UNIT 2 ANALYTIC FUNCTIONS

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2.1 INTRODUCTION

In Unit 1, we introduced you to functions of complex variables and their interpretation in terms of real valued functions of real variables. We also discussed there the concept of limit and continuity of the complex valued functions. We shall now discuss in this unit the differentiation of these complex valued functions.

We shall start by introducing in Sec. 2.2 the derivative of a function of complex variable at a point and discuss the difference between the derivative of a function of a real variable and that of a complex variable. Cauchy-Riemann equations which under certain conditions provide the necessary and sufficient condition for the differentiability of a function of complex variable at a point are discussed in Sec. 2.3. The concept of analytic functions, which play an important role in complex analysis and is useful in many applications of the complex variable theory is discussed in Sec. 2.4. Finally, in Sec. 2.5 we shall introduce harmonic functions which play an important role in applied mathematics. The temperature \( T(x, y) \) in thin plates in the \( xy \)-plane and a function \( V(x, y) \) denoting an electrostatic potential in a region free of charges are some of the examples of harmonic function. We shall be discussing these applications later in Unit 10.

Objectives

After studying this unit, you should be able to:

- obtain the derivative of a complex valued function of a complex variable;
- use the Cauchy-Riemann equations to prove the differentiability of a function of complex variable at a point;
- check the analyticity of a given function at a given point;
- check if a given function is harmonic in its domain of definition;
- obtain the harmonic conjugate of a given harmonic function.

2.2 DIFFERENTIATION

In Unit 1 we defined the limit of a complex valued function at a point. We shall now use it to develop the concept of differentiation of complex valued function.
We start with the following limit definition of the derivative of a complex function \( f(z) \).

**Definition 1:** A complex function \( f : D \to \mathbb{C} \), where \( D \) is an open subset of \( \mathbb{C} \), is said to be **differentiable** (or complex differentiable) at \( z_0 \in D \), if

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}
\]

exists. We denote this limit by \( \frac{df}{dz}(z_0) \) or \( f'(z_0) \) and is called the **derivative** of \( f \) at the point \( z_0 \).

The number \( f'(z_0) \) is generally a complex number. The function \( f \) is said to be differentiable on \( D \) if it is differentiable at every point of \( D \). A function which is differentiable in the entire complex plane is called an **entire** function.

Now we make the following remarks.

**Remark 1:** Open set is an integral part of Definition 1. It takes care of the fact that when \( h \) is small then \( z + h \) is also in \( D \).

**Remark 2:** Definition 1 is similar to what you have learnt in your analysis course for the function of real variable. But you should remember that \( h \) here is a complex variable and therefore its approach to zero is not confined to one direction. Here \( h \) may approach zero in infinitely many ways.

**Remark 3:** Since we cannot graph a complex function in the way we graph a real valued function, \( f'(z_0) \) cannot be visualized as a ‘slope’ of some curve as we do in the real case.

We now illustrate Definition 1 through an example.

**Example 1:** Find the derivative of \( f(z) = z^2 + 4z \).

**Solution:** We have

\[
f(z + h) = (z + h)^2 + 4(z + h) = z^2 + 2zh + h^2 + 4z + 4h.
\]

\[
\therefore f(z + h) - f(z) = z^2 + 2zh + h^2 + 4z + 4h - z^2 - 4z = 2zh + h^2 + 4h.
\]

Eqn. (1) then gives

\[
f'(z) = \lim_{h \to 0} \frac{2zh + h^2 + 4h}{h} = \lim_{h \to 0} (2z + h + 4) = 2z + 4.
\]

Thus, function \( f(z) = z^2 + 4z \) is everywhere differentiable with derivative \( f'(z) = 2z + 4 \).

You know from your knowledge of functions of real variables that if a function of real variable is differentiable at a point then it is continuous at that point. This
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holds true for complex functions as well. We shall now prove it in the following theorem.

**Theorem 1:** If a complex valued function \( f \) is differentiable at a point \( z_0 \), then it is continuous at \( z_0 \).

**Proof:** We know that a function \( f \) is continuous at \( z_0 \) if

\[
\lim_{z \to z_0} \left( f(z) - f(z_0) \right) = 0.
\]

Now, for \( z \neq z_0 \)

\[
f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} (z - z_0) .
\]

Since \( f \) is differentiable at \( z_0 \),

\[
\lim_{z \to z_0} \left( \frac{f(z) - f(z_0)}{z - z_0} \right) = f'(z_0) .
\]

Also, we have \( \lim_{z \to z_0} (z - z_0) = 0 \).

Therefore, we get from Eqn. (2)

\[
\lim_{z \to z_0} [f(z) - f(z_0)] = 0 \quad \text{or} \quad \lim_{z \to z_0} f(z) = f(z_0) , \quad \text{i.e.,} \quad f \text{ is continuous at } z_0 .
\]

Hence, if \( f \) is differentiable at \( z_0 \) then it is continuous at \( z_0 \).

\[\square\]

As in the case of real variables, the continuity of complex valued function \( f \) does not necessarily imply the differentiability of \( f \) as illustrated in the following example.

**Example 2:** Show that the function \( f(z) = \overline{z} \) is continuous at the point \( z = 0 \) but not differentiable at \( z = 0 \). Is this function continuous and differentiable elsewhere?

**Solution:** Let \( f(z) = u(x, y) + iv(x, y) \)

\[
f(z) = \overline{z} = x - iy \text{ gives } u(x, y) = x , \text{ and } v(x, y) = -y
\]

Now \( \lim_{(x, y) \to (0,0)} u(x, y) = \lim_{(x, y) \to (0,0)} x = 0 \)

and \( \lim_{(x, y) \to (0,0)} v(x, y) = \lim_{(x, y) \to (0,0)} (-y) = 0 . \)

\[
\therefore \lim_{z \to 0} f(z) = 0 + 0 = 0 \quad \text{and also } \quad f(0) = 0 .
\]

Hence, \( \lim_{z \to 0} f(z) = f(0) \Rightarrow f \) is continuous at \( z = 0 . \)

In fact, \( f(z) = \overline{z} \) is continuous everywhere. For if \( z_0 \) be any given point then

\[
| f(z) - f(z_0) | = | z - \overline{z_0} | = | z - z_0 |.
\]

Thus for given \( \varepsilon > 0 \) such that \( | f(z) - f(z_0) | < \varepsilon \), we have \( | z - z_0 | < \delta = \varepsilon . \)

To check the differentiability of \( f \) let \( h = h_1 + ih_2 . \) Now at \( z = 0 \)

\[
\frac{f(0 + h) - f(0)}{h} = \frac{f(h)}{h} = \frac{\overline{h}}{h}.
\]
If the limit of \( \frac{\bar{h}}{h} \) exists, it may be found by letting the point \( h = (h_1, h_2) \) approach the origin in the \( h \)-plane in any manner. When \( h \) approaches the origin horizontally through the point \((h_1,0)\) along the real axis (see Fig. 1),
\[
\bar{h} = h_1 + i0 = h_1 + 0 = h_1 = 1.
\]
When \( h \) approaches the origin vertically through the point \((0,h_2)\) along the imaginary axis,
\[
\bar{h} = 0 + ih_2 = -ih_2 \quad \text{and} \quad \frac{\bar{h}}{h} = -1.
\]
Since limits are not unique, \( f(z) = \bar{z} \) is not differentiable at \( z = 0 \). In fact it is nowhere differentiable. Proceeding as above, you can easily check by considering an arbitrary point \( z_0 \) that
\[
\frac{f(z_0 + h) - f(z_0)}{h} = \frac{z_0 + \bar{h} - z_0}{h} = \frac{\bar{h}}{h}.
\]

Let us now recall the differentiation formulas for real valued functions of real variables. They hold for complex valued functions as well. We shall summarise the differentiation formulas for complex valued functions in the theorem given below.

**Theorem 2:** If the functions \( f \) and \( g \) are complex differentiable at \( z \) with derivatives \( \frac{df}{dz} = f'(z) \) and \( \frac{dg}{dz} = g'(z) \), then
\[
\frac{d}{dz} \left[ f(z) + g(z) \right] = f'(z) + g'(z)
\]
(3)
\[
\frac{d}{dz} \left[ f(z) g(z) \right] = f'(z) g(z) + f(z) g'(z)
\]
(4)
\[
\frac{d}{dz} \left( \frac{f(z)}{g(z)} \right) = \frac{f'(z) g(z) - f(z) g'(z)}{[g(z)]^2} \quad \text{when} \quad g(z) \neq 0
\]
(5)

**Proof:** These formulas can be derived easily by using Definition 1. We first consider the proof of Formula (4). Since \( f \) and \( g \) are differentiable at the point \( z \), they are also continuous at \( z \).

Therefore, we can write
\[
\lim_{h \to 0} g(z + h) = g(z)
\]
(6)
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z)
\]
(7)
and
\[
\lim_{h \to 0} \frac{g(z + h) - g(z)}{h} = g'(z)
\]
(8)

Now for \( h \neq 0 \), we have
\[
\frac{f(z + h) g(z + h) - f(z) g(z)}{h} = \frac{f(z + h) g(z + h) - f(z) g(z + h) + f(z) g(z + h) - f(z) g(z)}{h}
\]
\[
= \frac{g(z + h)[f(z + h) - f(z)] + f(z)[g(z + h) - g(z)]}{h}
\]
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\[ g(z + h) \frac{[f(z + h) - f(z)]}{h} + f(z) \frac{[g(z + h) - g(z)]}{h} \]

Now taking the limit as \( h \) tends to 0 and using Eqns. (6), (7) and (8), we get

\[ \frac{d}{dz} [f(z) g(z)] = f'(z) g(z) + f(z) g'(z). \]

Formulas (3) and (5) can be proved in the same manner. We are leaving for you to do it yourself.

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E1) Prove Formulas (3) and (5) of Theorem 2.

Continuing with the complex differentiation formulas we now state a theorem which gives a chain rule for differentiating composite functions of complex variables.

**Theorem 3:** (Chain Rule). If \( f \) has derivative at \( z \) and \( g \) has derivative at \( f(z) \) then the composite function \( g \circ f \) has derivative at \( z \) and it is given by \( (g \circ f)'(z) = g'[f(z)] f'(z) \) or in simple terms

\[ \frac{d}{dz} (g \circ f)(z) = \frac{d}{dw} g(w) \frac{d}{dz} w, \text{ where } w = f(z). \]  

(9)

We shall not be proving the theorem here but illustrate it through examples. You may try to prove it yourself.

---

E2) Prove Theorem 3.

We now take up some more examples to illustrate Theorems 1, 2 and 3.

**Example 3:** Show that the function \( f(z) = z^n \), where \( n \) is a positive integer is differentiable at every point in the complex plane.

