UNIT 10 HAMILTONIAN GRAPHS

Consider the problem of organising a tour to visit different places. If possible it may be better to chart out a roundtrip so as to visit every place exactly once, going from one place to another without passing through a place that has already been visited. The question is whether such a journey is possible or not. To model this system consider a graph with vertices correspond to places and an edges joining two vertices if the respective places are accessible from each other without passing through any other place of visit. The problem has a solution if the graph contains a spanning cycle. A spanning cycle in a graph is called a Hamiltonian cycle and a graph having a Hamiltonian cycle is called a Hamiltonian graph.

This concept is similar to that of Eulerian graph where the graph contains an Eulerian circuit.

From the definition of a line graph of a graph [Definition 4.2.18, page 168 Textbook DW], it appears intuitively that an Eulerian circuit in a graph G is a Hamiltonian cycle in the line graph of G. Unfortunately, there is no general easy way to check whether a graph is Hamiltonian or not. But there are many necessary conditions for a graph to be Hamiltonian. Sufficient conditions are also there to check whether a graph is Hamiltonian or not. Some of these conditions are discussed in this unit. But a condition which is both necessary and sufficient is yet to be solved. It is an open problem for several years.

Loops and multiple edges are irrelevant in the discussion and disconnected graphs forbid spanning cycles. So in this unit we consider connected simple graphs only.

Objectives

After studying this unit, you should be able to

- find Hamiltonian cycles in graphs of small order;
- state and apply necessary conditions for the existence of a Hamiltonian circuits;
- state and apply sufficient conditions for the existence of a Hamiltonian circuit;
10.2 NECESSARY CONDITIONS

As stated earlier there is a bunch of conditions that can be checked for a graph to be Hamiltonian or not. A few necessary conditions are discussed here.

Before we discuss these conditions, let us see some examples.

**Example 1:** In any complete graph with order greater than 2, it is easy to find a spanning cycle. Start with any vertex, trace through a neighbouring vertex that has not been passed previously, till all the vertices are covered and then pass on to the beginning vertex.

**Example 2:** For every \( n \), the cycle graph \( C_n \) (a cycle graph is a graph with single cycles) is Hamiltonian (See Fig 1 below).

**Example 3:** The dodecahedron given in Fig.2 is Hamiltonian.

Why don’t you try it by yourself? In fact this problem was turned into a game by one of the famous Mathematician of 1915 century Sir William Rowan Hamilton (1805-1865) with whom the term Hamiltonian cycles are attached. Hamiltonian, who described, in a letter to his friend Graves, a mathematical game on the dodecahedron (See Fig 2) in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a spanning cycle.

In the next example we will give an example of a graph which is not Hamiltonian.

**Example 4:** Consider the graph given on the next page. This graph is called Herschel graph. This graph is not Hamiltonian. You might have noticed that it is bipartite and has an odd number of vertices. Therefore it is not Hamiltonian.
Hamiltonian Graphs

You will learn more about Hamiltonian graph while reading the Textbook DW.

READ TEXT BOOK DW Chapter 7, Section 7.2 pages 286 line 13 to page 288 line 16

NOTES:

(i) Page 287 line 1

The concept of 2-connectedness has been studied by you in Unit 7 [refer page 161 of Textbook DW]. A characterization of 2-connected graphs is given in Theorem 4.2.2 of Textbook DW. It reads ‘A graph G having at least three vertices is 2-connected if and only if each pair of vertices u, v ∈ V(G) there exist internally disjoint u,v-paths in G,

In a Hamiltonian graph, there is a cycle containing all vertices, vertices. This provides two paths, one clockwise and other anticlockwise, joining any pair of vertices. Thus, every Hamiltonian graph is 2-connected or in other words, a block.

Even though this is not numbered as a result in the Textbook DW, it is a necessary condition, but not sufficient.

The graph in the figure below is 2-connected but not Hamiltonian.

