### UNIT 6 MATCHINGS AND COVERS

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### 6.1 INTRODUCTION

Matching theory was introduced by Philip Hall in 1935. This theory has application in solving assignment problem and many interesting combinatorial problems.

In this unit, we discuss the material for discussing the problem posed in Example 1.1.9 of Textbook DW. The problem is about possibility of filling up of job vacancies with suitably qualified candidates from among a set of applicants each of whom is qualified for one or more of the jobs. This situation is modeled using graphs. The question ‘Whether it is possible to find a satisfactory assignment?’ is represented. For, the concepts of matching, saturated vertex, perfect matching, alternating path, augmenting path and Hall’s condition are discussed. Related concepts of maximum matching, maximal matching, a min-max theorem useful in network theory, concepts of independent sets, vertex cover and some related graph parameters are also discussed.

In section 6.2, we give the definitions and illustrations of matching and perfect matching in a graph. Section 6.3 deals with maximal matching, maximum matching, alternating paths, augmenting paths and a theorem due to Berge. The main theorem – Hall’s theorem – is discussed in Section 6.4. The subject of section 6.5 is a min-max theorem which gives the relationship between the solutions to a maximization problem and a minimization problem. In Section 6.6 independent set, vertex cover, edge cover, etc. are discussed. This section deals with graph parameters namely maximum size of independent set, maximum size of matching, minimum size of vertex cover and minimum size of edge cover also.

Since loops and multiple edges are insignificant in the study of matchings, in this unit also we confine to simple graphs only unless otherwise stated.

Pre-requisites for this unit are fundamental knowledge of set theory including the concept of set inclusion, and graph theory including characteristics of bipartite graphs.
Objectives

After studying this unit, you shall be able to

- check whether a given graph possesses a perfect matching or not;
- confirm whether a bipartite graph contains a matching that saturates every vertex in one of its partite sets;
- recognise matchings, maximal matchings, maximum matchings and perfect matchings, if any in a given graph;
- check whether a given matching in a graph is a maximal matching/maximum matching/perfect matching;
- check a given matching is maximum or not;
- find vertex covers, edge covers and independent sets;
- estimate the parameters $\alpha(G)$, $\alpha'(G)$, $\beta(G)$ and $\beta'(G)$ for a given graph $G$;
- check for and find a suitable assignment in a given system.

6.2 MATCHING

In this section we shall introduce you to matching and some related concepts.

Let us look at the job assignment problem stated in page 4 of Textbook DW. It is a classical problem that can be solved using graph models.

The problem is stated as follows:

Consider a set $A$ of jobs a company would like to fill and $B$ a set of candidates applied for. The jobs may be of different nature and some of the candidates might have qualified for some of the jobs only, and not for some other. The questions to be considered are whether all the positions can be filled with suitably qualified candidates from among the applicants? And if so, how?

Let us model this problem using graphs:

Consider a graph $G$ with vertex set $A \cup B$ and an edge joins $a \in A$ to $b \in B$ if and only if the candidate corresponding to $b \in B$ is qualified for the job corresponding to $a \in A$. So there are no edges joining two vertices in $A$ or two vertices in $B$.

Now the problem is of finding a set of edges in $G$ containing exactly one edge incident at each vertex of $A$ and no pair of edges have a common end vertex in $B$. Such a set of edges is a matching.

For example in the following figure (Fig.1) the sets $M_1 = \{e_1, e_3, e_5, e_7\}$ and $M_2 = \{e_3, e_4, e_7, e_8\}$ are suitable choices. But no set of edges containing $e_2$ or $e_8$ will work.
**Definition 1:** A matching in a graph $G$ is a subset $M$ of the edge set $E(G)$ such that no pair of edges in $M$ have a common end vertex in $G$.

**Definition 2:** A vertex $u$ in a graph $G$ is said to be saturated by a matching $M$ in $G$ if $u$ is an end vertex of some edge in $G$.

**Definition 3:** If $M$ is a matching in a graph $G$ and if all vertices of $G$ are $M$-saturated then $M$ is called a perfect matching.

Let us consider some examples.

**Example 1:** Let $G$ be the graph model of the Königsberg bridge problem shown in page 2 of Textbook DW.

In this graph $M_1=\{e_1, e_7\}$ is a matching and no other subset of edges containing $e_1$ is a matching since every other vertex shares an edge with $e_1$.

Can you find some other matching in this graph? You might see that $M_2=\{e_3,e_6\}$, $M_3=\{e_4\}$ and $M_5=\{e_5\}$ are also matchings in the $G$.

