t_0 \) with derivative \( \alpha f'(t_0) \).

ii) Prove that, if \( f \) and \( g \) are differentiable at \( t_0 \), then \( f + g \) is differentiable at \( t_0 \) with derivative \( f'(t_0) + g'(t_0) \).

To prove Eqn. (12), observe that

\[
\lim_{t \to t_0} f(t)g(t) - f(t_0)g(t_0) = \lim_{t \to t_0} g(t)f(t) - f(t_0)g(t) = f(t_0)g(t_0) + \lim_{t \to t_0} g(t_0)f(t) - f(t_0)g(t)
\]

Now proceed along the lines of the proof of Eqn. (13).

E3) i) Let \( S = \left\{ \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z} \right\} \) and \( S_1 = \mathbb{R} \setminus S \). Then, \( \tan t \), considered as a function of \( t \) is defined and differentiable on every point of \( S_1 \). Let \( f:S_1 \to \mathbb{R} \) be defined by \( f(t) = \tan t \).

Similarly, \( g(t) = 1 + it^2 \) and differentiable on \( \mathbb{R} \). So \( 1 + i \tan^2 t \) is \( (g \circ f)(t) \). Check that

\[
(g \circ f)(t) = g'(f(t))f''(t) = 2i \tan t \sec^2 t
\]

ii) Let

\( S = \{ x \in \mathbb{C} \mid \text{Im}(x) = 0, \text{Re}(x) < 0 \} \) and \( S_1 = \mathbb{C} \setminus S \)

The \( f(t) = i + t^2 \), considered as a function of \( t \) is defined on the whole of \( \mathbb{R} \). Further \( f(t) \in S_1 \) for all \( t \in \mathbb{R} \). (Why?) The function \( g(z) = \log z \), considered as a function of \( z \) is defined on \( S_1 \). So, the expression \( \log(i + t^2) \), considered as a function of \( t \) is \( (g \circ f)(t) \). Check that

\[
(g \circ f)(t) = g'(f(t))f''(t) = \frac{1}{i + t^2} \cdot 2t
\]
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E4) Let us take \( \phi(x) : [a, b] \rightarrow [a, b] \) as the identity map \( \phi(x) = x \). Check that all the conditions in Definition 4 is satisfied. This shows that \( \sim \) is reflexive.

Suppose \( z_1 : [a, b] \rightarrow \mathbb{C} \) and \( z_2 : [c, d] \rightarrow \mathbb{C} \) are two arcs with \( z_1 \sim z_2 \).

Then, there is an onto function \( \phi : [c, d] \rightarrow [a, b] \) such that \( \phi(c) = a, \phi(d) = b, \phi(t) \) is strictly increasing and therefore \( 1 \sim 1 \). Since \( \phi(t) \) is also onto, there is an inverse \( \psi : [a, b] \rightarrow [c, d] \). We have \( \psi(a) = \psi(\phi(c)) = c \) and \( \psi(b) = \psi(\phi(d)) = d \). So, \( \sim \) is symmetric.

Suppose \( z_i : [a, b] \rightarrow \mathbb{C} \) and \( z_j : [c, d] \rightarrow \mathbb{C} \) are two arcs with \( z_i \sim z_j \).

Then, there is an onto function \( \phi : [a, b] \rightarrow [c, d] \) such that \( \phi(0) = z_i(0) = z_j(0) \) and \( \phi(1) = z_i(1) = z_j(1) \). Let \( \psi : [c, d] \rightarrow [a, b] \).

Since \( \phi(t) \) is also onto, there is an inverse \( \phi^{-1} : [a, b] \rightarrow [c, d] \) such that \( \phi^{-1}(0) = z_i(0) = z_j(0) \) and \( \phi^{-1}(1) = z_i(1) = z_j(1) \). So, \( \sim \) is symmetric.

Suppose \( z_i : [a, b] \rightarrow \mathbb{C} \) and \( z_j : [c, d] \rightarrow \mathbb{C} \) are two arcs with \( z_i \sim z_j \).

Then, there is an onto function \( \phi : [a, b] \rightarrow [c, d] \) such that \( \phi(0) = z_i(0) = z_j(0) \) and \( \phi(1) = z_i(1) = z_j(1) \). Let \( \psi : [c, d] \rightarrow [a, b] \).