**Solution:** Let \( z \) be any point in the complex plane. Then

\[ \frac{f(z + h) - f(z)}{h} = \frac{(z + h)^n - z^n}{h} \]

\[ = \frac{1}{h} \left[ \binom{n}{1} z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \cdots + \binom{n}{n} h^n \right] \]

\[ = \binom{n}{1} z^{n-1} + \binom{n}{2} z^{n-2} h + \cdots + \binom{n}{n} h^{n-1} \]

(10)

Taking limit as \( h \to 0 \) on both the sides of Eqn. (10), we obtain

\[ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = n z^{n-1} \]

Therefore, \( \frac{d}{dz} (z^n) = nz^{n-1} \),

(11)

where \( z \) is any point in the complex plane.
**Remark 4:** Since \( f(z) = z^n \), \( n \) a positive integer, is differentiable at every point in the complex plane, any polynomial \( P_n(z) \) in \( z \) is also differentiable at every point in the complex plane.

**Example 4:** Show that a real valued function of a complex variable is either nowhere differentiable or if it is differentiable then its derivative is zero.

**Solution:** Let \( f \) be the real valued function of complex variable \( z \). Suppose \( f \) is differentiable at \( z_0 \) with derivative \( f'(z_0) \). Then
\[
f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ where } h = (h_1, h_2) \in \mathbb{C} \text{ with } h_1, h_2 \text{ being real.}
\]
If we take the limit \( h \to 0 \) through the point \((h_1, 0)\) along the real axis then \( f'(z_0) \) is purely a real number.

If we take the limit \( h \to 0 \) through the point \((0, h_2)\) along the imaginary axis then \( f'(z_0) \) is purely an imaginary number.

Since \( f'(z_0) \) exists, we must have \( f'(z_0) = 0 \).

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Let us consider another example.

**Example 5:** Compute the derivative of function \( f \) given by
\[
f(z) = \frac{(1 + z^2)^4}{z^2} (z \neq 0).
\]

**Solution:** Putting \( w = 1 + z^2 \) and using Chain rule (9) and Formula (3), we have
\[
\frac{d}{dz}(1 + z^2)^4 = \frac{d}{dw}(w^4) \frac{d}{dz}(1 + z^2) = 4w^3 \left( \frac{d}{dz}1 + 2z \right) = 4(1 + z^2)^3 2z = 8z(1 + z^2)^3.
\]
Now using Formula (5), we get
\[
\frac{d}{dz} f(z) = \frac{d}{dz} \frac{(1 + z^2)^4}{z^2} = \frac{z^2 \frac{d}{dz}(1 + z^2)^4 - (1 + z^2)^4 \frac{d}{dz}z^2}{z^4} = \frac{8z^3(1 + z^2)^3 - 2z(1 + z^2)^4}{z^4} = 2(1 + z^2)^3 \left( \frac{3z^2 - 1}{z^3} \right).
\]

***

You may now try the following exercises.

E3) Prove that the expression given by Eqn. (11) for the derivative of \( z^n \) remains valid when \( n \) is a negative integer \( (n = -1, -2, \ldots) \), provided that \( z \neq 0 \).
E4) Show that \( f'(z) \) does not exist at any point when

(i) \( f(z) = \text{Im } z \)  
(ii) \( f(z) = \text{Re } z \).

E5) Show that \( f(z) = |z| \) is not differentiable at \( z = 0 \) but is continuous at \( z = 0 \). Is \( f \) differentiable at other points in \( \mathbb{C} \)?

E6) Give an example of a continuous complex valued function which is nowhere differentiable except at the origin.

You have thus seen that a complex valued function may have derivative at every point, only at a fixed point or at no point in the finite complex plane. The natural question which may occur to you is: Is there any formula/method to check the differentiability of a given function with an ease? A partial answer to this question is given by the Cauchy-Riemann equations (C-R equations), which are a pair of equations involving first order partial derivatives. We shall now be discussing these equations in the next section.

Cauchy-Riemann equations are named in honour of the French mathematician, Augustin-Louis Cauchy (1789-1857), who discovered and used them, and in honour of the German mathematician, George Friedrich Bernhard Riemann (1826-1866), who made them fundamental in his development of the theory of complex variables.

### 2.3 CAUCHY-RIEMANN EQUATIONS

Let us look at complex differentiability of a function once again.

Let \( f \) be a complex differentiable function at the point \( z = x + iy = (x, y) \) and \( f(z) = u(x, y) + iv(x, y) \), where \( u(x, y) \) and \( v(x, y) \) are real valued functions of the real variables \( x \) and \( y \). Then by the definition of complex differentiability the derivative of \( f, f'(z) \) is given by

\[
f'(z) = \lim_{{h \to 0}} \frac{{f(z + h) - f(z)}}{h}
\]

where \( h \) is a complex number not equal to 0. This limit is independent of the choices of \( h \) approaching 0. In simple words, whatever way \( h \) approaches 0, the outcome will remain \( f'(z) \). Let us consider two approaches as follows:

**Approach 1:** Let \( h \in \mathbb{R} \) (\( h \) approaches 0 along the real axis). We then have

\[
f'(z) = \lim_{{h \to 0}} \frac{{f(z + h) - f(z)}}{h}
= \lim_{{h \to 0}} \frac{{f(x + iy + h) - f(x + iy)}}{h}
= \lim_{{h \to 0}} \frac{{f(x + h, y) - f(x, y)}}{h}
= \lim_{{h \to 0}} \frac{{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}}{h}
= \lim_{{h \to 0}} \frac{{u(x + h, y) - u(x, y)}}{h} + i \lim_{{h \to 0}} \frac{{v(x + h, y) - v(x, y)}}{h}
= \frac{{\partial}}{{\partial x}}u(x, y) + i \frac{{\partial}}{{\partial y}}v(x, y)
= u_x (x, y) + iv_x (x, y)
\]

(12)
where \( u_x(x, y) \) and \( v_y(x, y) \) denote the first order partial derivatives with respect to \( x \) of the functions \( u \) and \( v \), respectively at \( (x, y) \).

**Approach 2:** Now consider \( h \) as purely imaginary. Put \( h = ik, k \in \mathbb{R} \) (\( h \) approaches 0 along imaginary axis). In this case we have

\[
f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{k \to 0} \frac{f(x + iy + ik) - f(x + iy)}{ik} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{ik} = \lim_{k \to 0} \frac{u(x, y + k) + iv(x, y + k) - u(x, y) - iv(x, y)}{ik} = \lim_{k \to 0} \frac{u(x, y + k) - u(x, y)}{ik} + i \lim_{k \to 0} \frac{v(x, y + k) - v(x, y)}{ik} = -i \lim_{k \to 0} \frac{u(x, y + k) - u(x, y)}{k} + \lim_{k \to 0} \frac{v(x, y + k) - v(x, y)}{k} = -i u_y(x, y) + v_y(x, y)
\]

where \( u_y \) and \( v_y \) are the partial derivatives of \( u \) and \( v \), respectively with respect to \( y \). If \( f'(z) \) exists, then we must have from Eqns. (12) and (13)

\[
u_y(x, y) + iv_x(x, y) = -i u_y(x, y) + v_y(x, y).
\]

Comparing the real and the imaginary parts of Eqn. (14) we conclude that if the function \( f(z) = u(x, y) + iv(x, y) \) is complex differentiable at a point \( z = (x, y) \) then its real and imaginary parts necessarily satisfy, at that point, the pair of partial differential equations

\[
u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y).
\]

Partial differential Eqns. (15) are known as **Cauchy-Riemann equations**. In short, they are written as C-R equations. For instance, in Example 1, we showed that the function \( f(z) = z^2 + 4z = x^2 - y^2 + 4x + i(2xy + 4y) \) is differentiable everywhere. To verify that the Cauchy-Riemann equations are satisfied by the function everywhere, we see here that \( u(x, y) = x^2 - y^2 + 4x, v(x, y) = 2xy + 4y \) satisfy Eqn. (15) since \( u_x = 2x + 4 = v_y \) and \( u_y = -2y = -v_x \).

Let us now summarise the result proved above in the form of the following theorem.

**Theorem 4:** Let \( U \) be an open subset of complex numbers and \( f : U \to \mathbb{C} \) be a complex function. If \( f \) is differentiable at \( z_0 = (x_0, y_0) \in U \) then the first order partial derivatives of the real and imaginary parts of \( f \) (i.e., \( u(x, y) \) and \( v(x, y) \), respectively) satisfy the C-R equations

\[
u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).
\]

Also \( f'(z_0) \) can be written as

\[
f'(z_0) = u_x + iv_x
\]
**Remark 5:** You may note that the C-R equations are merely necessary condition for the complex differentiability of a function at a point and not the sufficient one. That is, if a complex function does not satisfy the C-R equations at a point then it is not complex differentiable at that point but if the C-R equations are satisfied at a point it may not guarantee its differentiability thereat. The function may or may not be differentiable at that point.

We now consider some examples to illustrate Remark 5.

**Example 6:** Show that the function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$f(z) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

satisfies the C-R equations at $z = 0$ but is not differentiable there.

**Solution:** If we put $f(z)$ in the form $u(x, y) + iv(x, y)$ then it can be seen easily (since $f(z)$ is real valued) that

$$u(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ 0 & \text{if } y = 0, \\ 1 & \text{otherwise}. \end{cases} \quad v(x, y) = 0 \quad (\text{everywhere}).$$

Clearly, all partial derivatives $u_x, u_y, v_x$ and $v_y$ vanish everywhere. Therefore the C-R equations are satisfied everywhere. Also for complex number $h \neq 0$

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} 0 & \text{if } h \text{ is purely real,} \\ 0 & \text{if } h \text{ is purely imaginary,} \\ 1 & \text{if } h = k + ik, k \text{ is non-zero purely real number.} \end{cases}$$

It clearly shows that $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist. Hence the function is not complex differentiable at the origin even if the C-R equations are satisfied thereat.

**Example 7:** Prove that $f$ defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

satisfies the C-R equations at $z = 0$ but is not differentiable there.

**Solution:** Put $z = x + iy, x, y \in \mathbb{R}$ then

$$z^5 = (x + iy)^5 = (x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)$$

Let $f(z) = u(x, y) + iv(x, y)$ then for $z \neq 0$ we have

$$u(x, y) = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2 + y^2)^2}, \quad v(x, y) = \frac{5x^4y - 10x^2y^3 + y^5}{(x^2 + y^2)^2}.$$ 

Now we compute the partial derivatives at $(0,0)$:

$$\frac{\partial}{\partial x} u(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x^5}{x} = 1,$$
\[ \frac{\partial}{\partial y} u(0, 0) = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = 0, \]
\[ \frac{\partial}{\partial y} v(0, 0) = \lim_{y \to 0} \frac{v(x, 0) - v(0, 0)}{x} = 0, \quad \text{and} \]
\[ \frac{\partial}{\partial y} v(0, 0) = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{x - y} = \lim_{y \to 0} \frac{y^5}{y} = 1. \]

This implies that \( u_x(0, 0) = v_y(0, 0), u_y(0, 0) = -v_x(0, 0) \). Therefore the C-R equations are satisfied at \( z = 0 \).

Now we show that \( f \) is not differentiable at \( z = 0 \). We compute the following limits taking two different paths, one along the line \( y = x \) and the other along the line \( y = -x \) as shown in Fig. 2.

