$$e^{i\alpha}\frac{z+z_0}{z+\bar{z}_0},\ e^{i\alpha}\frac{iz-z_0}{iz-\bar{z}_0} \text{ and } e^{i\alpha}\frac{iz+z_0}{iz+\bar{z}_0}$$

respectively, where $\alpha \in \mathbb{R}$ and $Im\, z_0 > 0$.

E33) Writing $z = x + iy$, $w = \sin z = u + iv$ and making use of Eqn. (55), the image of the horizontal line segment $y = c$, $-\pi \le x \le \pi$ is

$$u = \sin x \cosh c \text{ and } v = \cos x \sinh c.$$

It is easily seen that u and v satisfy the equation of ellipse

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = \sin^2 x + \cos^2 x = 1$$

having its foci at $\pm\sqrt{\cosh^2 c - \sinh^2 c} = \pm 1$ (by Eqn. (30)). Consider the following four cases:

Case 1: If $-\pi \le x \le -\pi/2$, then $u < 0$ and $v < 0$. Also, u is a decreasing function of x and v is an increasing function of x. Thus as the point (x, c) moves along the line segment in the increasing direction, its image point (u, v) transverses an ellipse in the clockwise direction in the third quadrant.

Case 2: If $-\pi/2 \le x \le 0$, then $u < 0$ and $v > 0$. In this case, both u and v are increasing functions of x. Thus the point (u, v) traces an ellipse in the clockwise direction in the second quadrant as the point (x, c) moves along the line segment in the increasing direction.

Case 3: For $0 \le x \le \pi/2$, both u and v are positive. Since u is an increasing function of x and v is a decreasing function of x. Consequently, as the point (x, c) moves along this line segment, its image point (u, v) transverses an ellipse in the clockwise direction in the first quadrant.

Case 4: In the case $\pi/2 \le x \le \pi$, $u > 0$ and $v < 0$ are decreasing functions of x; and the image (u, v) traces an ellipse lying in the fourth quadrant in the clockwise sense.

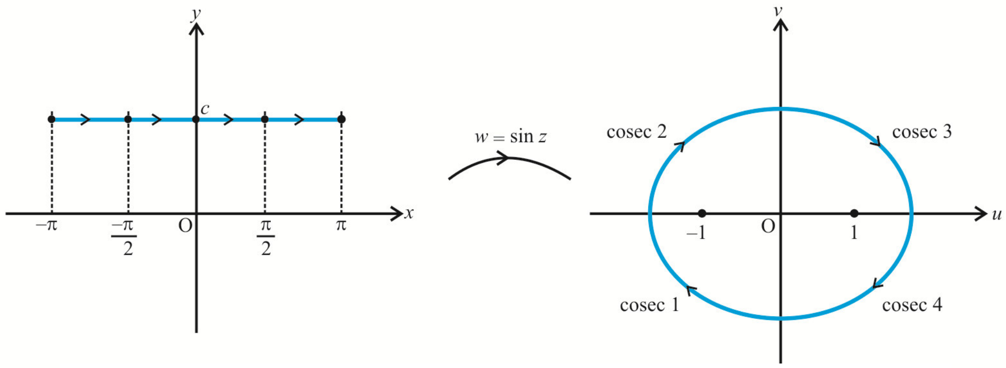

Image of the horizontal line $y = c\,(-\pi \le x \le \pi)$ under the mapping $w = \sin z$

Fig. 10

All these four cases are depicted in Fig. 10. Hence the transformation $w = \sin z$ maps the horizontal line segments $y = c\,(c > 0)$ and $-\pi \le x \le \pi$ in a one-to-one manner onto an ellipse with foci at ± 1 traversed in the clockwise direction.

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