E5) i) Let \( \phi : [c, d] \rightarrow [a, b] \) be such that \( z_1(\phi(t)) = z_2(t) \). If \( z_1 \) is simple, it is \( 1 \sim 1 \). Since \( \phi(t) \) is \( 1 \sim 1 \) and \( z_2(t) = (z_1 \circ \phi)(t) \), it follows that is also \( 1 \sim 1 \), i.e. \( z_2 \) is also simple.

ii) If \( z_1 \) is closed \( z_1(a) = z_1(b) \). Note that \( \phi \) is strictly increasing and onto is follows that \( \phi(c) = a, \phi(d) = b \). \( \psi \) is also closed.

iii) If \( z_1 \) is smooth, then \( z_2(t) = (z_1 \circ \phi)(t) \) is differentiable by chain rule and \( z_2'(t) = z_1'(\phi(t)) \phi'(t) \).

E6) Let \( P = \{a = t_0 < t_1 < t_2 < \ldots < t_n = b\} \) a partition such that \( z(t) \) is smooth in \( [t_i, t_{i+1}] \). Write

\[
L_p(z) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |z'(t)| \, dt
\]

The steps are as follows:

i) First, note that if \( y \neq t_i \) for all \( i \) and \( y_j \in [t_k, t_{k+1}] \) for some fixed \( k, 0 \leq k \leq n-1 \), we have

\[
\int_{t_k}^{t_{k+1}} |z'(t)| \, dt = \int_{t_k}^{y_j} |z'(t)| \, dt + \int_{y_j}^{t_{k+1}} |z'(t)| \, dt
\]

by properties of integrals. So, it follows that if \( P' = P \cup y \), then \( L_{P'}(z) = L_p(z) \).

ii) Use induction to show the following: Let \( \{a = t_0 < t_1 < t_2 \ldots < t_{i-1} < t_i < \ldots < t_{m-1} < t_m = b\} \) be such that \( t_i \neq y_j (y_j \in [a, b]) \) for any \( i, j, 0 \leq i \leq n, 1 \leq j \leq m \). Consider the
partition $P'$ obtained by taking the union
$P \cup \{a = y_0 < y_1 < \cdots < y_{i-1} < y_i < \cdots < y_{m-1} < y_m = b\}$. Then,

$$L_{P'}(z) = L_P(z)$$

iii) Let $P_1$ and $P_2$ be arbitrary partitions such that $z[t]$ is smooth in
each of the subintervals of the partition and suppose $S_1 = (P_2 \setminus P_1)$
and $S_2 = (P_1 \setminus P_2)$. Then, $P_1 \cup S_1 = P_2 \cup S_2 = P$ (Say). Then,

$$L_{P'}(z) = L_{P'}(z) = L_{P_2}(z).$$

E7) See Fig. 19: An obvious choice for the partition $P$ is $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$
because $z(t)$ is linear in the subintervals $\left[0, \frac{1}{3}\right]$, $\left[\frac{1}{3}, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and
therefore continuously differentiable.

![Fig. 19](image)

Further, the derivative is non-zero in these intervals. Check that $z(t)$ is
continuous at $\frac{1}{3}$ and $\frac{2}{3}$.

To complete the checking, you have to find the appropriate left and right
derivatives at the points in $P$. At 0, we check

$$\lim_{t \to 0} \frac{z(t) - z(0)}{t - 0} = \lim_{t \to 0} \frac{(9 - 3i)t - (1 + i) - (-1 + i)}{t - 0} = (9 - 3i).$$

So, $z'(0^+)$ exists. Also,

$$\lim_{t \to 0^+} z'(t) = 9 - 3i$$

Similarly, at $\frac{1}{3}$, we have

$$\lim_{t \to \frac{1}{3}} \frac{z(t) - z\left(\frac{1}{3}\right)}{t - \frac{1}{3}} = \lim_{t \to \frac{1}{3}} \frac{(9 - 3i)t - (1 + i) - \left(9 - 3i\right)\frac{1}{3} - (1 + i)}{t - \frac{1}{3}}$$

$$= \lim_{t \to \frac{1}{3}} \frac{(9 - 3i)\left(t - \frac{1}{3}\right)}{t - \frac{1}{3}} = 9 - 3i.$$