![Fig. 2](image)

Along \( y = x \) (let \( h = k + ik \))
\[ \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{k \to 0} \frac{f(k + ik)}{k + ik} \]
\[ = \lim_{k \to 0} \frac{(k + ik)^5}{(k + ik)(k^4 + k^2 + 1)} \]
\[ = \frac{(1 + i)^4}{4}. \]

Along \( y = -x \) (let \( h = k - ik \))
\[ \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{k \to 0} \frac{f(k - ik)}{k - ik} \]
\[ = \lim_{k \to 0} \frac{(k - ik)^5}{(k - ik)(k + ik)(k^4 + k^2 + 1)} \]
\[ = \frac{(1 - i)^4}{4}. \]

Thus, we observe that the two different paths result in two different limiting values of the quotient and hence the function is not differentiable at \( z = 0 \).

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As you have seen that the Cauchy-Riemann equations are necessary conditions for the existence of the derivative of a function \( f \) at a point \( z_0 \), they can be used
to locate points where \( f \) is not differentiable. For instance, for the function 
\[ f(z) = |z|^2, \]
we have \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 0 \). Now if the C-R equations are to be satisfied at a point \((x, y)\), then we must have \( x = y = 0 \).
However, you already know that function \( f(z) = |z|^2 \) is not differentiable at any non-zero point (ref. E6)). It is differentiable only at the origin.

You may now check your understanding of the Cauchy-Riemann equations while doing the following exercises.

**E7)** Prove that the Cauchy-Riemann equations are satisfied for the function
\[ f(z) = \sqrt{|x+yi|} \] at the point \( z = 0 \) but the function is not differentiable at that point.

**E8)** Use the Cauchy-Riemann equations to show that the following functions are nowhere differentiable:

(i) \( f(z) = z + \bar{z} \)
(ii) \( f(z) = e^y \cos x + ie^y \sin x \)
(iii) \( f(z) = xy + iy \)
(iv) \( f(z) = 2xy + i(x^2 - y^2) \).

**E9)** Show that when \( f(z) = x^3 + i(1 - y)^3 \), then we can write 
\[ f'(z) = u_x + iv_x = 3x^2 \] only when \( z = i \).

**E10)** Let \( u \) and \( v \) denote the real and imaginary components of the function \( f \) defined by the equations
\[ f(z) = \begin{cases} \frac{z^2}{2} & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases} \]
Verify that the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \) are satisfied at the origin.

**E11)** i) Recall that if \( z = x + iy \) then \( x = \frac{z + \bar{z}}{2} \) and \( y = \frac{z - \bar{z}}{2i} \).

By applying the chain rule in calculus to a function \( F(x, y) \) of two real variables, derive the expression
\[ \frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right). \]

ii) Using the expression derived in part i), show that if the real and imaginary parts of the function \( f(z) = u(x, y) + iv(x, y) \) satisfy the C-R equations, then
\[ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)] = 0. \]

Using the polar coordinate system we can also derive the polar form of the Cauchy-Riemann equations. You know from Unit 1 that using the coordinate transformation \( x = r \cos \theta \), \( y = r \sin \theta \), a function \( w = f(z) \) of the complex variable \( z = re^{i\theta} \) can be written in terms of polar coordinates in the form
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\[ w = f(z) = u(r, \theta) + iv(r, \theta) \]  
(18)

where \( u \) and \( v \) are real valued functions of the real variables \( r \) and \( \theta \). Using the chain rule for differentiating \( u \) and \( v \) with respect to \( r \) and \( \theta \) and simplifying, the Cauchy-Riemann equations in polar form become

\[ ru_r = v_\theta, \quad u_\theta = -r v_r. \]  
(19)

The derivative \( f'(z) \) at a point \( z \) is then

\[ f'(z) = e^{i\theta}(u_r + iv_r). \]  
(20)

We are not going into the details of the derivation of Eqns. (19) here. We are leaving it as an exercise for you to do it yourself.

E12) Derive the Cauchy-Riemann Eqns. (19) in polar form.

We shall now illustrate how this change of coordinates to polar form sometimes make the computations easier.

Example 8: Show that \( f(z) = 1/z^4 \) is differentiable at \( z \neq 0 \). Also obtain \( f'(z) \).

Solution: Consider \( z = x + iy \) where \( x = r \cos \theta, y = r \sin \theta (r > 0) \).

Now \( z = re^{i\theta} \) and \( f(z) \) can be written as \( f(z) = \frac{1}{r^4}e^{-4i\theta} = u(r, \theta) + iv(r, \theta) \).

Clearly, \( u(r, \theta) = \frac{1}{r^4} \cos 4\theta \) and \( v(r, \theta) = \frac{-1}{r^4} \sin 4\theta \).

\((u(r, \theta) \) and \( v(r, \theta) \) are continuous and differentiable at every non-zero point \( z = re^{i\theta} \).)

Now we compute the partial derivatives:

\[ u_r = \frac{-4 \cos 4\theta}{r^5}, \quad u_\theta = \frac{-4 \sin 4\theta}{r^4} \]
\[ v_r = \frac{4 \sin 4\theta}{r^5}, \quad v_\theta = \frac{-4 \cos 4\theta}{r^4} \]

We have

\[ ru_r = \frac{-4 \cos 4\theta}{r^4} = v_\theta \quad \text{and} \quad u_\theta = \frac{-4 \sin 4\theta}{r^4} = -r \left( \frac{4 \sin 4\theta}{r^5} \right) = -r v_r. \]

Therefore, the C-R equations are satisfied at every non-zero point \( z = re^{i\theta} \) and all the partial derivatives with respect to \( r \) and \( \theta \) are continuous. Hence \( f \) is differentiable everywhere when \( z \neq 0 \). The derivative \( f'(z) \) is given by

\[ f'(z) = e^{-i\theta}(u_r + iv_r) \]
\[ = e^{-i\theta} \left( \frac{-4 \cos 4\theta}{r^5} + i \frac{4 \sin 4\theta}{r^5} \right) \]
\[ = \frac{-4 e^{-i\theta}}{r^5}(\cos 4\theta - i \sin 4\theta) = -\frac{4}{r^5} e^{-i\theta} e^{-i4\theta} \]
\[ = -\frac{4}{r^5} e^{-i\theta} e^{-i\theta} = -\frac{4}{r^5} \left( r e^{i\theta} \right)^{-5} = -\frac{4}{z^5} \]
\[ \Rightarrow f'(z) = -4z^{-5}. \]

***
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It follows from the discussion so far that for a given complex valued function \( f \) satisfaction of the Cauchy-Riemann equations at a point is not sufficient to ensure the existence of the derivative of the function \( f \) at that point. It is thus clear that we need some additional hypothesis for the function \( f \). Surprisingly, what we require is nothing more than certain continuity conditions as given by the following theorem.

**Theorem 5:** The function \( f(z) = u(x, y) + iv(x, y) \) is differentiable at a point \( z = x + iy \) if the partial derivatives \( u_x, u_y, v_x, v_y \) are continuous and satisfy the Cauchy-Riemann equations \( u_x = v_y, u_y = -v_x \) there. The derivative is then given by \( f'(z) = u_x + iv_x \).

**Proof:** Consider the point \( z = x + iy \) and define \( \Delta z = \Delta x + i\Delta y \). Then
\[
\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y), \quad \Delta v = v(x + \Delta x, y + \Delta y) - v(x, y).
\]
You may now recall from your knowledge of two variable calculus (the mean value theorem) that because of the continuity of \( u_x, u_y, v_x, v_y \) at a point \((x, y)\), we may write
\[
0 \leq \lim_{\Delta z \to 0} \frac{|\Delta u|}{|\Delta z|} = \lim_{\Delta z \to 0} |\Delta u|_{\Delta z} |\Delta z| = 0
\]
and
\[
0 \leq \lim_{\Delta z \to 0} \frac{|\Delta v|}{|\Delta z|} = \lim_{\Delta z \to 0} |\Delta v|_{\Delta z} |\Delta z| = 0
\]
Now let \( w = f(z) \), then
\[
\Delta w = f(z + \Delta z) - f(z) = \Delta u + i\Delta v
\]
Now substituting the expressions from Eqns. (21) and (22) into Eqn. (23), we get
\[
\Delta w = (u_x \Delta x + u_y \Delta y + (\epsilon_1 + i\epsilon_2))|\Delta z|
\]
Further, using the Cauchy-Riemann Equations in the above equation, we get
\[
\Delta w = (u_x + iv_x)(\Delta x + i\Delta y) + (\epsilon_1 + i\epsilon_2)|\Delta z|
\]
\[\therefore \frac{\Delta w}{\Delta z} = (u_x + iv_x) + (\epsilon_1 + i\epsilon_2)\frac{|\Delta z|}{\Delta z} \]
Putting \( \epsilon = (\epsilon_1 + i\epsilon_2) \) and noting that \( \lim_{\Delta z \to 0} \frac{|\Delta z|}{\Delta z} = 1 \), we get that \( w = f(z) \) is differentiable with derivative \( f'(z) = u_x + iv_x \).

---

Theorem 5 gives the sufficient condition for the differentiability of a complex valued function at a point.

As an illustration to Theorem 5, consider the following example.

**Example 9:** Show that \( f'(z) \) exists everywhere when \( f(z) = e^z \) and find its value.

**Solution:** The given function is
\[
f(z) = e^{-z} = e^{-x}e^{iy}, \quad (z = x + iy).
\]
We can write it in the form
\[
f(z) = e^{-x}(\cos y - i\sin y).
\]
Here $u(x, y) = e^{-x} \cos y$ and $v(x, y) = -e^{-x} \sin y$.

Since we have $u_x = v_y = -e^{-x} \cos y$ and $u_y = -v_x = -e^{-x} \sin y$ everywhere and these derivatives are everywhere continuous, the conditions in Theorem 5 are satisfied at all points in the complex plane. Thus, $f'(z)$ exists everywhere and

$$f'(z) = -e^{-x} \cos y + i e^{-x} \sin y$$

$$= -e^{-z} (\cos y - i \sin y)$$

$$= -e^{-z} = -f(z).$$

You may now try the following exercises.

---

**E13)** Determine where $f'(z)$ exists when $f(z) = z \text{Im} z$ and find its value.

**E14)** Show that the function $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$ is differentiable only at points that lie on the coordinate axes.

**E15)** Let $f(z) = (\ln r)^2 - \theta^2 + i2\theta (\ln r)$ where $r > 0$ and $-\pi < \theta \leq \pi$. Show that $f$ is differentiable for $r > 0, -\pi < \theta < \pi$ and find $f'(z)$.

---

In the next section we shall study about a class of functions, called analytic functions, which are differentiable not only at a single point but have derivatives at all the points in a neighbourhood of that point.

### 2.4 Analyticity

Let us now discuss the concept of analyticity of the complex valued functions. We start with the following definitions:

**Definition 2:** A complex valued function $f$ is said to be analytic (or holomorphic) at a point $z_0$ if $f$ is differentiable at $z_0$ and at every point in some neighbourhood of $z_0$.