Here all the four vertices are $M_1$-saturated as well as $M_2$-saturated and hence they are perfect matchings. But the vertices $x$ and $y$ are not $M_3$-saturated and $x$ and $z$ are not $M_5$ saturated. So $M_3$ and $M_5$ are not perfect matchings.

We shall consider another example.

**Example 2:** Let us consider the Petersen graph defined in 1.1.36, page 13 of Textbook DW with vertices labeled $v_1$, $v_2$, $v_3$, ..., $v_{10}$ and edges labeled $e_1$, $e_2$, $e_3$, ..., $e_{15}$ as in Fig. 3.
M_1={e_6, e_7, e_8, e_9, e_{10}}, M_2={e_1, e_3, e_{10}, e_{12}, e_{13}} and M_3={e_2, e_5, e_9} are matchings. But no three of the five edges on the outer cycle form a matching. Here M_1 and M_2 are perfect matchings as each of them saturates every vertex. But M_3 is not perfect.

You can start reading **Textbook DW**.

**NOTES:**

(i) **Page 107 line 1:**

**The problem of pairing room mates:** Consider a set of people to be paired to accommodate two each in a room. Here the constraint is not on the number of rooms or number of peoples but the individuals’ willingness to share rooms with others. If everybody is ready to share a room with any other there is no difficulty, if the number of people is even. But that may not be the real situation. Consider the graph model of this problem. The vertex set of the graph corresponds the set of people and edge joins two vertices if and only if the respective persons are ready to share a room. This graph need not be bipartite. Then the question is of finding a matching in a graph.

An obvious necessary condition is that the number of people must be even. The Hall’s theorem can be applied here only if it is possible to find out a spanning bipartite subgraph with partite sets of equal size.

(ii) **Page 107 line 7**

Matching is defined as a set of edges with certain conditions. But you should convince yourself that there is no existence for an edge...
in a graph without its end vertices. So, for every mention about edges, the presence of end-vertices is understood. Because of this, a matching in a graph $G$ can be viewed as a sub-graph of $G$ whose components are $K_2$, the complete graph of order two. Refer to the Examples 1 and 2 above.

(iii) **Page 107 line 8**

Sharing end points means the set of end vertices of any two edges are disjoint subsets of the vertex set. A vertex $v$ is incident to an edge $e$ means that $v$ is an end vertex of the edge $e$.

[refer page 6 lines 11 and 10 from the bottom of Textbook DW]

(iv) **Page 107 lines 9 and 10**

If $G$ is a graph and $M$ a matching in it then a vertex $v$ of $G$ is $M$-saturated if $M$ contains an edge $e$ with $v$ as one of its end vertices. If no such edge is in $M$ then $v$ is $M$-unsaturated.

(v) **Page 107 from line 12 to line 14**:

Recall the definition of $K_{n,n}$, the complete bipartite graph [page 9 lines 26 and 27 of text book DW ]. It has $n$ vertices in each partite set and every vertex in each partite set is adjacent to all vertices in the other partite set. So it is easy to find a perfect matching in it. For example $M=\{x_1 y_1, x_2 y_2, \ldots, x_n y_n\}$ is a perfect matching.

For finding all perfect matchings in $K_{n,n}$, consider any permutation $P$ of the set $Y$ and take the edge with end vertex $x_i$ and the $i^{th}$ vertex in $P$, for $i=1,2,3,\ldots, n$. Among these the perfect matching $\{x_i y_i / i = 1,2,3,\ldots,n\}$ correspond the identity permutation. Since the order of arrangement of elements of $Y$ is different in distinct permutations of it, we get distinct matchings. These are the only perfect matchings possible since there are only $n$ edges incident with each vertex in $K_{n,n}$ so $n!$ matchings.

An alternate computation of the number of perfect matchings in $K_{n,n}$ is given below:

There are $n$ edges incident at $x_1$. So we have $n$ choices to select an edge incident at $x_1$. Let it be $x_1 y_i$. After selecting it, for an edge incident at $x_2$ we have exactly $n-1$ choices, i.e. except $x_2 y_i$.

Similarly, for edges at $x_3, x_4, \ldots$ etc. we have $n-2, n-3, \ldots$, choices for successive selection of edges incident at. Hence, by the principle of counting the total choices for perfect matching in $K_{n,n}$ is $n!$

Clearly each matching provides a one-to-one correspondence between elements of $X$ and elements of $Y$. So a perfect matching is a bijection between $X$ and $Y$. 
Graph Theory

(vi) **Page 107 line 15**

The symbol \([n]\) stands for the set of \(n\) distinct objects, probably first \(n\) natural numbers.

(vii) **Page 108 line 1**

As mentioned above, a graph can have a perfect matching only if it is of even order. This is true for complete graphs also. So a complete graph has a perfect matching only if it is of even order. Fortunately, this is sufficient also – a complete graph has a perfect matching if and only if it is of even order. To find a matching in a complete graph of even order, consider a spanning cycle and delete alternate edges.