(ii) Page 287 lines 4 to 8

Consider the complete bipartite graph Km,n with partite sets X and Y. A Hamiltonian cycle in Km,n must contain vertices alternatively from the partite sets X and Y since there are no edges joining
vertices in a partite set. Hence a necessary condition is \(|X| = |Y|\), i.e. \(m = n\). If this holds, starting from any vertex, it is easy to find spanning cycle in \(K_{n,n}\) since every vertex in \(X\) is adjacent to every vertex in \(Y\). So a complete bipartite graph is Hamiltonian only if it is of even order.

(iii)  **Page 287 lines 9 to 13**

The proposition states that for any Hamiltonian graph \(G\) and a non-empty set \(S \subset V(G)\), \(S\) has at most \(|S|\) components.

This fact will be more clear to you if you consider the following graph.

![Fig. 5](image)

Let us look at the proof.

Let \(G\) be a Hamiltonian graph of order \(n\) and let \(C = v_1, v_2, \ldots, v_n\) be a Hamiltonian cycle. Consider any non-empty subset \(S\) of \(V(G)\) and let \(G_1, G_2, \ldots, G_k\) be the components of \(G - S\).

Consider tracing along the Hamiltonian cycle \(C\) from vertex to vertex along edges joining them. To trace from one component to another we have to pass through \(S\). No pair of components has common vertices or edges joining the vertices in them. So, to trace from one component to another we have to pass through a vertex in \(S\) [various situations are shown in the figure]. But the constraints of Hamiltonian cycles prohibit passing through a vertex more than once. However, we have to trace all the components and back to the starting vertex. Thus, \(S\) must contain at least \(k\) vertices. This completes the proof.

But the condition given in the theorem is not sufficient. For example, consider the graph given in Fig. 1. It satisfies the condition of properties 7.2.3, but it is not Hamiltonian as there is no Hamiltonian cycle.

(iv)  **Page 287 lines 18 to 20**

Example 7.2.5 in *Textbook DW* gives two graphs. The first fails to satisfy the above condition because if deletion of the set (See Fig. 6 below) \(S = \{u, v\}\) from the graph results in three components. \(|S| < 3\), number of components.
The second graph (See Fig. 7 below) satisfies the condition, but not Hamiltonian.

This can be clarified in the following terms: There are three vertices of degree 2 in the graph and they are adjacent to the central vertex [the vertex of degree 6]. While tracing through the vertices of the graph for a Hamiltonian cycle, we need two visits at the central vertex more than once. This is not allowed in a Hamiltonian cycle. Hence, this graph is non-Hamiltonian.

(v)  Page 288 lines 1 to 3

The case of Petersen graph is also so.

(vi)  Page 288 lines 4 to 13

This remark is about the strengthening of necessary condition to get sufficiency. Research on strengthening of the condition given in proposition 7.2.3 is going on and a brief history and a definition [toughness] are given in these lines.
Now we make some general remark.

**Remark:** Whenever we have a necessary condition, we have to look whether it is sufficient or not. If not, strengthening of a necessary condition can be considered. In the case of Hamiltonian graphs, no nice necessary condition becomes sufficient but some sufficient conditions are arrived by strengthening necessary conditions or otherwise.

Why don’t you try some exercises now?

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E1) Which of the following graphs are Hamiltonian? Give reasons for your answer. Also write down the Hamiltonian cycle wherever possible.

i)

![Fig.9](image)

ii)

![Fig.10](image)

iii)

![Fig.11](image)
iv)

E2) Prove that the graph $K_{n,n}$ is Hamiltonian cycle and also write a Hamiltonian cycle in it.

The next section deals with sufficient condition.

### 10.3 SUFFICIENT CONDITIONS

Now we turn to sufficient conditions for a graph to be Hamiltonian. A characterisation of Hamiltonian graphs among a special class of graphs also is discussed in this section.

You can start reading the Textbook D.W.

**NOTES:**

(i) **Page 288 lines 20 and 21**

Here is a simple sufficient condition. If the minimum degree $\partial(G)$ is not less than half of the order of $G$ then the graph is Hamiltonian [Theorem 7.2.8 Textbook D.W]. The bound is sharp in the sense that if the minimum degree is lower than half of order, then the graph need not be Hamiltonian. Families of such a graphs [counter examples] are described in Example 7.2.7 Textbook D.W. The examples are obtained by ‘pasting’ two cliques at one of their vertices. The figure below shows such graph pasted at the vertex $v$. [A clique is a complete sub graph, see Definition 1.1.8 page 4 Textbook D.W].