**Definition 3:** A complex valued function $f$ is said to be analytic in an open set $U$ if it is differentiable at every point of $U$.

From the above definitions, it is clear that a function $f$ is analytic at a point $z_0$ if it is analytic on some open set containing the point $z_0$ or you can say, there exists a positive real number $r$ such that $f$ is analytic on the open disc $D(z_0, r)$. Accordingly, $f$ is analytic on a set of complex numbers if it is analytic on an open set containing that set.

You may notice here that analyticity at a point is not the same as differentiability at that point. It is the differentiability at all points in some $\varepsilon$-nbd of that point.
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We may thus say that being analytic is stronger than being differentiable as illustrated in the example given below.

Example 10: Show that the function \( f(z) = |z|^2 \) is differentiable at \( z = 0 \) but it is nowhere analytic.

Solution: We see that
\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^2}{h} = \lim_{h \to 0} \frac{h\overline{h}}{h} = \lim_{h \to 0} \overline{h} = 0 = f'(0) \quad (\because h \to 0 \Rightarrow \overline{h} \to 0)
\]

Thus, \( f \) is differentiable at \( z = 0 \).

Now putting \( z = x + iy \) and \( f(z) = u(x, y) + iv(x, y) \) where \( u(x, y) = x^2 + y^2, v(x, y) = 0 \), we find that
\[
\frac{\partial}{\partial x} u(x, y) = 2x, \quad \frac{\partial}{\partial y} u(x, y) = 2y, \quad \frac{\partial}{\partial x} v(x, y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y} v(x, y) = 0.
\]

Clearly, the Cauchy-Riemann equations are satisfied only at \( z = 0 \). That means function is nowhere differentiable excepts at \( z = 0 \). It cannot therefore be analytic at any point.

***

We now take up some more examples.

Example 11: Show that the function \( f(z) = x^2 + y^2 + 2i xy \) is nowhere analytic.

Solution: We identify the functions \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 2xy \). The equation \( u_x = v_y \) becomes \( 2x = 2x \), which holds everywhere. But the equation \( u_y = -v_x \) becomes \( 2y = -2y \), which holds only when \( y = 0 \). Thus \( f \) is differentiable only at points that lie on the real axis. However, for any point \( z = x_0 + i0 \) on the real axis and any \( \delta \)-neighbourhood of \( z_0 \), the point \( z_1 = x_0 + i\delta/2 \) is a point at which \( f \) is not differentiable (see Fig. 3). Therefore, \( f \) is not differentiable in any neighbourhod of \( z_0 \) and consequently, it is not analytic at \( z_0 \).

***

Example 12: Let \( f(z) \) be analytic in a domain \( D \), and let \( D \) be the image of \( D \) with respect to \( x\)-axis, \( (D = \{z : \overline{z} \in D \}) \) see Fig. 4. Show that \( F(z) = f(\overline{z}) \) is analytic in \( D \).

Solution: Let \( \alpha \) be an arbitrary point of \( D \). We have to show that \( F'(\alpha) \) exists. If \( \alpha = b, \overline{\alpha} = \omega \) then
\[
\frac{F(z) - F(\alpha)}{z - \alpha} = \frac{f(\overline{z}) - f(\overline{b})}{\overline{z} - \overline{b}}
\]

\[
= \frac{f(\overline{z}) - f(\overline{b})}{\overline{z} - \overline{b}}
\]

\[
= \frac{f(\overline{\omega}) - f(\overline{\alpha})}{\overline{\omega} - \overline{\alpha}}.
\]
Now if \( z, a \in \overline{D} \) then \( \bar{z} = \omega \in D, \omega = b \in D \). Keeping in view the analyticity of \( f(z) \) and the fact that if \( z \to a \) in \( \overline{D} \) we have \( \omega (z) \to b (\bar{a}) \) in \( D \) (as conjugation is a continuous function).

Therefore,
\[
\lim_{z \to a} \frac{F(z) - F(a)}{z - a} = \lim_{\omega \to b} \frac{f(\omega) - f(b)}{\omega - b} = f'(b) = \overline{f'(a)}
\]
\[
\Rightarrow \; F'(a) = \overline{f'(a)}.
\]
It shows that \( F(z) \) is analytic in \( D \).

Before we proceed further, you may try the following exercises.

---

**E16** Find the points of the complex plane at which the following functions are analytic

i) \( f(z) = 2xy + i(x^2 - y^2) \)

ii) \( f(z) = \frac{1}{z} \).

**E17** Verify that the function

\( g(z) = \ln r + i \theta \quad (r > 0, 0 < \theta < 2\pi) \)

is analytic in the indicated domain of definition, with derivative \( g'(z) = \frac{1}{z} \).

Then show that the composite function \( G(z) = g(z^2 + 1) \) is analytic in the quadrant \( x > 0, y > 0 \), with derivative \( G'(z) = \frac{2z}{z^2 + 1} \).

---

There is a class of functions which are analytic at every point in the finite complex plane and are defined as follows:

**Definition 4:** A function differentiable/analytic everywhere in the finite complex plane is called an **entire function**.

For instance, **every complex polynomial is an entire function** as the derivative of a polynomial exists everywhere. Also, you can see that elementary **trigonometric functions** \( \sin z, \cos z \), **exponential function** \( e^z \) are differentiable at every point of the complex plane and hence **are entire functions**. Following are examples of some more entire functions.

**Example 13:** Verify that each of these functions is entire

i) \( f(z) = 3x + y + i(3y - x) \)

ii) \( f(z) = e^{-\gamma} \sin x - ie^{-\gamma} \cos x \).

**Solution:** Here we will use Theorem 5, the sufficient condition of differentiability, to show that the given functions are entire. If we put function \( f(z) \) in the form \( u(x, y) + iv(x, y) \) then we have

i) \( u(x, y) = 3x + y \) and \( v(x, y) = 3y - x \). Now the partial derivatives are given by
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\[ u(x, y) = 3, \quad v(x, y) = 1, \quad u_x(x, y) = -1 \quad \text{and} \quad v_y(x, y) = 3. \]

It can be seen that all the partial derivatives exist and are continuous (in fact they are constant) at every point of the complex plane and they satisfy the C-R equations \( u_x = v_y \) and \( u_y = -v_x \). Therefore by Theorem 5, \( f \) is differentiable at every point of the complex plane with derivative \( f'(z) = u_x + iv_x = 3 - i \).

ii) \( u(x, y) = e^{-y} \sin x \) and \( v(x, y) = -e^{-y} \cos x \). You can easily compute the partial derivatives

\[ u_x(x, y) = e^{-y} \cos x, \quad u_y(x, y) = -e^{-y} \sin x \]

and \( v_x(x, y) = e^{-y} \sin x, \quad v_y(x, y) = e^{-y} \cos x \).

Since \( e^{-y}, \sin x \) and \( \cos x \) are defined for all real numbers \( x \) and \( y \) and are real differentiable everywhere, these partial derivatives are defined and continuous at every point of the complex plane. Moreover, you may observe that \( u_x(x, y) = v_y(x, y), u_y(x, y) = -v_x(x, y) \) for all \( (x, y) \). Thus they satisfy the C-R equations everywhere. By Theorem 5, \( f(z) \) is differentiable at every point of the complex plane with derivative \( f'(z) = e^{-y} \cos x + ie^{-y} \sin x \).

Alternatively, we have

i) \( f(z) = 3x + y + i(3y - x) \)

\[ = 3(x + iy) - i(x + iy) \]

\[ = 3z - iz = (3 - i)z \]

Obviously, \( f \) is analytic in the entire complex plane with derivative \( f'(z) = 3 - i \).

ii) \( f(z) = e^{-y} \sin x - ie^{-y} \cos x = e^{-y} (\sin x - i \cos x) \)

\[ = -ie^{-y} (\cos x + i \sin x) \]

\[ = -ie^{-y} e^{ix} \]

\[ = -ie^{i(x+iy)} = -ie^{iz}. \]

Again, \( f \) is analytic in the entire complex plane with derivative \( f'(z) = e^{iz} = e^{-y} \cos x + ie^{-y} \sin x \).

***

Let us consider the two functions \( f(z) = |z|^2 \) and \( f(z) = \frac{1}{z} \). From Example 10 you know that \( f(z) = |z|^2 \) is nowhere analytic where as, \( f(z) = \frac{1}{z} \) is analytic at every point in the finite plane except at the point \( z = 0 \), i.e. at the origin (ref. E16 ii)). Such a point is called a singular point, or singularity, of \( f(z) = \frac{1}{z} \).

Formally, we give the following definition.

**Definition 5:** A point \( z_0 \) is called a **singular point** (or singularity) of a function \( f \) if \( f \) is analytic at some point in every neighbourhood of \( z_0 \) but fails to be analytic at \( z_0 \).
We can thus say that \( z_0 = 0 \) is a singular point of the function \( f(z) = \frac{1}{z} \) whereas, function \( f(z) = |z|^2 \) has no singular point since it is nowhere analytic. There exists no neighbourhood (about any point of \( \mathbb{C} \)) which contains a point at which the function \( f(z) = |z|^2 \) is analytic, so the function has no singularity in \( \mathbb{C} \). We shall be discussing about the singularities of complex valued functions in detail in Unit 7.

Before moving further, you may recall a result from your real analysis course about the constancy of a differentiable real valued function. Vanishing of the first derivative of the function throughout the domain of definition implies the constancy of the function. We now prove a proposition giving a similar result for complex valued functions. But before that let us consider a lemma regarding real-valued functions on a domain of complex plane. The result proved in the lemma will be useful for further discussion.

**Lemma 1:** Suppose \( u(x, y) \) has partial derivatives \( u_x \) and \( u_y \) that vanish at every point of the domain \( D \). Then \( u \) is a constant throughout \( D \).

**Proof:** Proof of the lemma requires you to remember the following two results.

1. Any two points in the domain can be joined by a polygonal line consisting of finite number of line segments joined end to end lying entirely in \( D \).
2. The mean value theorem of two variable calculus.

For the sake of simplicity (there is no loss of generality) we consider two points \((a, b)\) and \((c, d)\) in the domain joined by a polygonal path consisting of one horizontal (parallel to \( x \)-axis) and one vertical (parallel to \( y \)-axis) line segments as shown in Fig. 5.

![Fig. 5](image)

We are given that the partial derivatives \( u_x, u_y \) are everywhere vanishing. Now considering the line segment connecting the points \((a, b)\) and \((c, b)\) and using the mean value theorem (MVT), we have

\[
\frac{u(c, b) - u(a, b)}{c - a} = u_x(\alpha, b) = 0 \quad (a < \alpha < c). \tag{24}
\]

It then follows from Eqn. (24) that

\[
u(a, b) = u(c, b) \tag{25}
\]
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Similarly, considering the line segment connecting the points \((c, b)\) and \((c, d)\) and applying MVT we observe that
\[
\frac{u(c, d) - u(c, b)}{d - b} = u_y(c, \beta) = 0 \quad (b < \beta < d).
\] (26)

which gives
\[
u(c, b) = u(c, d).
\] (27)

From Eqns. (25) and (27), we then have
\[u(a, b) = u(c, d)\]
i.e., the value of \(u\) at the points \((a, b)\) and \((c, d)\) remains same.