Computation of the number of perfect matchings in \(K_{2n}\) is given in page 108 of the *Textbook DW* [example 3.1.3].

(viii) **Page 108 after line 16**

![Fig.4](image_url)

Even though the graph shown above is of even order (6), it contains no perfect matching. A perfect matching in this graph needs three edges, to saturate six vertices. But, each of the vertices \(x\) and \(w\) is common to three edges. Since the graph contains five edges only, in any set of three or more edges in this graph, at least two of must share either \(x\) or \(w\). So, a matching in this graph can have at most two edges only. This justifies the claim.

By now you must have understood the concept of matching.

We now consider more examples

**Example 3:** Let us consider the graph given in the next page (see Fig.5).

In the graph we observe that \(M_1 = \{e_1, e_3, e_5, e_{15}\}\), \(M_2 = \{e_2, e_6, e_{10}, e_{11}\}\), \(M_3 = \{e_7, e_{14}\}\) and \(M_4 = \{e_2, e_4, e_6\}\) are matchings.
You can also observe that none of these matchings is perfect matching. This is a graph of even order 10. A perfect matching in this graph must contain five edges. $M_1$ and $M_2$ contain four edges only and $M_3$ contains two only. It is possible to find more matchings of size four, possibly not disjoint from already we have. You can try to find a perfect matching by adding any one of the remaining edges in the graph. But certainly fail. So this graph also has no perfect matching. Try to find a formal proof.

\[\square\square\square\]

Note that the above figure is a re-drawing of the Petersen graph. [See page 13 of Textbook DW for a formal definition of the Petersen graph and page 17 exercise 1.1.24 for different drawings.]

You know that an edge can be represented by its own label or using the labels of its end vertices. In the following examples we adopt the later.

**Example 4:** Let us consider the following graph.

Consider the graph represented by the diagram shown above. Then $M_1 = \{v_1v_2, v_3v_4\}$, $M_2 = \{v_1v_2, v_6v_3, v_4v_5\}$ and $M_3 = \{v_1v_6, v_2v_3, v_4v_5\}$ are matchings. $M_1$ is not perfect [why?]. Both $M_2$ and $M_3$ are perfect.

From this example we can observe that a graph can have more than one perfect matchings.
E1) Label the edges of the graph given in the figure below, prepare a list of matchings and identify perfect matchings, if any.

![Fig.7](image)

E2) Label the vertices of the graph given in the figure below, prepare a list of matchings and identify perfect matchings, if any.

![Fig.8](image)

From examples above, we have seen that a graph can have more than one perfect matchings. A perfect matching has to saturate all the vertices in the graph and hence every perfect matching must of size equal to half of the number of vertices in the graph. So, perfect matchings can exist only in graphs of even order. However it is not a sufficient condition as we have seen in note (viii) above. Moreover a proper subset of a perfect matching is not at all a perfect matching. But it is true that every subset of a matching is a matching.

From Example 1, it is clear that two matchings in a graph may be disjoint or may have some edges in common. $M_1$ and $M_3$ are disjoint, $M_2$ and $M_4$ have the edges $e_2$ and $e_6$ in common. It is possible to obtain new matchings from $M_3$ and $M_4$ by providing more edges. But this is not possible with $M_1$ and $M_2$. This lead to the concepts of maximal matchings. But there may be maximal matchings of different sizes as we see later. This leads to the concept of maximum matchings. We discuss these two concepts in the next section.
6.3 MAXIMAL AND MAXIMUM MATCHINGS

Maximal matching, maximum matching are two optimal concept resulting from the comparison of the matchings in graph with respect to set inclusion and size of matchings.

Let us start with the formal definitions.

**Definition 1**: A matching $M$ in a graph $G$ is called a *maximal matching* if it is not a proper subset of any other matching in $G$.

**Definition 2**: A matching $M$ in a graph $G$ is called a *maximum matching* if there is no matching in $G$ of bigger size.

**Definition 3**: Let $M$ be a matching in a graph $G$. Then a path $P$ in the graph $G$ is an $M$-alternating path if the edges of $G$ appearing along $P$ are alternatively in $M$ and not in $M$.

**Definition 4**: A path $P$ is $M$-augmenting if $P$ is $M$-alternating and both the end vertices of $P$ are not saturated by $M$.