$$\lim_{t \to \frac{1}{3}} z'(t) = 9 - 3i.$$

Similarly, check that:
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i) \[ z'(\frac{1^+}{3}) = -(6 + 15i), \quad \lim_{t \to \frac{1}{3}} z'(t) = (-6 + 15i). \]

ii) \[ z'(\frac{2^-}{3}) = -(6 + 15i), \quad \lim_{t \to \frac{2}{3}} z'(t) = (-6 + 15i). \]

iii) \[ z'(\frac{2^+}{3}) = -(3 + 12i), \quad \lim_{t \to \frac{2}{3}} z'(t) = -(3 + 12i). \]

iv) \[ z'(1^+) = -(3 + 12i), \quad \lim_{t \to 1^-} z'(t) = -(3 + 12i). \]

So, \( z(t) \) is piecewise smooth and therefore a contour. We have,

\[
L(z) = \int_0^1 19 - 3i \, dt + \int_{\frac{2}{3}}^1 1 - 6 + 15i \, dt + \int_{\frac{3}{2}}^1 1 - 3 + 12i \, dt = \frac{1}{3} \left\{ 3\sqrt{10} + \sqrt{261} + \sqrt{153} \right\}.
\]

E8) See Fig. 20 We take the partition \( \left\{ 0, \frac{1}{2}, 1 \right\} \)

We have \( z'(t) = \begin{cases} 
2\pi i e^{2\pi it} & 0 < t < \frac{1}{2} \\
4 & \frac{1}{2} < t < 1 
\end{cases} \)

So, \( z'(t) \) is continuous in \( 0 < t < \frac{1}{2} \) and \( \frac{1}{2} < t < 1 \)

Check that \( z'(0^+) = \lim_{t \to 0^+} \frac{z(t) - z(0)}{t} = \lim_{t \to 0^+} \frac{e^{2\pi it} - 1}{t} \)

\[
= \lim_{t \to 0^+} \frac{2\pi it + (\frac{(2\pi it)^2}{2!} + \frac{(2\pi it)^3}{3!} + \cdots)}{t} = 2\pi i \]

\[
L(t \to 0^+) z'(t) = L(t \to 0^+) 2\pi i e^{2\pi it} = 2\pi i. \] At \( t = \frac{1}{2} \) \( L(t \to \frac{1}{2}) \frac{e^{2\pi it} - e^{\pi i}}{t - \frac{1}{2}} = L(t \to \frac{1}{2}) \frac{e^{2\pi it} + 1}{t - \frac{1}{2}} \)
Complex Integration-I

\[
\begin{align*}
Lt_{t \to \frac{\pi}{2}} \cos 2\pi t + i Lt_{t \to \frac{\pi}{2}} \sin 2\pi t &= \frac{-2\pi \sin 2\pi t}{1} + i \frac{2\pi \cos 2\pi t}{1} \\
\text{by L’Hospital’s rule.} \quad \therefore Lt_{t \to \frac{\pi}{2}} \frac{e^{2\pi it} + 1}{t - \frac{\pi}{2}} &= -2\pi i.
\end{align*}
\]

We leave it to you to check that \( Lt_{t \to \frac{\pi}{2}} z'(t) = z\left(\frac{1}{2}\right) \) and \( Lt_{t \to 0^+} z'(t) = z'(1^-) \).

E9) i) At the points where \( C \) intersects the \( x \)-axis \( t^3 \sin \left(\frac{\pi}{t}\right) = 0 \), so

\[
t = 0 \text{ or } \left(\frac{\pi}{t}\right) = 0. \quad t = 0 \text{ corresponds to the point } Z = 0. \text{ If } \sin \left(\frac{\pi}{t}\right) = 0, \frac{\pi}{t} = n\pi, \text{ so } t = 1/n \text{ where } n \text{ is a natural number since } t \geq 0.
\]

ii) We leave it to you to check that \( z(t) \) is continuous. Note that, at all the points other 0 it is easy to see that the arc is smooth. We have \( z'(t) = 1 + i3\pi t^2 \sin \left(\frac{\pi}{t}\right) + \pi t \cos \left(\frac{\pi}{t}\right) \). We have

\[
\left| i3\pi t^2 \sin \left(\frac{\pi}{t}\right) + \pi t \cos \left(\frac{\pi}{t}\right) \right| \leq 3t^2 + \pi t. \quad \text{So,}
\]

\[
i3\pi t^2 \sin \left(\frac{\pi}{t}\right) + \pi t \cos \left(\frac{\pi}{t}\right) \to 0 \text{ as } t \to 0.
\]

So, \( 1 + i3\pi t^2 \sin \left(\frac{\pi}{t}\right) + \pi t \cos \left(\frac{\pi}{t}\right) \to 1 \text{ as } t \to 0 \).