Since \((a, b)\) and \((c, d)\) are arbitrarily chosen points in the domain \(D\) we can say that the value of \(u\) at every point of \(D\) remains same. That is, \(u\) is constant throughout \(D\).

We now consider the following proposition:

**Proposition 1:** Suppose that \(f\) is an analytic function in a domain \(D\). Then \(f\) will be a constant function if any of the following conditions are satisfied in \(D\):

(i) \(f'(z) = 0\) \((\forall \ z \in D);\)

(ii) \(|f(z)|\) is constant in \(D;\)

(iii) \(f(z)\) is real for all \(z \in D;\)

(iv) \(\arg f(z)\) is constant in \(D;\)

(v) \(Re f\) is constant in \(D;\)

(vi) \(Im f\) is constant in \(D;\)

**Proof:** Let \(f(z) = u(x, y) + iv(x, y)\) where \(u\) and \(v\) are real valued functions of the variables \(x\) and \(y\).

(i) If \(f'(z) = 0\) for all \(z \in D\) then clearly all partial derivatives \(u_x, u_y, v_x\) and \(v_y\) vanish at every point of the domain \(D\). By Lemma 1, \(u\) and \(v\) are constant and hence \(f\) is a constant.

(ii) If \(|f(z)| = 0\) then obviously \(f(z)\) is identically zero and hence a constant.

Suppose \(|f(z)| = |u + iv| = c \neq 0\).

Then
\[
u^2 + v^2 = c^2.
\] (28)

Now differentiating Eqn. (28) partially with respect to \(x\) and \(y\), we get
\[
u u_x + v v_x = 0 \quad \text{and} \quad u u_y + v v_y = 0.
\]

Solving the above equations and using the C-R equations, we obtain
\[u_x = 0, v_x = 0, u_y = 0 \quad \text{and} \quad v_y = 0.
\]

By Lemma 1, \(u\) and \(v\) are constant which implies that function \(f\) is a constant.

(iii) As \(f(z)\) is real for all \(z \in D, v_x = 0 = v_y\). Since the C-R equations are satisfied we get, \(u_x = 0 = u_y\). As the partial derivatives \(u_x, u_y, v_x, v_y\) are zero, we conclude that \(f\) is a constant.
(iv) We are given that \( \tan^{-1} \frac{v}{u} = c \) (a constant) which implies \( v = u \tan c \).

Differentiating \( v \) with respect to \( x \) and \( y \), we obtain
\[
v_x = u_x \tan c, \quad v_y = u_y \tan c.
\] (29)

Using the C-R equations, Eqns. (29) reduces to
\[
-u_y = u_x \tan c, \quad u_x = u_y \tan c.
\] (30)

Solving Eqns. (30), we get \( u_x = 0 = u_y \) and from Eqns. (29), we get \( v_x = 0 = v_y \) and thus \( f \) is a constant.

(v) Given that \( u = c \) (a constant). This implies that \( u_x = 0 = u_y \) and using the C-R equations, we get that \( v_x = 0 = v_y \). We conclude that \( f \) is a constant.

(vi) Proceed as in (v) above (interchange \( u \) and \( v \)).

Remark 6: You may note here that the connectedness of \( D \) is essential for Proposition 1 to hold. It will not hold true if the domain \( D \) is merely an open set. For example, consider an open set \( U \) which is the union of two disjoint open discs \( D_1 = \{ z : |z| + 2|z| < 1 \} \) and \( D_2 = \{ z : |z| - 2|z| < 1 \} \). Define a function \( f \) as
\[
f(z) = \begin{cases} 1 & \text{if } z \in D_1 \\ -1 & \text{if } z \in D_2 \end{cases}
\]

Obviously, function \( f \) is analytic in \( U \). It satisfies all the conditions listed in Proposition 1 but it is not a constant function in \( U \). However, the function is constant on each connected subset of \( D \).

You may now test your understanding of the concepts discussed above while doing the following exercises:

---

E18) State why a composition of two entire functions is entire. Also state why any linear combination \( c_1 f_1(z) + c_2 f_2(z) \) of two entire functions, where \( c_1 \) and \( c_2 \) are complex constants, is entire.

E19) Determine the singular points of the function \( f(z) = \frac{z^2 + 1}{(z+2)(z^2 + 2z + 2)} \) and state why the function is analytic everywhere except at those points.

E20) If \( f'(z) = 0 \) everywhere in a disc \( D = \{ z : |z| \leq R \} \), then \( f(z) \) must be constant throughout \( D \).

E21) If \( f(z) \) and \( g(z) = \overline{f(z)} \) are analytic functions in a domain \( D \) then show that \( f(z) \) is constant in \( D \).

---

Remark 7: A function \( f \) such that \( \overline{f} \) is analytic is called anti-analytic. Thus from E21) above, you may observe that a function which is analytic as well as anti-analytic has to be a constant function.
In the next section, we deal with harmonic functions which are real-valued functions of two real variables which satisfy the Laplace’s equation. Laplace equation is named after a French mathematician, Pierre, Simon de Laplace (1749-1827) who first studied its properties. This equation is of fundamental importance in the mathematical modelling of 2-dimensional physical problems concerning fluid flow, steady state heat conduction, electrostatics and other phenomenon.

### 2.5 HARMONIC FUNCTIONS

We begin with a formal definition of harmonic function.

**Definition 6:** A real valued function \( H \) of two real variables, defined on a domain \( D \) of \( xy \)-plane that is,

\[
H : D \subset \mathbb{R}^2 \to \mathbb{R},
\]

is called a **harmonic function** if the first and second order partial derivatives of \( H \) with respect to the variables \( x \) and \( y \) exists and are continuous everywhere in \( D \) (that is, \( H \) is a \( C^2 \)-function) and these second order partial derivatives satisfy the differential equation

\[
\nabla^2 H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0, \quad (x, y) \in D,
\]

known as **Laplace’s equation**. Here \( \nabla^2 \) is the second order differential operator called the **Laplacian**.

Harmonic functions occur widely in applied mathematics and are solutions to many physical problems. They occur in the study of temperature in thin plates in the \( xy \)-plane and to study variations in an electrostatic potential in the interior of a region in three-dimensional space. We shall be discussing some of these applications later in Unit 10.

We now take up some examples of harmonic functions.

**Example 14:** Show that the function \( H : \mathbb{R}^2 \to \mathbb{R} \) given by \( H(x, y) = 2xy \) is harmonic.

**Solution:** Here \( H \) is a polynomial in two variables therefore all its partial derivatives of all order exists and are continuous throughout the \( xy \)-plane. Moreover,

\[
\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 + 0 = 0.
\]

i.e., \( H \) satisfies the Laplace’s equation also and hence it is harmonic. It is one of the simplest example of non-constant harmonic function defined on the whole plane.

***

**Example 15:** Show that the functions \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \) are harmonic.

**Solution:** You may easily check that first and second order partial derivatives of \( u(x, y) \) and \( v(x, y) \), with respect to the variables \( x \) and \( y \), exists and are continuous. You may also recall from your knowledge of undergraduate real
analysis course that the real valued functions \( u(x, y) \) and \( v(x, y) \) are \( C^2 \)-functions. In fact their partial derivatives of all order exists.

Further, observe that

\[
\frac{\partial^2 u(x, y)}{\partial x^2} = e^x \cos y \quad \text{and} \quad \frac{\partial^2 u(x, y)}{\partial y^2} = -e^x \cos y
\]

which implies that \( \nabla^2 u = 0 \). Similarly, you can see that \( \nabla^2 v = 0 \).

Thus the functions \( u(x, y) \) and \( v(x, y) \) are harmonic.

\*

You may note that in Example 15, the real-valued functions \( u(x, y) \) and \( v(x, y) \) are the component functions of the exponential function \( e^z = e^x (\cos y + i \sin y) \) which is an entire function. In fact, Example 15 serves as an illustration to a theorem which we are stating now.

**Theorem 6:** If a function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \), then its component functions \( u \) and \( v \) are harmonic in \( D \).

**Proof:** To prove this theorem we need a result namely, if a function of complex variable is analytic at a point, then its derivatives of all order exist at that point and are, all analytic there. We shall be proving the result later in Unit 5 (see Theorem 5, Sec. 5.3).

Let us assume that \( f \) is analytic in \( D \). Then the real and imaginary components of \( f \) satisfy the Cauchy-Riemann equations

\[
u_x = v_y, \quad u_y = -v_x \quad (31)
\]

throughout \( D \). Differentiating both sides of Eqns. (31) with respect to \( x \), we have

\[
u_{xx} = v_{yx}, \quad u_{xx} = -v_{xx}. \quad (32)
\]

Similarly, differentiating both sides of Eqn. (31) with respect to \( y \), we get

\[
u_{xy} = v_{yy}, \quad u_{xy} = -v_{xy}. \quad (33)
\]

By a theorem in advanced calculus course, which you would have studied at your undergraduate level, the continuity of the partial derivatives of \( u \) and \( v \) gives

\[
u_{yx} = u_{xy} \quad \text{and} \quad v_{xy} = v_{yx}. \]

Using them in Eqns. (32) and (33), we obtain

\[
u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.
\]

Thus, \( u \) and \( v \) are harmonic in \( D \).

\*

As an illustration to the theorem, consider the function \( f : D \to \mathbb{C} \) defined as

\[f(z) = \frac{1}{z}.
\]

You have already seen in E 16 ii) that the function \( f \) is an analytic function in the domain \( D = \mathbb{C} \setminus \{0\} \). You can easily check that its real and imaginary parts \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \), respectively are harmonic in the \( xy \)-plane except at the origin.
E22) Show that the functions \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \) are harmonic in the \( xy \)-plane except at the origin.

You may observe here that we cannot choose two arbitrary harmonic functions \( u \) and \( v \) and claim that the resulting function \( f = u + iv \) is analytic. For instance, \( u(x, y) = x \) and \( v(x, y) = -y \) are harmonic functions in \( \mathbb{C} \), but \( f = u + iv = x - iy = \bar{z} \) is nowhere analytic. Infact, it is nowhere differentiable (ref. Example 2). Similarly, consider the function \( f = e^\imath (\sin y + i \cos y) \), where we have interchanged the real and imaginary components of the complex exponential function \( e^\imath z \). You have seen in Example 15 that both the component functions are harmonic but the function is not analytic. You can check that C-R equations are nowhere satisfied.

If two harmonic functions \( u(x, y) \) and \( v(x, y) \) in a domain \( D \) satisfy the C-R equations throughout \( D \) then \( v(x, y) \) is called the harmonic conjugate function of \( u(x, y) \). For instance, in Example 15, \( v(x, y) = e^x \sin y \) is the harmonic conjugate of \( u(x, y) = e^x \cos y \).