Consider the Petersen graph given in Fig. 3. Sec 6.2. $M_1=\{e_1, e_3, e_{10}\}$ is a matching. But it is not a maximal matching since the $M_2=\{e_1, e_3, e_{10}, e_{15}\}$ is a matching containing $M_1$. Now try to include more edges to $M_2$. The edges $e_2$ and $e_4$ are sharing end vertices with $e_3 \in M_2$; $e_5$, $e_6$ and $e_7$ are sharing end vertices with $e_1$; $e_8$ and $e_9$ are sharing with $e_3$; $e_{11}$ is sharing with $e_{10}$; $e_{12}$ and $e_{13}$ are sharing with $e_{15}$ and sharing with $e_{10}$. So $M_2$ is maximal matching.

Similarly you can find more maximal matchings starting with $M_1$ and choosing an edge other than $e_{15}$ to include. Consider $M_3=\{e_1, e_3, e_{10}, e_{12}\}$. This is not maximal since $M_4=\{e_1, e_3, e_{10}, e_{13}\}$ is a matching containing it. No more edge can include since the graph has only ten vertices. So this is also a maximal matching. But as already observed, size of a matching in this graph will not exceed five. So this is a matching of maximum size – a maximal matching. Moreover this is a perfect matching also.

Consider again the matching $M=\{e_1, e_3, e_{10}\}$. The paths $P_1= v_1v_2v_3v_4v_5$, $P_2= v_6v_1v_2v_3v_4v_5$ and $P_3= v_6v_1v_2v_3v_4v_5v_10v_7v_9$ are $M$-alternating paths, $P_3$ is an $M$-augmenting path while $P_1$ and $P_2$ are not since the vertex $v_5$ is $M$-saturated due to $e_{10}$ in $M$. $P_4= v_6v_1v_2v_3v_4v_5v_10v_7v_9$ is not an $M$-alternating path since $e_{11}$ and $e_{13}$ are not in $M$. 
NOTES:

(i) **Page 108 lines 21 to 23**

As mentioned earlier, maximal is related to set inclusion and maximum is related to size of the matching. Maximum matchings are maximal also, since no matching of bigger size exists in the graph. But, maximal need not be maximum. A perfect matching is always maximum and hence maximal.

Consider from matchings given by $M_1 = \{v_1v_2, v_3v_4\}$, $M_2 = \{v_1v_2, v_3v_6, v_4v_5\}$, $M_3 = \{v_1v_6, v_2v_3, v_4v_8\}$ and $M_4 = \{v_2v_6, v_3v_5\}$ in the graph whose diagram is given. Here both $M_2$ and $M_3$ are maximum as well as maximal. $M_4$ is a maximal matching since it is impossible to find another matching in $G$ properly containing $M_4$ but it is not a maximum matching. $M_1$ is not maximal since $M_5 = \{v_1v_2, v_3v_4, v_5v_6\}$ is a matching in $G$ properly containing $M_1$.

(ii) **Page 109 lines 3 to 5**

In $M$-augmenting path, both end vertices are $M$-unsaturated. So the pendant edges are not in $M$ and alternate edges are in $M$. An immediate consequence is that the length of an $M$-augmenting path is odd.

The concept of augmenting path is helpful in checking maximum matching and in obtaining a matching of bigger size from a matching, if possible.

In the above figure, consider the matching $M = \{v_1v_2, v_3v_4\}$. 
The path $v_1v_2v_3v_4v_5$ is an M-alternating path and not an M-augmenting path since the end vertex $v_1$ is M-saturated. The path $v_6v_1v_2v_3v_4v_5$ is an M-augmenting path since it is M-alternating and both its end vertices are M-unsaturated.

(iii) **Page 109 lines 14 and 15**

The symmetric difference $A\Delta B$ of two sets $A$ and $B$ is given by $A\Delta B = (A \cup B) - (A \cap B)$ and also by $A\Delta B = (A - B') \cup (B - A')$ where the prime (') stands for set theoretic complementation.

(iv) **Page 109 lines 23 and 24**

This lemma is used to prove theorem 3.1.10 page 109 of Textbook DW.

(v) **Page 109 lines 14 and 15**

Theorem 3.1.10 characterises maximum matching in terms of augmenting paths.

Let us consider more examples

**Example 5:** Let us consider the following graph.

![Fig.11](image)

Consider the matchings $M_1 = \{e_3, e_7\}$, $M_2 = \{e_1, e_3, e_{10}\}$ in this tree. Both $M_1$ and $M_2$ are maximal and $M_2$ is maximum. There is no perfect matching in this graph [why?]. The path $P_1 = v_{10}v_8v_3v_4v_5$ is an $M_1$-augmenting path but not $M_2$-augmenting.