Also, \( z'(0^+) = Lt_{t \to 0^+} \frac{t + it^3 \sin \left(\frac{\pi}{t}\right)}{t} = 1 \)

since \( \left| t^2 \sin \left(\frac{\pi}{t}\right) \right| \leq t^2 \text{ and } t^2 \to 0 \text{ as } t \to 0^+ \).

E10) Write \( f_2(t) = u_1(t) + iv_1(t), f_2(t) = u_2(t) + iv_2(t) \) where \( u_1, u_2, v_1 \) and \( v_2 \) are real valued integrable functions. Then, by the definition of integral of complex valued functions, we have

\[
\int_a^b (f_1 + f_2) = \int_a^b (u_1 + u_2) + i \int_a^b (v_1 + v_2).
\]

Using properties of real valued integrable functions, we get

\[
\int_a^b (f_1 + f_2) = \int_a^b u_1 + \int_a^b u_2 + i \int_a^b v_1 + i \int_a^b v_2.
\]

Regroup again to get the result.

E11) Write

\[
\int_a^b w = \int_a^b u + i \int_a^b v. \quad (42)
\]

We have \( U'(t) = u(t) \) and \( V'(t) = v(t) \). Now, apply the Fundamental Theorem of Calculus to the integrals in the RHS of Eqn. (42) to get
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\[ \int_a^b u(t) = U(a) - U(b) \] and \[ \int_a^b v(t) = V(a) - V(b) \].

Substitute these values in Eqn. (42) and regroup.

E12) Observe that
\[ \frac{d}{dt} (i \cos it) = i (-i \sin it) = \sin it. \]
So, applying Fundamental Theorem of Calculus, we get
\[ \int_0^{\frac{\pi}{2}} \sin it = i \cos \left( \frac{\pi}{2} \right) - \cos 0 = -i. \]

E13) i) \[ \int_x^a \frac{dx}{x^r} = \frac{a^{1-r}}{1-r} - \frac{1}{1-r} \]
\[ \lim_{x \to a} \int_x^a \frac{dx}{x^r} = \lim_{x \to a} \left\{ \frac{1}{a^{1-r}} \cdot \frac{1}{1-r} - \frac{1}{1-r} \right\} = -\frac{1}{r-1} \text{ since } r-1 > 0 \]

\[ \int_{-r}^0 \frac{dx}{1+x^2} = \tan^{-1}(0) - \tan^{-1}(r). \]
\[ \lim_{r \to -r} \int_{-r}^0 \frac{dx}{1+x^2} = (0) - \left( -\frac{\pi}{2} \right) \]
\[ \int_{-r}^0 \frac{dx}{1+x^2} = \tan^{-1}(r) - \tan^{-1}(0). \]
\[ \lim_{r \to -r} \int_{-r}^0 \frac{dx}{1+x^2} = \frac{\pi}{2} - 0. \therefore \int_{-\infty}^0 \frac{dx}{1+x^2} = \pi. \]

E14) We have
\[ \int_a^b e^{i\theta x} f(x) = \int_a^b \text{Re}(e^{i\theta x} f(x)) + i \text{Im}(e^{i\theta x} f(x)) \]
\[ = \int_a^b \text{Re}(e^{i\theta x} f(x)) + i \int_a^b \text{Im}(e^{i\theta x} f(x)) \]
where \( \int_a^b \text{Re}(e^{i\theta x} f(x)) \) and \( \int_a^b \text{Im}(e^{i\theta x} f(x)) \) are real numbers. So,
\[ \text{Re} \left( \int_a^b e^{i\theta x} f(x) \right) = \text{Re} \left( \int_a^b \text{Re}(e^{i\theta x} f(x)) + i \int_a^b \text{Im}(e^{i\theta x} f(x)) \right) \]
\[ = \int_a^b \text{Re}(e^{i\theta x} f(x)) \]