The following is a special case $n=11$
This graph is not Hamiltonian since \( v \) is a cut vertex and
\[
\partial(G) = 4 = \frac{n(G)}{2} - 1
\]
This justifies the claim above since \( \partial(G) \) is just 1 less than half of the order of \( G \), but \( G \) is non-Hamiltonian.

(ii) Page 288 lines 23 and 25

An example [corresponding to \( n = 1 \)] of the graph described in Example 7.2.7 Textbook DW is given in the figure.

(iii) Page 288 lines 4 and 3 from below

This theorem due to Dirac proves the sufficiency of the condition. Here we shall give a full proof of the theorem.

Proof: Here we assume that \( n(G) \geq 3 \) for no graph of order 2 in Hamiltonian. The method of proof is assuming the other case and arrive at a contradiction. Suppose there is a non-Hamiltonian graph with the given conditions that there is a non-Hamiltonian graph \( G \) with \( \delta(G) \geq \frac{n(G)}{2} \). Let \( G \) be a maximal with their properties. That is if we add same edge to \( G \), it will be come Hamiltonian. Hence, there is a Hamiltonian path \( v_1 - v_2 - \cdots - v_n \) with \( v_1 \) not adjacent to \( v_n \). Note that every spanning cycle is \( G + v_n v_1 \) containing the edge \( v_n v_1 \). Suppose there is an \( i \in \{2, 3, \ldots, n - 1\} \) such that \( v_1 \) adjacent to \( v_{i+1} \) and \( v_1 \) adjacent \( v_i \) is \( G \).

Then \( v_1 - v_{i+1} - v_{i+2} - \cdots - v_n - v_{i} - v_{i-1} \cdot v_2 - v \), is a Hamiltonian cycle is \( G \). So it is enough to prove such on \( i \) exist.

Let \( S = \{i: v_1 \text{ is adjacent to } v_{i+1}\} \)
\( T = \{i: v_n \text{ is adjacent to } v_i\} \)
Since \( \delta(G) \geq \frac{n(G)}{2} \), we have \( |S| + |T| \geq n \). Note that \( n \notin S \), for \( v_1 \) is not adjacent to \( v_1 \). Also \( n \notin T \). Hence \( n \notin S \cup T \).

But we have \( |S \cup T| = |S| + |T| - |S \cap T| \). Hence \( |S \cap T| \geq 1 \) since \( S \) and \( T \) has a common element. Therefore there is an \( i \) such that \( v_i \) is adjacent to \( v_{i+1} \) and \( v_n \) is adjacent to \( v_i \). Hence, there is a Hamiltonian cycle.

(iv) **Page 289 line 3**

Maximal non-Hamiltonian means that the graph becomes Hamiltonian if we introduce one edge joining any two non-adjacent vertices in it.

(v) **Page 289 line 6**

The symbol \( u \leftrightarrow v \) stands for ‘\( u \) is adjacent to \( v \)’. [See Textbook DW page 2 line 3 from bottom] and the symbol given here is the negation, \( u \) and \( v \) are not adjacent.

(vi) **Page 289 lines 8 to 9**

By the argument explained in note (iv), we have a Hamiltonian cycle in the graph obtained by adding a new edge in a maximal non-Hamiltonian graph. In the proof we construct a Hamiltonian cycle in the former graph using a Hamiltonian cycle in the latter.

(vii) **Page 289 lines 24 to 26**

The lemma 7.2.9 by Ore is a simple characterisation of Hamiltonian graphs among a special class of graphs.

Let us look at the details of the proof of Lemma 7.2.9.

**Proof lemma 7.2.9**: If \( G \) is Hamiltonian then it contains a Hamiltonian cycle and the cycle is a Hamiltonian cycle in \( G+uv \) also.