**Example 6:** Let us consider the following graph.

![Fig.12](image)
In the figure consider a matching \( M=\{v_1v_3, v_4v_6, v_5v_9, v_8v_{11}\} \). Then the path \( P_1=v_2v_4v_6v_9v_5v_8v_{11}v_7 \) is an \( M \)-augmenting path. But \( P_2=v_2v_4v_6v_8v_7 \) is not \( M \)-augmenting since it is not an \( M \)-alternating path.

Try to do the following exercises:

E3) If \( M \) is a perfect matching in a graph \( G \), then is it possible to have an \( M \)-augmenting path in the graph? Justify your answer.

E4) Determine the minimum size of a maximal matching and size of a maximum matching in the cycle \( C_n \).

**Remark 1**: Note that the concept of maximal matching is a set theoretic property in the sense that there is no ‘bigger’ matching in the graph while that of maximum matching is related to the size in the sense that the graph contains no matching of bigger size.

**Remark 2**: You can note that a graph may have more than one maximum matchings and more than one maximal matchings. All maximum matchings will be of same size but maximal matchings may differ in size. Further, note that every perfect matching is maximum as well as maximal and every maximum matching is maximal. A maximal matching need not be a maximum matching. If there exist a perfect matching in a graph \( G \), then maximum matchings are precisely the perfect matchings. So if you have a perfect matching in a graph and if it is not perfect then the graph has no perfect matchings. This idea also will help in checking and finding perfect matchings. Theorem 3.1.10 in page 109 of the textbook DW is helpful in checking maximum matchings.

We summarise the above discussion by highlighting the following points:

- a subset of a matching is also matching
- a graph can have more than one maximal matching
- maximal matchings in a graph may be of different size
- a graph can have more than one maximum matchings
- all maximum matchings in a graph must be of same size
- a maximum matching is maximal also
- a maximal matching need not be maximum
- every non-empty graph have a non-empty matching
- a graph need not have a perfect matching
- a perfect matching, if exists, is maximum and hence maximal as well
a subset of a maximal/maximum/perfect matching is not a maximal/maximum/perfect matching

Under the terminology we studied in this section, the assignment problem is to find a matching in the graph model, that saturates every vertex in the partite set A. Further we look at some more conditions of the graph model, as given below. We note that

a) the number of applicants must not be less than the number of job positions, i.e. $|B| \geq |A|$ 

b) there must be at least one candidate qualified for each job, i.e. every vertex in A has non-zero degree.

c) for every collection of k jobs there must be at least k candidates qualified for those k jobs. i.e. $|N(S)| \geq |S| \forall S \subseteq A$

Note that the first two conditions obviously follow from the third. In fact, in 1935 an English Mathematician Phillip Hall [1904-1982] proved that the condition $|N(S)| \geq |S| \forall S \subseteq A$ is sufficient for a graph to have the required matching. Thereafter the condition is called Hall’s Matching condition in honour of him.

This is discussed in the next section.

6.4 HALL’S MATCHING CONDITION

Here we shall illustrate the Hall’s Matching condition. We shall first look at the assignment problem. We have seen that the solution of the problem requires the existence of a matching that saturates every vertex in a partite set X in a bipartite graph G with partite sets X and Y with $|X| \leq |Y|$. This was answered by Hall. In this section we discuss the related theorem due to Hall.

You may read the portion marked in the box.

READ THE BOOK DW chapter 3, section 1 FROM PAGE 110 line 10 to page 111 last line.

NOTES:

(i) Page 110, line 18

The condition $|N(S)| \geq |S| \forall S \subseteq X$ is the Hall’s condition for the existence of a matching that saturates every vertex in X.
Even though this condition seems to be simple, every subset of the partite set $A$ in the graph $G$ must satisfy this condition. This is important when one use the condition for verification.

(ii) \textit{Page 110, line 20}

Theorem 3.1.11 in text DW page 110 is called Hall’s marriage theorem when $|X|=|Y|$. [refer line 24 page 111 of Textbook DW].

(iii) \textit{Page 110, lines 23 and 24}

In the sufficiency part, we have to prove existence of matching that saturates every vertex in $X$. Instead the contrapositive is proved in Theorem 3.1.11 Textbook DW.

The statement to be proved is that ‘if $\forall S \subseteq X \exists S \subseteq X$ such that $|N(S)| < |S|$ , then the X,Y-bipartite graph has a matching that saturates $X'$. The contra positive of this statement is that ‘if an X,Y-bipartite graph does not have a matching that saturates $X$, then must exist a subset $S$ of $X$ such that $|N(S)| < |S|$ .

(iv) \textit{Page 111, lines 24}

A classical problem called the \textbf{Marriage problem} is a special case of the assignment problem, in which the graph in the model has partite sets of equal size. This has been settled by Frobenius in 1917, may be without using graph theory.