E15) Let \( x = \int_a^b |f(x)|, n \in \mathbb{N} \). Then, \( x_n \leq x_{n+1} \) for all \( n \in \mathbb{N} \), i.e. \( \{x_n\} \) is a monotonic non-decreasing sequence. (Why?) By definition of improper integral
\[ \lim_{n \to \infty} x_n = \int_\infty^b |f| . \]
Therefore,
\[ \int_a^b |f| = x_n \leq \int_\infty^b |f| \]
Again,
\[ y_n = \int_a^b |f| \leq \int_\infty^b |f| \]
by Eqn. (34). Further,
\[ \lim_{n \to \infty} y_n = \int_\infty^b |f| \]
So, Eqn. (35) follows. (Use the fact that, if \( a_n \) converges to \( \ell \) as \( n \to \infty \) and \( a_n \leq c \) for \( n \geq n_0 \), then \( \ell \leq c \).)
E16) \[ \int f(w)\,dw = \int_0^1 \left( e^{2\pi i t} + 2 \right) 2\pi i e^{2\pi i t} \,dt \]
\[ + \int_0^1 \left( (4t - 3)^2 + 2 \right) 4dt \]
\[ = \int_0^1 2\pi i e^{2\pi i t} \,dt + \int_0^1 4\pi i e^{2\pi i t} \,dt + \int_0^1 \left( 16t^2 + 11 - 24t \right) 4dt \]
\[ = \frac{1}{3} \left. \left( e^{2\pi i t} \right) \right|_0^1 + 2 \left. \left( e^{2\pi i t} \right) \right|_0^1 + \left\{ \frac{64}{3} \frac{1}{8} + 22 - 12 \right\} \]
\[ = \frac{1}{3} \left( -1 - 1 \right) + 2 \left( -1 - 1 \right) + \left( \frac{56}{3} - 14 \right) = -\frac{2}{3} - 4 + \frac{56}{3} - 14 = 0 \]

E17) \[ \int f = \int_a^b f(y(t)) (y'(t)) \,dt = \int_a^b f(y(a + b - t)) (-y'(a + b - t)) \,dt \]
Putting \( u = a + b - t \) \( du = -dt \). When \( t = a, u = b \). When \( t = b, u = a \). So,
the integral becomes \[ \int_a^b f(y(u)) y'(u) \,du = \int_0^b f(y(u)) y'(u) \,du \]
\[ = -\int_a^b f(y(u)) y'(u) \,du = -\int f. \]

E18) Clearly from Fig. 21, for all \( z \) lying on \( C \),
\[ |z| \geq \frac{1 + i}{2} \Rightarrow |z| \geq \frac{1}{\sqrt{2}} \]
\[ \Rightarrow \frac{1}{|z|} \leq \sqrt{2} \Rightarrow \frac{1}{|z|^4} \leq 4 \]
\[ L = \text{length of } C = \sqrt{2}. \]
Therefore, \[ \int_C \frac{dz}{z^4} \leq 4\sqrt{2}. \]

E19) Let \( C \) be given by \( z : [a, b] \rightarrow \mathbb{C} \) where \( z \) is a smooth function and
\[ z(a) = z_1, z(b) = z_2. \]
Then, \[ \int_C f(z) \,dz = \int_a^b z'(t) \,dt = z(b) - z(a) = z_2 - z_1. \]
Suppose \( C \) is piecewise smooth given by \( z : [a, b] \rightarrow \mathbb{C} \) such that \( z(t) \) is smooth on each of the subintervals of the partition \( \{a = a_0 < a_1 < \cdots < a_n = b\} \). Also, suppose that contours \( C_1, C_2, \ldots, C_n \)
are such that \( C_k \) is given by \( z : [a_k, a_{k+1}] \rightarrow \mathbb{C} \).
\[ \int_C f(z) \,dz = \sum_{i=1}^n \int_{C_i} f(z) \,dz = \sum_{i=1}^n \int_{a_i}^{a_{i+1}} z'(t) \,dt \]
\[ = \sum_{i=1}^n (z(a_{i+1}) - z(a_i)) = z(a_n) - z(a_0) = z(b) - z(a) = z_1 - z_2. \]