Note that the word conjugate here is not the same as conjugate of a complex number \( z \). Further, the harmonic conjugate \( v(x, y) \) of \( u(x, y) \) is unique and is determined up to an additive constant. But then the question is how to find the harmonic conjugate of a given function?

Before we give you a method of finding the conjugate harmonic function of a given function consider the following theorem which gives a useful result about conjugate harmonic functions.

**Theorem 7:** A function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if and only if \( v \) is harmonic conjugate of \( u \).

**Proof:** Let us first assume that \( v \) is the harmonic conjugate of \( u \) in \( D \). Then by Theorem 5, \( f \) is analytic in \( D \). Conversely, if \( f \) is analytic in \( D \) then we know from Theorem 6 above that \( u \) and \( v \) are harmonic in \( D \) and by virtue of Theorem 4 they satisfy the Cauchy-Riemann equations.

This completes the proof of the theorem.

We now take up a few examples and illustrate the method of finding the conjugate harmonic function of a given function.

**Example 16:** If the function \( u(x, y) = x^2 - y^2 \) is harmonic find its harmonic conjugate.

**Solution:** We have \( \frac{\partial u}{\partial x} = 2x \) and \( \frac{\partial u}{\partial y} = -2y \). Then from the C-R equations we know that the conjugate function \( v(x, y) \) satisfies

\[
\frac{\partial v}{\partial x} = 2y, \quad \text{and} \quad \frac{\partial v}{\partial y} = 2x
\]
Integrating the first equation of (34) with respect to $x$, keeping $y$ fixed, we get
\[ v = 2xy + g(y), \text{ where } g(y) \text{ is a function of } y \text{ only.} \] (35)
Differentiating Eqn. (35) with respect to $y$ and substituting the value of $\frac{\partial v}{\partial y}$ in the second equation of Eqns. (34), we get $g'(y) = 0$. Thus $g(y)$ is a constant and the conjugate function $v$ is $v = 2xy + c$, where $c$ is a constant of integration.

***

You may note that in Example 16, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are the real and imaginary components, respectively of the entire function $f(z) = x^2 - y^2 + i2xy = z^2$ and $v$ is harmonic conjugate of $u$ throughout the complex plane. However, in view of Theorem 7, $u$ cannot be the harmonic conjugate of $v$ since the function $f(z) = 2xy + i(x^2 - y^2)$ is not analytic anywhere (See E16 i)). Thus if $v$ is a harmonic conjugate of $u$ in some domain than it is not necessary that $u$ is a harmonic conjugate of $v$ there.

**Example 17:** Find the conjugate harmonic function $v(x, y)$ corresponding to $u(x, y) = x^2 - y^2 + x$ on the domain $|z| < \infty$ (whole complex plane).

**Solution:** Obviously $u(x, y)$ being a polynomial in variables $x$ and $y$ is continuously differentiable upto any order (remember we need only upto order 2 and clearly here partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ exist and are continuous on $|z| < \infty$). Also we have
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0. \]
Thus $u(x, y)$ is harmonic function on domain $|z| < \infty$. If $v(x, y)$ is conjugate to $u(x, y)$ then they satisfy the C-R equations and we have
\[ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y. \] (36)
Integrating the first equation of Eqns. (36) with respect to $y$, treating $x$ as a constant, we obtain
\[ v = 2xy + y + c(x) \] (37)
where $c(x)$ is an arbitrary function of $x$ only.
Differentiating Eqn. (37) with respect $x$ and using the second equation of Eqns. (36), we get
\[ \frac{\partial v}{\partial x} = 2y = 2y + c'(x) \Rightarrow c'(x) = 0 \]
which gives $c(x) = c$, where $c$ is an arbitrary complex constant. Therefore, from Eqn. (37) the conjugate harmonic function is given by
\[ v(x, y) = 2xy + y + c = y(2x + 1) + c \]

***

Summing up the discussion above, we say that if a real valued function $v(x, y)$ is the harmonic conjugate of function $u(x, y)$ in a given domain then

(i) both $u(x, y)$ and $v(x, y)$ satisfy the Laplace’s equation.
(ii) \( u(x, y) \) and \( v(x, y) \) satisfy the C-R equations, \( u_x = v_y, u_y = -v_x \) in the domain.

(iii) \( f(z) = u(x, y) + iv(x, y) \) is analytic in the domain.

You may now try the following exercises.

E23) Find the harmonic conjugate of the function \( u(x, y) = e^x \cos y \).

E24) Show that if \( v \) and \( V \) are harmonic conjugates of \( u \) in a domain \( D \), then \( v(x, y) \) and \( V(x, y) \) can differ at most by an additive constant.

E25) Suppose that \( v \) is the harmonic conjugate of \( u \) in a domain \( D \) and also that \( u \) is the harmonic conjugate of \( v \) in \( D \). Show that both \( u(x, y) \) and \( v(x, y) \) must be constant throughout \( D \).

E26) Show that \( u(x, y) \) is harmonic in some domain and find the harmonic conjugate \( v(x, y) \) when

i) \( u(x, y) = \sinh x \sin y \)

ii) \( u(x, y) = \frac{y}{x^2 + y^2} \).

E27) Let the function \( f(z) = u(r, \theta) + iv(r, \theta) \) be analytic in a domain \( D \) that does not include the origin. Using the Cauchy-Riemann equations in polar form and assuming the continuity of the partial derivatives, show that throughout \( D \), the function \( u(r, \theta) \) satisfies the partial differential equation

\[
r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0
\]

which is the polar form of the Laplace’s equation. Further, show that the same is true for the function \( v(r, \theta) \).

We now end this unit by giving a summary of what we have covered in it.

### 2.6 SUMMARY

In this unit we have covered the following:

1) Differentiation of complex valued functions of complex variables.

2) The Cauchy-Riemann equations are the pair of partial differential equations

\[
\begin{align*}
u_x(x_0, y_0) &= v_y(x_0, y_0) \\
u_y(x_0, y_0) &= -v_x(x_0, y_0)
\end{align*}
\]

which the first-order partial derivatives of the component functions \( u \) and \( v \) of a function \( f(z) = u(x, y) + iv(x, y) \) must satisfy at a point \( z_0 = (x_0, y_0) \) when the derivative of \( f \) exists there.

3) The Cauchy-Riemann equations provide the necessary conditions for the existence of the derivative of a function \( f \) at a point. But their satisfaction
at a point is not sufficient to ensure the existence of the derivatives of a function $f$ at that point.

4) For a function $f(z) = u(x, y) + iv(x, y)$ which is defined on an open set $U$, if $f$ is differentiable at a point $z_0 \in U$, then $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations at $z_0$. Conversely, if the first order partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous and satisfy the Cauchy-Riemann equations at a point $z_0 \in U$ then $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \in U$.

5) A function $f$ is said to be analytic at a point $z_0$ if it is analytic on some open set containing the point $z_0$. $f$ is said to be analytic on a subset of complex numbers if it is analytic on an open set containing that subset.

6) A real valued function $H$ of two real variables $x$ and $y$, defined on a domain $D$ of the $xy$-plane is a harmonic function if the first and the second order partial derivatives of $H$, with respect to the variables $x$ and $y$, exists and are continuous everywhere in $D$. Further, these second order partial derivatives satisfy the Laplacian

$$\Delta^2 H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0, \forall (x, y) \in D.$$ 

7) The real and the imaginary parts of an analytic function are harmonic functions.

### 2.7 SOLUTIONS/ANSWERS

E1) i) We have for $h \neq 0$

$$\frac{[f(z+h)+g(z+h)]-[f(z)+g(z)]}{h} = \frac{f(z+h)-f(z)}{h} + \frac{g(z+h)-g(z)}{h}$$

Now taking the limit as $h \to 0$, we have

$$\lim_{h \to 0} \frac{[f(z+h)+g(z+h)]-[f(z)+g(z)]}{h} = f'(z) + g'(z).$$

Therefore, the limit of the right hand side expression exists and we have

$$\frac{d}{dz} [f(z)+g(z)] = f'(z) + g'(z).$$

which proves Formula (3).

ii) Let $k(z) = \frac{f(z)}{g(z)}$. We then have

$$\frac{k(z+h) - k(z)}{h} = \frac{g(z)f(z+h) - f(z)g(z+h)}{g(z)g(z+h)h} = \frac{g(z)f(z+h) - f(z)g(z) + f(z)g(z) - f(z)g(z+h)}{g(z)g(z+h)h}$$


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\[ \frac{g(z)[f(z+h) - f(z)]}{g(z)g(z+h)h} - \frac{f(z)[g(z+h) - g(z)]}{g(z)g(z+h)h} = g(z)f'(z) - f(z)g'(z) \]

Now taking the limit \( h \to 0 \) and noting that the function \( g(z) \) being differentiable is continuous as well, thus \( g(z+h) \to g(z) \) as \( h \to 0 \), and we have

\[ \lim_{h \to 0} \frac{k(z+h) - k(z)}{h} = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \]

or \( k'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \)

or \( \frac{d}{dz} \frac{f(z)}{g(z)} = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \).

E2) Let \( f \) and \( g \) satisfy the hypothesis of Theorem 3 with \( \zeta = f(z) \) and \( w = g(\zeta) \). Then the differentiability of \( f \) at \( z \) and of \( g \) at \( f(z) \) gives

\[ \Delta w = g'(\zeta) \Delta \zeta + \epsilon_1 |\Delta \zeta|, \lim_{\Delta \zeta \to 0} \epsilon_1 = 0 \]

and

\[ \Delta \zeta = f'(z) \Delta z + \epsilon_2 |\Delta z|, \lim_{\Delta z \to 0} \epsilon_2 = 0 \].

From these two equations we deduce that

\[ \Delta w = g'(\zeta) f'(z) \Delta z + g'(\zeta) \epsilon_1 |\Delta z| + \epsilon_2 |\Delta \zeta| \]

Now dividing the above equation by \( \Delta z \) and letting \( \Delta z \to 0 \) and noting that as \( \Delta z \to 0 \) \( \Delta \zeta \to 0 \), and hence \( \epsilon_1 \to 0 \) as well as \( \epsilon_2 \to 0 \), we obtain

\[ \frac{dw}{dz} = g'(\zeta) f'(z) \Rightarrow (g \circ f)'(z) = g'(f(z)) f'(z) \).

E3) Write \( m = -n \). Then \( f(z) = z^n = -z^{-m} = \frac{1}{z^m} (m > 0 \text{ and } z \neq 0) \).

Using Eqns. (5) and (11), we have

\[ \frac{d}{dz} \left( \frac{1}{z^m} \right) = \frac{d}{dz} \left( \frac{1}{z^m} \right) \cdot \frac{z^m - 1}{d(z^m)} = \frac{0 - mz^{m-1}}{z^{2m}} = -\frac{mz^{-m-1}}{z^{2m}} \]

\[ = -nz^{-n-1} \]

\[ \Rightarrow \frac{d}{dz} (z^n) = nz^{n-1} \]

E4) i) Here \( \frac{f(z+h) - f(z)}{h} = \frac{Im(z+h) - Im(z)}{h} = \frac{Im(z) + Im(h) - Im(z)}{h} \)

\[ = \frac{Im(h)}{h} \].

