E21) i) We have $\frac{d}{dz}\left(\frac{1}{\pi}e^{\pi z}\right) = e^{\pi z}$. So,

$$\int_{i}^{\frac{i}{2}} e^{\pi z} dz = \frac{1}{\pi}\left(e^{\frac{i\pi}{2}} - e^{i\pi}\right) = \frac{1}{\pi}(i-(-1)) = \frac{i+1}{\pi}.$$

ii) We have $\frac{d}{dz}\left(2\sin\left(\frac{z}{2}\right)\right) = = \cos\left(\frac{z}{2}\right)$.

$$\int_{0}^{\pi+2i} \cos\left(\frac{z}{2}\right)dz = 2\sin\left(\frac{\pi+2i}{2}\right) = 2\cos(i) = 2\left(\frac{e^{i(i)}+e^{-i(i)}}{2}\right) = \frac{1}{e} + e.$$

The third part is similar.

E22) i) Since z and e^{-z} are entire functions therefore, $f(z) = z e^{-z}$ satisfies conditions of Cauchy-Goursat Theorem and therefore $\int z e^{-z} dz = 0$.

ii) $z^2 + 2z + 2 = [z - (-1+i)\ (z - (-1-i)]$.

Clearly, $f(z) = \frac{1}{z^2 + 2z + 2}$ is analytic at all points interior to and on simple closed contour $|z| = 1$.

$$\therefore \int_C \frac{dz}{z^2 + 2z + z} = 0.$$

E23) i) $f(z) = \frac{1}{3z^2 + 1}$. Now $3z^2 + 1 = 0 \Rightarrow 3z^2 = -1 \Rightarrow z^2 = -1/3 \Rightarrow z = \pm\frac{i}{\sqrt{3}}$.

Clearly, singularities of $f(z)$ are inside the contour C_2.

$f(z) = \frac{1}{3z^2 + 1}$ is analytic in the closed region consisting of contours C_1 and C_2 and all points between them as shown by the shaded region in Fig. 17. From Corollary 2 to Theorem 7,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

ii) $f(z) = \frac{z}{1 - e^z}$.

$1 - e^z = 0 \Leftrightarrow e^z = 1$

$\Leftrightarrow e^{x+iy} = 1.e^{2\pi i}$

$\Leftrightarrow e^x = 1$ and $y = 2\pi + 2n\pi = 2(n+1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

$\Leftrightarrow x = 0$ and $y = 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$

$\Rightarrow$ $z = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$ are the points of singularities of $f(z)$ which lie outside the closed region between contour C_1 and C_2 and hence $f(z)$ is analytic in closed region in Fig. 17 and the result follows from corollary 2 to Theorem 7.

E24) Given that $f(z)$ is analytic (in fact entire).

From Fig. 18(a), $C = C_3 + (-C_1)$ is simple closed contour and by Cauchy-Goursat theorem

$$\int_C f(z) = 0 \Rightarrow \int_{C_3} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0 \Rightarrow \int_{C_3} f(z)\,dz = \int_{C_1} f(z)\,dz$$

Fig. 18(b) gives: $\int_{C_2} f(z)\,dz + \int_{C_3} f(z)\ dz = 0 \Rightarrow \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz$.

If $C = C_1 + C_2$ then conclude the result yourself.