\textbf{The marriage problem}

Consider a set of $n$ girls and a set of $n$ boys. Some of the girls love some of the boys and every girl has at least one boy whom she loves. Is it possible to find a one-to-one alliance among these boys and girls?

In the light of Hall’s condition such an alliance is possible only if there is set of $k$ boys to match with every set of $k$ girls. A graph model of this problem is a bipartite graph with the set of girls and the set of boys as partite sets and an edge between two vertices in these two sets if and only if the respective girl loves the boy represented by the other vertex. Now the question is about the existence of a perfect matching in the bipartite graph.

(v) \textit{Page 111, lines 30}

A $k$-regular $X,Y$ – bigraph means a bipartite graph with partite sets $X$ and $Y$ with every vertex is of degree $k$. 

\begin{quote}
\textit{If $p$ and $q$ are two statements, then the implication \textbf{‘$p$ is true whenever $q$ is’} is equivalent to \textbf{‘$q$ is not true whenever $p$ is not true’}. The latter implication is called the \textbf{contrapositive of the former}. The method of contrapositive proof is common in mathematics.}
\end{quote}
Example 7: Let us consider the following graph.

![Fig. 13](image)

For \(S = \{x_1, x_2, x_3\}\) in the figure above \(N(S) = \{y_1, y_2\}\). So \(|N(S)| < |S|\) and hence by Hall’s theorem this graphs has no matching that saturates the partite set \(X = \{x_1, x_2, x_3, x_4\}\).

Example 8: Let us consider the following graph.

![Fig. 14](image)

In this graph \(M = \{x_1y_1, x_2y_3, x_3y_4, x_4y_2\}\) is a matching that saturates \(X = \{x_1, x_2, x_3, x_4\}\). You can verify Hall’s condition for every subset of \(X\).

Here are some exercises for you.

E5) Let \(M\) and \(N\) be matchings in an \(X,Y\)-bigraph \(G\). Suppose \(M\) saturates a subset \(S\) of \(X\) and \(N\) saturates a subset \(T\) of \(Y\). Prove that \(G\) contain a matching that saturates \(S \cup T\).

E6) Try the above exercise when \(G\) is any graph and \(S\) and \(T\) are any two disjoint subsets of \(V(G)\).

E7) Try the above exercise when \(G\) is any graph and \(S\) and \(T\) are any two subsets of \(V(G)\).

From the Hall’s theorem, it is clear that the assignment problem and its special case, the marriage problem, have a solutions if and only if the condition \(|N(S)| \geq |S| \forall S \subseteq X\) holds. There are algorithms to find out such an assignment [matching]. We are not included this here as it is not part of the syllabus here. Some algorithms are discussed in Section 3.2 of Textbook DW.

6.5 A MIN-MAX THEOREM

In this section we shall discuss the min-max theorem is, due to Konig and Egervary, which establishes the equality of the size of maximum matching and minimum vertex cover in any bipartite graph. There are other min-max theorems elsewhere.
You can start reading Textbook DW.

READ TEXT BOOK DW from page 112 line 1 to page 113 line 28

NOTES:

(i) Page 112 lines 10 and 11

A vertex cover of a graph G is a subset of V(G) such that it contains at least one end vertex of each edge in G. Note that any subset of V(G) that contains a vertex cover is also a vertex cover and trivially V(G) itself is a vertex cover of G.

The following figure illustrates this

Fig.15

In figure A= \{v_1\} and B= \{v_2, v_3, v_4, v_5, v_6, v_7\} are vertex covers of G. Any set of vertices containing v_1 is a vertex cover however no subset of B is a vertex cover.

Any set of \(n-1\) vertices in \(K_{n,n}\) is vertex cover but no subset of less vertices is a vertex cover since no such set will cover the edge joining the exempted vertices.

(ii) Page 112 line 19

If we look at Example 3.1.15 in page 112 textbook DW you will motivate to Theorem 3.1.16 page 112 textbook DW. In the figures, the vertices belonging to vertex cover are marked by enclosing them in squares and the edges is the matching are highlighted with bold face.

(iii) Page 112 line 25

The theorem assures the existence of a vertex cover same as in size that of a maximum matching and a matching of size same as that of a minimum vertex cover in a bipartite graph. The proof is by exhibiting the required sets assuming the hypothesis [maximum matching / minimum vertex cover]. This theorem used in a proof of an important theorem, Menger’s theorem [page 167 Textbook DW]. Theorem 3.1.16, page 112, Textbook DW hold good in bipartite graphs only and not true in general graphs.
The second graph in the figure for Example 3.1.15 page 112 of the Textbook DW is a counter example. In the graph a vertex cover need at least three edges since every vertex is of degree 2, two vertices can cover only four edges but the graph has five edges. A maximum matching contains only two edges since for having a matching with three edges it need at least six vertices in the graph.