Let \( h = (h_1, h_2) \). If \( h \) approaches the origin horizontally through \( (h, 0) \), we get \( Im(h) = 0 \). Thus \( \frac{f(z+h) - f(z)}{h} = 0 \). Hence, if the limit exists, its value must be zero.
However, when $h$ approaches the origin vertically through $(0, h_2) \Rightarrow Im(h) = h_2$ and $f(z + h) - f(z) = 1$. Thus $f'(z)$ does not exist.

ii) Proceed as in i) above.

E5) $f(z) = |z|$ is continuous (ref. Example 15, Unit 1).

If $h = (h_1, h_2)$, then

$$f(h) - f(0) = \frac{|h|}{h} \to \begin{cases} 1 & \text{for } h = h_1 + 0i \text{ for } h_1 \to 0 \text{ when } h_1 > 0 \\ -1 & \text{for } h = h_1 + 0i \text{ for } h_1 \to 0 \text{ when } h_1 < 0 \end{cases}$$

Thus $|z|$ is not differentiable at $z = 0$. In fact $f(z) = |z|$ is nowhere differentiable in $\mathbb{C}$.

E6) Show that $f(z) = |z|^2$ is a continuous function which is differentiable only at the origin.

E7) Here $v(x, y) = 0$, $v_x = 0 = v_y$. Also

$$u_x = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = 0.$$  

On similar lines, $u_y = 0$. Thus, the C-R equations are satisfied at $z = 0$. On the other hand, $f(z) - f(0) = \sqrt{|xy|}/(z - 0)$. Using the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ in the above equation, we get

$$f(z) - f(0) = r\sqrt{\cos \theta \sin \theta}.$$  

The right hand side of the above equation is independent of $r$ but depends on $\theta$. For $\theta = 0$ or $\theta = \frac{\pi}{2}$, the limit value is 0 but for $\theta = \frac{\pi}{4}$ the limiting value is $\frac{1-i}{2}$. Therefore, $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist. Hence $f(z)$ is not differentiable at $z = 0$.

E8) i) We have $f(z) = z + \bar{z} = 2x$. Thus $u(x, y) = 2x$ and $v(x, y) = 0$.

Therefore, $u_x(x, y) = 2$ and $v_y(x, y) = 0$.

Since $u_x \neq v_y$, the C-R equations are not satisfied at any point of $\mathbb{C}$. Thus $f$ is nowhere differentiable.

ii) $f(z) = e^y \cos x + ie^y \sin x$. Here $u(x, y) = e^y \cos x$ and $v(x, y) = e^y \sin x$. Therefore, $u_x(x, y) = -e^y \sin x$ and $v_y(x, y) = e^y \sin x$.

Thus, the C-R equations are not satisfied at any point and therefore it cannot be differentiable at any point.

Part iii) and iv) can be shown on the similar lines.
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E9) \( f(z) = x^3 + i(1 - y)^3. \)

Here \( u(x, y) = x^3, \ v(x, y) = (1 - y)^3. \)

Now the C-R equations are satisfied only if \( 3x^2 = -3(1 - y)^2 \) and \( 0 = 0 \)
which imply \( x = 0 \) and \( y = 1. \) Thus, the C-R equations are satisfied only at \( (0, 1), \) i.e., \( z = i. \) Also observe that in every neighbourhood of \( z = i, \) partial derivatives exist and are continuous. Therefore \( f(z) \) is differentiable only at \( z = i. \) We can therefore write \( f'(z) = u_x + i v_x = 3x^2 + i.0 = 3x^2. \)

E10) \( f(z) = \begin{cases} \frac{\overline{z}^2}{z^2}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases}. \)

We can write \( \frac{\overline{z}^2}{z^2} = \frac{\overline{z}^3}{z \overline{z}}. \)

Hence, \( u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}, \)

and \( v(x, y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}. \)

Now \( u_x(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{h^3}{h} = 1. \)

Likewise, \( u_y(0, 0) = 0 = v_x(0, 0) \) and \( v_y(0, 0) = 1. \)

Thus, the C-R equations are satisfied at \( (0, 0). \)

E11) i) We express \( F(x, y) \) as follows:
\( F = F[x(z, \overline{z}), y(z, \overline{z})]. \) \( x \) and \( y \) are treated as functions of \( z \) and \( \overline{z}. \)

Now differentiating \( F \) w.r.t \( \overline{z} \) and using the chain rule we get,
\( \frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}} \)
\( = \frac{1}{2} \frac{\partial F}{\partial x} - \frac{1}{2i} \frac{\partial F}{\partial y} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right). \)

ii) For a function \( f(z) = u(x, y) + iv(x, y) \) we are given that \( u_x = v_y \) and \( u_y = -v_x. \)

Now \( \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \) [by part i]
\( = \frac{1}{2} [u_x + iv_x + i(u_y + iv_y)] \)
\( = \frac{1}{2} [(u_x - v_y) + i (v_x + u_y)] \)
\( = \frac{1}{2} (0 + i.0) = 0 \) [using the C-R equations]
Thus, \( \frac{\partial f}{\partial z} = 0 \). \[ \text{complex form of the C-R equations} \]

E12) We are given that Cauchy-Riemann equations are satisfied at some point \( z_0 \neq 0 \). As \( x \) and \( y \) are functions of \( r \) and \( \theta \), using the chain rule for differentiating real valued functions of two real variables, we obtain

\[
\begin{align*}
    u_r &= u_x x_r + u_y y_r \\
    u_\theta &= u_x x_\theta + u_y y_\theta
\end{align*}
\]

As \( x_r = \cos \theta \), \( y_r = \sin \theta \), \( x_\theta = -r \sin \theta \) and \( y_\theta = r \cos \theta \), we obtain

\[
\begin{align*}
    u_r &= u_x \cos \theta + u_y \sin \theta \\
    u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\
    v_r &= v_x \cos \theta + v_y \sin \theta \\
    v_\theta &= -v_x r \sin \theta + v_y r \cos \theta.
\end{align*}
\]

As the C-R equations are satisfied, \( u_x = v_y \) and \( u_y = -v_x \) and Eqn. (39) becomes

\[
v_r = -u_x \cos \theta + u_y \sin \theta, \quad v_\theta = u_x r \sin \theta + u_y r \cos \theta
\]

at the point \( z_0 \). It now follows from Eqns. (38) and (40) that \( ru_r = v_\theta \) and \( u_\theta = -rv_r \), at the point \( z_0 \) which are the required C-R equations in polar form.

E13) \( f(z) = z \text{Im} z \).

We express \( f(z) \) as \( f(z) = (x + iy) y = xy + iy^2 \).

Thus, \( u(x, y) = xy \) and \( v(x, y) = y^2 \).

First we try to find out those points at which the C-R equations are satisfied i.e., \( u_x = v_y \) and \( u_y = -v_x \).

We get, \( y = 2y \) and \( x = 0 \Rightarrow y = 0 \) and \( x = 0 \).

Thus the C-R equations are satisfied only at \( (0, 0) \), i.e. at \( z = 0 \). We see that being the polynomial functions \( u \) and \( v \) are differentiable everywhere, i.e., there is a neighbourhood of \( z = 0 \) where the partial derivatives exist and are continuous. Thus, according to Theorem 5, \( f(z) \) is differentiable only at \( z = 0 \) with \( f'(0) = u_x(0) + iv_x(0) = 0 + i0 = 0 \).

E14) \( f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2 y) \).

Here \( u(x, y) = x^3 + 3xy^2 \) and \( v(x, y) = y^3 + 3x^2 y \).

Now \( u_x = 3x^2 + 3y^2, u_y = 6xy, v_x = 6xy \) and \( v_y = 3y^2 + 3x^2 \).

If \( f(z) \) is differentiable at \( (x, y) \) then it must satisfy the C-R equations:

\[
\begin{align*}
    u_x &= v_y \\
    u_y &= -v_x
\end{align*}
\]

\( \Rightarrow 3(x^2 + y^2) = 3(x^2 + y^2) \) [which is identically satisfied] and \( 6xy = -6xy \)

\( \Rightarrow xy = 0 \Rightarrow x = 0 \) or \( y = 0 \).

Thus, if \( f(z) \) has to be differentiable then the points of differentiability are only those points which lie along coordinate axes. Since the partial derivatives are continuous everywhere, therefore \( f(z) \) is differentiable along coordinates axes, i.e., \( f(z) \) is differentiable at points of the form \((x,0)\) or \((0,y)\) only.
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E15) We have \( u(r, \theta) = (\ln r)^2 - \theta^2 \), \( v(r, \theta) = 2\theta (\ln r) \).
Clearly, \( u \) and \( v \) are defined in the indicated domain. We compute the partial derivatives
\[
u_r = \frac{2}{r}(\ln r), \quad v_r = -2\theta, \quad v_\theta = 2(\ln r),
\]
which exist in the domain \( \{ r, \theta \} : r > 0, -\pi < \theta < \pi \} \) and are also continuous there.

Also, \( u_r = \frac{2}{r} \ln r = 2(\ln r) \), \( v_\theta = \frac{2}{r} \ln r = v_\theta \),

and \( u_\theta = -2\theta = -r \left( \frac{2\theta}{r} \right) = -r v_r \). The C-R equations are satisfied.

Hence, \( f'(z) \) exists for all \( z \) in the indicated domain and
\[
f'(z) = e^{-i\theta}(u_\theta + i v_r) = e^{-i\theta} \left( \frac{2}{r}(\ln r) + i \frac{2}{r} \theta \right) = \frac{2}{r} e^{-i\theta}(\ln r + i \theta).\]

E16) i) \( f(z) = 2xy + i (x^2 - y^2) \).

We have, \( u(x, y) = 2xy \), \( v(x, y) = x^2 - y^2 \).

Therefore, \( u_x = 2y, u_y = 2x, v_x = 2x, v_y = -2y \) and so the C-R equations are satisfied only at \((0, 0)\). Thus \( f(z) \) is nowhere analytic.

**Note** that \( f'(0) \) exists.

ii) \( f(z) = \frac{\pi}{z} = \frac{x + iy}{x^2 + y^2} \).

We have, \( u(x, y) = \frac{x}{x^2 + y^2}, v(x, y) = \frac{-y}{x^2 + y^2} \).

Therefore, \( u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, u_y = \frac{-2xy}{(x^2 + y^2)^2} \)
\[
v_x = \frac{-2xy}{(x^2 + y^2)^2}, v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

The C-R equations are satisfied everywhere except at the origin. Also the first order partial derivatives exist and are continuous (polynomial functions) at all points except at the origin.

Function \( f \) is analytic in the domain \( D = \mathbb{C} \setminus \{0\} \).

E17) We have \( g(z) = \ln r + i\theta \) \((r > 0, 0 < \theta < 2\pi) \) \(( z \neq 0) \). Component functions are: \( u(r, \theta) = \ln r \) and \( v(r, \theta) = \theta \).