Conversely, let \( G + uv \) be Hamiltonian. Then \( G + uv \) contains a Hamiltonian cycle. If \( G + uv \) has a Hamiltonian cycle \( C \), not containing the edge \( uv \) then \( C \) is a Hamiltonian cycle in \( G \) also. Otherwise every Hamiltonian cycle in \( G + uv \) contains the edge \( uv \) and we can produce a Hamiltonian cycle from \( C \) as described in pages 7 to 18 page 289 Textbook DW.

\[\square\square\square\]

This lemma is not applicable to all graphs since it demands a condition ‘\( d(u) + d(v) \geq n(G) \)' for every non-adjacent vertices \( u \) and \( v \). This condition is not at all necessary for a graph to be Hamiltonian since every cycle is Hamiltonian but no cycle with more than four vertices possesses this condition.
Moreover, the conclusion depends on the Hamiltonicity of another graph. However, this gave rise to the concept of Hamiltonian closure, due to Bondy and Chvatal, of a graph defined in 7.2.10 page 289 Textbook DW and Theorem 7.2.11 on page 290 Textbook DW.

(viii) Page 290 lines 3 and 4

Theorem 7.2.11 requires no proof since it directly follows from lemma 7.2.9 and definition 7.2.10.

(ix) Page 290 line 7

This lemma assures the uniqueness of the closure of a graph.

Now we make a remark.

**Remark 1:** If the condition that \( d(u) + d(v) \geq n(G) \) is dropped, then \( G \) need not be Hamiltonian even if \( G + uv \) is Hamiltonian.

The following graph illustrates this.

![Fig. 15](image)

Here \( G + uv \) is Hamiltonian for any two non adjacent vertices in \( G \), but this not Hamiltonian.

**Example 5:** Let us find the closure of the graph given below:

![Fig. 16](image)
Let us denote the given graph by $G_1$. Then the closure is given by the following iteration:

i) ISt iteration

![Figure 17](image)

$G_2$

ii) IInd Iteration

![Figure 18](image)

$G_3$

iii) IIIrd Iteration

![Figure 19](image)

$G_4$

Hence $G_4 = C(G) = K_8$
You can try some exercises now.

E3) Show that if the closure of a graph is complete, then it is Hamiltonian. Show that the converse is not true.

E4) Use Ore’s theorem to show that the following graph is not Hamiltonian.

![Fig.20]

10.4 SUMMARY

In this unit, we have studied

1) concepts of Hamiltonian cycles and Hamiltonian graphs.
2) two necessary conditions for graph to be Hamiltonian.
3) two sufficient conditions for a graph to be Hamiltonian.
4) two characterizations of Hamiltonian graphs.
5) concept of the closure of a graph.

10.5 HINTS/SOLUTIONS

E1) i) This is a Hamiltonian cycle. The cycle is given by $v_1-v_n-v_2-v_5-v_3-v_f-v_1$.

ii) Not Hamiltonian-since it is a bipartite graph with odd number of vertices.

iii) The Not Hamiltonian since this is a true and therefore has no cycle.

iv) This is Hamiltonian-the cycle is given by $v_1-v_2-v_3-v_4-v_6-v_5-v_1$.

E2) Let $x = \{u_1, u_2, \ldots, u_n\}$, $y = \{v_1, v_2, \ldots, V_n\}$ be the bipartition of $K_{n,n}$. Then the cycle $u_1-v_1-u_2-v_2, \ldots, u_n-v_n-v_1$ is a Hamiltonian cycle.

E3) If the closure of a graph $G$ is complete, then the closure is Hamiltonian because any complete graph is Hamiltonian. Therefore $G$ is Hamiltonian. The converse of the statement is that if the closure of a
graph is Hamiltonian, then the graph is complete. This is not-true for a counter example, any the cyclic graph with more than four vertices is Hamiltonian and is its own closure and therefore is not complete.

E4) We note that \( d(u) + d(v) \geq 5 \) for each pair of non-adjacent vertices \( u \) and \( v \). Therefore by Ore’s theorem the given graph is Hamiltonian.