Now we see an example:

**Example 9:** Let us consider the following graph.

![Graph](image)

Here \(\{x_1y_3, x_3y_1, x_4y_4\}\) is a maximum matching \(\{x_1, x_4, y_3\}\) is a minimum vertex cover. Since the given graph is bipartite, by Theorem 3.1.16, the maximum size of a matching equals minimum size of a vertex cover of the graph.

We have seen that a matching \(M\) is a subset of the edge set whose elements have no common end vertex and a vertex cover is subset \(S\) of the vertex set such that every vertex in the graph is adjacent to some vertex in \(S\). Recall the concept of independent sets you have studied in Unit 1, an independent set is a subset of the vertex set such that no pair of them form an edge in the graph. This is dual to the concept of vertex cover. In comparison with this, a matching can be considered as an independent set of edges. Then a natural question of an edge cover arises. This is also of interest in graph theory and its applications.

The concepts of independent set, edge cover, some of the relations between them and graph parameter related to there are discussed in the next section.

### 6.6 INDEPENDENT SETS AND COVERS

Recall that a subset \(S\) of the vertex set \(V(G)\) of a graph \(G\) is independent if the subgraph induced by \(S\) is an empty graph, i.e. no two vertices in \(S\) form an edge in \(G\). This is similar to the concept of matching since matching is a subset of \(E(G)\) such that no pair shares a vertex and independent set a subset of \(V(G)\) without sharing of edges. One can view a matching as an independent set of edges. Edge cover is a concept similar to vertex cover.

You will learn about these concepts after reading the relevant portions of the Textbook DW mentioned in the box given in the next page.
Recall the definition of independent set in a graph; refer to Textbook DW page 4 lines 3 and 4. This concept is dual to the concept of vertex cover defined in the previous section; refer to Textbook DW page 112 line 9. An independent set in a graph $G$ is a subset $S$ of the vertex set $V(G)$ such that no edge in $G$ joins any pair of vertices in $S$. Note that matching can be considered an independent set of edges. Naturally, any subset of an independent set is also an independent set. Hence, as in matching, the questions of maximal and maximum independent sets arise.

The independence number $\alpha(G)$ of a graph $G$ is the maximum size of an independent set in $G$. Convince yourself that independence number of any complete graph is 1 since each vertex is adjacent to every other vertex. Independence number of a complete bipartite graph is the cardinality of the bigger partite set. Independence number of an even cycle of length $2n$ is $n$ and that of an odd length $2n+1$ is also $n$.

Let $G$ be the Petersen graph shown above. Let $S_1 = \{v_1, v_3, v_7\}$. Because the presence of $v_1$ forbids $v_2$, $v_5$ and $v_6$ from $S_1$, being adjacent to it, presence of $v_3$ forbids $v_2$, $v_4$ and $v_8$ and $v_7$ forbids $v_2$, $v_9$ and $v_{10}$. But $S_1$ is not a maximum independent set since $S_2 = \{v_1, v_3, v_9, v_{10}\}$ is an independent set containing more elements. But $G$ contains no independent set containing more than four elements. Since an independent set in $G$ can have only at most two vertices each from the cycles $v_1v_2v_3v_4v_5v_1$ and $v_6v_9v_7v_{10}v_8v_6$.

So the independence number of the Petersen graph is 4.
(iii) Page 114 line 1

The graph in the figure is bipartite but not complete and the observation in the example after note (ii) is not contrary to that in Example 3.1.18 page 114 Textbook DW.

(iv) Page 114 line 6

The concept of edge cover is the counter part of vertex cover.

Moreover, if L is an edge cover of a graph G then every vertex in G is an end vertex of some edge in L. So every edge in G must share end vertices with at least one edge in L. Hence concept of edge cover can be considered as a dual to the concept of matching. An edge cover of a graph of order n must contain at least \[ \left\lceil \frac{n}{2} \right\rceil \] edges.

where \( \left\lceil x \right\rceil \) stands for the least integer greater than or equal to n.

Any sub set of E(G) containing an edge cover of G is also an edge cover of G. This motivates the concept of minimum edge cover.

(v) Page 114 lines 11 and 12

A perfect matching in a graph contains one edge each adjacent to every vertex and hence is an edge cover.

(vi) Sufficient explanation is given in textbook DW related to definition 3.1.20 and matter thereafter in page 114 and 115

The following example provides an illustration.

Example 10: Let us consider the following graph.