We have,
\[
u_r = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1.
\]

Observe that the partial derivatives exists in the indicated domain and are continuous there. The C-R equations are also satisfied as
\[
r u_r = 1 = v_\theta \quad \text{and} \quad u_\theta = 0 = -r v_r.
\]

Hence, \( g(z) \) is analytic in the indicated domain.

Again,
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\[ g'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i.0\right) \]

\[ = e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}. \]

Applying the chain rule and taking \( g'(z) = \frac{1}{z} \), we get

\[ G'(z) = g'(z^2 + 1). 2z = \frac{1}{z^2 + 1}. 2z = \frac{2z}{z^2 + 1}. \]

E18) Let \( f(z) \) and \( g(z) \) be any two entire functions. Then by definitions
\[
\frac{d}{dz} f(z) \text{ and } \frac{d}{dz} g(z) \text{ exists everywhere in the complex plane. Clearly, } \\
\ f \circ g \text{ and } g \circ f \text{ are defined in the plane and } (f \circ g)(z) = f(g(z)) \ \forall z \text{ and } \\
\ (g \circ f)(z) = g(f(z)) \ \forall z. \\
\text{Then by the chain rule:} \\
\frac{d}{dz} (gof)(z) = \frac{d}{dz} g(f(z)) = \frac{d}{dz} g(u) \\
\quad = \frac{du}{dz} = \frac{d}{zu} g(u) \\
\quad = g'(f(z)) \frac{d}{dz} (f(z)) \\
\quad = g'(f(z)). f'(z) \ \forall z. \\
\Rightarrow g \circ f \text{ is analytic everywhere. You may complete the remaining part yourself.}
\]

E19) \( f(z) = \frac{z^2 + 1}{(z+2)(z^2 + 2z + 2)}. \)

Singular points are the solutions of the equation:
\( (z+2)(z^2 + 2z + 2) = 0. \)
\( \Rightarrow z = -2, -1 \pm i. \)

Since \( f(z) = \frac{P(z)}{Q(z)} \), where \( P(z) \) and \( Q(z) \) are polynomials we know from

quotient rule of differentiations that \( f'(z) \) exists everywhere except at
\( z = -2, -1 \pm i \) because the function is not defined at these points.

E20) If \( f(z) = u(x, y) + i v(x, y) \) then \( f'(z) = 0 \) implies
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0, \text{ at every point of } D. \quad \text{[using the C-R equations]} \\
\]

Let \( p = a + ib \) and \( q = c + id \) be points in \( D \) (see Fig. 6). Then at least one
of \( p_1 = a + id \) and \( q_1 = c + ib \) lies in \( D \) because
\( p, q \in D \Rightarrow a^2 + b^2 < R^2 \), \( c^2 + d^2 < R^2 \)
\( \Rightarrow a^2 + b^2 + c^2 + d^2 < 2R^2 \)
\( \Rightarrow (a^2 + d^2) + (c^2 + b^2) < 2R^2 \)
\( \Rightarrow \text{either } a^2 + d^2 < R^2 \text{ or, } c^2 + b^2 < R^2 \) (at least)
\( \Rightarrow \text{either } p_1 \in D \text{ or, } q_1 \in D. \)
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Suppose \( p \in D \), then \( f(x) = u(x, d) \) and \( g(x) = u(a, y) \) are real functions with zero derivative and so they are constant. Thus 
\( u(a, d) = u(a, b) = u(c, d) \) and similarly \( v(a, b) = v(c, d) \). Thus \( f(z) \) is constant \([u(a, b) = u(p), u(c, d) = u(q)]\).

E21) We are given that \( f = u + iv \) is analytic. Therefore by the C-R equations, we get \( u_x = v_y \) and \( u_y = -v_x \). Also \( g = \bar{f} = v + iv = u - iv \) is analytic. Therefore by the C-R equations for \( g \), we get \( u_x = -v_y \) and \( u_y = v_x \). Using these equations we get \( 2u_x = 0 \) and \( 2u_y = 0 \), which implies that \( u \) is independent of \( x \) and \( y \). Thus \( u \) is a constant. Similarly, show that \( v \) is a constant and hence \( f = u + iv \) is a constant.

E22) \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \) are real valued functions of two real variables \( x \) and \( y \) defined on the whole \( xy \)-plane except the origin. Their partial derivatives of all order exist in the \( xy \)-plane except at the origin. Also
\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^3} + \frac{2y(y^2 - x^2)}{(x^2 + y^2)^3} = 0.
\]
Similarly, show that \( v_{xx} + v_{yy} = 0 \). Thus, functions \( u(x, y) \) and \( v(x, y) \) are harmonic in the \( xy \)-plane except at the origin.

E23) From Example 15, \( u(x, y) \) is harmonic. \( \frac{\partial u}{\partial x} = e^x \cos y \) and \( \frac{\partial u}{\partial y} = -e^x \sin y \). The conjugate function satisfies
\[
\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y.
\]
Integrating first equation we get \( v = e^x \sin y + \phi(y) \), where \( \phi(y) \) is a function of \( y \) alone. Substituting this value of \( v \) in the second equation, we get \( \phi'(y) = 0 \). Thus \( \phi(y) \) is a constant, and the conjugate function of \( u \) is \( v = e^x \sin y + c \), where \( c \) is a constant.

E24) Since \( v \) and \( V \), if exist, are harmonic conjugate of \( u \) in a domain \( D \). \( f(z) = u + iv \) and \( f_1(z) = u + iV \) are analytic functions in \( D \). Now by the C-R equations:
\[
\begin{align*}
    u_x &= v_y, \quad u_y = -v_x \quad \text{and} \quad u_x = V_y, \quad u_y = -V_x, \\
    v_y &= V_y \quad \text{and} \quad v_x = V_x, \\
    (v-V)_y = 0 \quad \text{and} \quad (v-V)_x = 0 \\
    v-V &= \text{const}.
\end{align*}
\]
i.e. \( v = V + C \), where \( C \) is a constant.

E25) Since \( v \) is harmonic conjugate of \( u \), we have from the C-R equations,
\[
\begin{align*}
    u_x(x, y) &= v_x(x, y), \quad u_x(x, y) = -v_y(x, y), \quad (x, y) \in D. \quad (41)
\end{align*}
\]
Since \( u \) is harmonic conjugate of \( v \), we have,
From Eqns. (41) and (42), we obtain that \( u_x(x, y) = 0, u_y(x, y) = 0 \), \( v_x(x, y) = 0 \) and \( v_y(x, y) = 0 \) in \( D \). Then by Lemma 1 \( u(x, y) \) and \( v(x, y) \) are constant in \( D \). (you may note that \( D \) is a domain, an open and connected set).

E26) i) We know that \( \sinh x = \frac{e^x - e^{-x}}{2} \) and \( \cosh x = \frac{e^x + e^{-x}}{2} \). Therefore,
\[
\frac{d}{dx} \sinh x = \cosh x \quad \text{and} \quad \frac{d}{dx} \cosh x = \sinh x \quad \text{for all} \quad x \in \mathbb{R}.
\]
Also, we have
\[
\frac{\partial u}{\partial x} = u_x(x, y) = \cosh x \sin y, \quad \frac{\partial^2 u}{\partial x^2} = u_{xx}(x, y) = \sinh x \sin y,
\]
and
\[
\frac{\partial u}{\partial y} = u_y(x, y) = \sinh x \cos y, \quad \frac{\partial^2 u}{\partial y^2} = u_{yy}(x, y) = -\sinh x \sin y.
\]
Thus \( u_{xx}(x, y) + u_{yy}(x, y) = 0, (x, y) \in \mathbb{R}^2 \) which implies that \( u(x, y) \) is a harmonic function in a domain which is the whole complex plane. If \( v(x, y) \) is the harmonic conjugate of \( u(x, y) \) then they must satisfy the C-R equations
\[
u_x(x, y) = \cosh x \sin y = v_y(x, y) \quad (43) \]
\[
u_y(x, y) = \sinh x \cos y = -v_x(x, y). \quad (44)
\]
Now integrating Eqn. (43) with respect to \( y \), treating \( x \) constant, we get
\[
\nu(x, y) = -\cosh x \cos y + c(x), \quad (45)
\]
where \( c(x) \) is function of \( x \) only.
Differentiating Eqn. (45) partially with respect to \( x \) and then comparing with Eqn. (44), we get \( c'(x) = 0 \) which implies that \( c(x) = c \) (an arbitrary complex constant).
Thus, the harmonic conjugate of \( u(x, y) \) is
\[
\nu(x, y) = -\cosh x \cos y + c.
\]
In particular, if we choose \( c = 0 \) then
\[
\nu(x, y) = -\cosh x \cos y \quad \text{is the harmonic conjugate of} \quad u(x, y).
\]

ii) It can be seen easily that domain in this case is the punctured complex plane \( D = \mathbb{C} \setminus \{0\} \). Now to check whether \( u(x, y) \) is harmonic we compute
\[
u_x(x, y) = \frac{-2xy}{(x^2 + y^2)^2}, \quad \nu_{xx}(x, y) = \frac{-2y(y^2 - 3x^2)}{(x^2 + y^2)^3},
\]
and
\[
u_y(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad \nu_{yy}(x, y) = \frac{-2y(3x^2 - y^2)}{(x^2 + y^2)^3}.
\]
These expressions imply that \( u_{xx}(x, y) + u_{yy}(x, y) = 0 \) for all \( (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) and therefore \( u(x, y) \) is harmonic in the punctured complex plane \( D \). From the C-R equations, we obtain
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\[ v_y(x, y) = \frac{-2xy}{(x^2 + y^2)^2} \quad (46) \]
\[ v_x(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad (47) \]

Integrating Eqn. (46) with respect to \( y \), we get

\[ v(x, y) = \frac{x}{x^2 + y^2} + c(x) \text{ where } c(x) \text{ is function of } x \text{ only.} \quad (48) \]

Differentiating Eqn. (48) partially with respect to \( x \) and comparing with Eqn. (47), we get \( c'(x) = 0 \) which after integration gives \( c(x) = c \) where \( c \) is an arbitrary complex constant. Thus, the harmonic conjugate of \( u(x, y) \) is

\[ v(x, y) = \frac{x}{x^2 + y^2} + c. \]

If we take \( c = 0 \), we get \( \frac{x}{x^2 + y^2} \) as the harmonic conjugate of \( u(x, y) \).

27) We have the C-R equations in polar form as:

\[ r \ u_r = v_\theta \quad (49) \]
\[ u_\theta = -r \ v_r. \quad (50) \]

Differentiating Eqn. (49) w.r.t. \( r \), we get

\[ u_r + r \ u_{rr} = v_{\theta r}. \quad (51) \]

Differentiating Eqn. (50) w.r.t. \( \theta \), we get

\[ u_{\theta \theta} = -r \ v_{r \theta}. \quad (52) \]

Multiplying Eqn. (51) by \( r \) and adding to Eqn. (52), we get

\[ r \ u_r + r^2 \ u_{rr} + u_{\theta \theta} = 0, \quad (53) \]