E20) Suppose \( C \) is parametrised by the smooth contour \( z : [a, b] \rightarrow \mathbb{C} \). We
have \[ \int_C z^n \,dz = \int_a^b z^n(t) z'(t) \,dt. \]
\[ \text{We have } \frac{d}{dt} \left( \frac{1}{n+1} z^{n+1}(t) \right) = z^n(t) z'(t). \]
The result now follows from the Fundamental Theorem of Calculus.
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E21) i) We have \( \frac{d}{dz} \left( \frac{1}{\pi} e^{z \pi i} \right) = e^{z \pi i}. \) So,
\[
\int e^{z \pi i} dz = \frac{1}{\pi} \left( e^{z \pi i} - e^{i \pi} \right) = \frac{1}{\pi} (i - (-1)) = \frac{i + 1}{\pi}.
\]

ii) We have \( \frac{d}{dz} \left( 2 \sin \left( \frac{z}{2} \right) \right) = -\cos \left( \frac{z}{2} \right). \)
\[
\int_{0}^{\pi} \cos \left( \frac{z}{2} \right) dz = 2 \sin \left( \frac{\pi + 2i}{2} \right) = 2 \cos(i) = 2 \left( \frac{e^{(i)} + e^{-(-i)}}{2} \right) = \frac{1}{e} + e.
\]
The third part is similar.

E22) i) Since \( z \) and \( e^{-z} \) are entire functions therefore, \( f(z) = z e^{-z} \) satisfies conditions of Cauchy-Goursat Theorem and therefore \( \int z e^{-z} dz = 0. \)

ii) \( z^2 + 2z + 2 = [z(-1+i)(z-(-1-i))]. \)
Clearly, \( f(z) = \frac{1}{z^2 + 2z + 2} \) is analytic at all points interior to and on simple closed contour \( |z| = 1. \)
\[
\therefore \int_{C} \frac{dz}{z^2 + 2z + z} = 0.
\]

E23) i) \( f(z) = \frac{1}{3z^2 + 1}. \) Now \( 3z^2 + 1 = 0 \Rightarrow 3z^2 = -1 \Rightarrow z^2 = -1/3 \Rightarrow z = \pm \frac{i}{\sqrt{3}}. \)
Clearly, singularities of \( f(z) \) are inside the contour \( C_2. \)
\( f(z) = \frac{1}{3z^2 + 1} \) is analytic in the closed region consisting of contours \( C_1 \) and \( C_2 \) and all points between them as shown by the shaded region in Fig. 17. From Corollary 2 to Theorem 7,
\[
\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.
\]

ii) \( f(z) = \frac{z}{1 - e^z}. \)
\( 1 - e^z = 0 \Rightarrow e^z = 1 \)
\[
\Rightarrow e^{x+iy} = 1 e^{2\pi i}
\]
\( \Rightarrow e^y = 1 \) and \( y = 2\pi + 2n\pi = 2(n+1)\pi, \) \( n = 0, \pm 1, \pm 2, \ldots \)
\( \Rightarrow x = 0 \) and \( y = 2n\pi \) \( n = 0, \pm 1, \pm 2, \ldots \)
\( \Rightarrow z = 2n\pi i \) \( n = 0, \pm 1, \pm 2, \ldots \) are the points of singularities of \( f(z) \) which lie outside the closed region between contour \( C_1 \) and \( C_2 \) and hence \( f(z) \) is analytic in closed region in Fig. 17 and the result follows from corollary 2 to Theorem 7.

E24) Given that \( f(z) \) is analytic (in fact entire).
From Fig. 18(a), \( C = C_1 + (-C_1) \) is simple closed contour and by Cauchy-Goursat theorem
\[
\int_{C} f(z) dz = \int_{C_1} f(z) dz + \int_{-C_1} f(z) dz = 0 \Rightarrow \int_{C_1} f(z) dz = \int_{C_1} f(z) dz.
\]
Fig. 18(b) gives:
\[
\int_{C_1} f(z) dz + \int_{C_2} f(z) dz = 0 \Rightarrow \int_{C_2} f(z) dz = -\int_{C_1} f(z) dz.
\]
If \( C = C_1 + C_2 \) then conclude the result yourself.