In the graph shown above \{v_1\} and \{v_3\} are independent sets. The set \{v_1\} is a maximal independent set but not the set \{v_3\}. Also the set \{v_3, v_5, v_7\} is a maximum independent set and the set \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_1v_7\} is an edge cover of G. The set \{v_1v_2, v_3v_4, v_5v_6, v_7v_1\} is also an edge cover of G.
This is a minimum edge cover of $G$. An edge cover of this graph must be of size at least four, since $G$ has seven vertices.

With this we come to an end of our discussion on independent sets and covers and also to this unit.

Let us now summaries the points discussed in this unit.

### 6.7 SUMMARY

In this unit, we have covered the following:

1. The concepts of matching, perfect matching, maximal matching, maximum matching, vertex saturated by a matching, alternating and augmenting paths, vertex cover, edge cover and independent set.

2. The number of perfect matchings of $K_{n,n}$ is $n!$.

3. The number of perfect matchings of $K_n$ is $(2n)!/2^n n!$.

4. A maximal matching need not be a maximum matching.

5. A perfect matching is maximal and maximum also.

6. Every component of the symmetric difference of two matchings is a path or an even cycle.

7. A matching $M$ in a graph $G$ is a Maximum matching in $G$ if and only if $G$ has no $M$-augmenting path.

8. Berge’s characterization of a maximum matching in terms of augmenting path.

9. If $(X, Y)$ is a bipartite graph with $|X| \leq |Y|$, a necessary and sufficient condition for the existence of a matching $M$ that saturates all vertices of $X$ is the Hall’s condition.

10. Every $k$-regular bipartite graph has a perfect matching.

11. In a bipartite graph, the maximum size of a matching equals minimum size of a vertex cover.

12. Duality of matching and covering.

13. The four parameters $\alpha(G)$, the maximum size of independent set in a graph $G$; $\alpha'(G)$, the maximum size of a matching in $G$; $\beta(G)$ minimum size of a vertex cover and $\beta'(G)$, the minimum size of an edge cover in $G$ and the following relations among these parameters:

   (i) $\alpha'(G) = \beta(G)$ if $G$ is bipartite,

   (ii) $\alpha(G) = \beta'(G)$ if $G$ is bipartite and without isolated vertices.
(iii) \( \alpha(G) + \beta(G) = n(G) \) in any graph

(iv) \( \alpha'(G) + \beta'(G) = n(G) \) if \( G \) has no isolated vertices

### 6.8 HINTS/SOLUTIONS

**E1)** Try it by yourself

**E2)** Try it by yourself

**E3)** No, Augmenting path need unsaturated vertices. But a perfect matching saturates every vertex.

**E4)** Consider \( C_n \) figure below (see Fig.19)

![Fig.19](image)

Label the edges of \( C_n \) consecutively by \( e_1, e_2, e_3, \ldots, e_n \) as in figure. Consider the matching \( M = \{ e_1, e_4, e_7, \ldots \} \) the last vertex in \( M \) is \( e_{n-2} \) if \( n \) is a multiple of 3, otherwise \( e_{n-1} \).

Then \( M \) is a maximal matching since every other edge share common vertex with one edge in \( M \). However no set of less edges will be a maximum matching.

For this look at the following special cases,

**Case 1:** Consider \( C_4 \)

![Fig.20](image)

Then \( \{ v_1v_2, v_3v_4 \} \) is a maximal matching, but no set of one edge is maximal.

**Case 2:** Consider \( C_5 \)

![Fig.21](image)
Then \{v_1v_2, v_3v_4\} is a maximal matching but no set of one edge is maximal.

**Case 3:** Consider \(C_6\)

![](image22)

Then \{v_1v_2, v_4v_5\} is a maximal matching but no set of one edge is maximal.

So, if \(n = 0 \text{ (mod 3)}\), then the minimum size of a maximal matching = \(n/3\). if \(n = 1 \text{ (mod 3)}\) or \(2\text{(mod 3)}\), then the minimum size of a maximal matching = \(\left\lceil \frac{n}{3} \right\rceil + 1\)

**E5)** **Hint:** Consider some convenient subset of \(M \cup N\).

**E6)** Consider the situation in which an edge in \(M\) and an edge in \(N\) have a common end vertex \(u \notin S \cup T\).

**E7)** Same argument as in E6

**Additional Problems**

**Problem 1:** For any graph \(G\), let odd \((G)\) denote the number of odd components of \(G\). If \(G\) has a perfect matching then, prove that odd \((G-S)\) \(\leq \mid S \mid \forall S \subseteq V(G)\).

**Solution:** Let \(G\) have a perfect matching \(M\) Let \(S \subseteq V(G)\)

Let odd \((G-S) = k\) say \(G_1, G_2, \ldots, G_k\). Since each \(G_i\), \(i = 1\) to \(k\) has odd number of vertices, at least one vertex of each \(G_i\) must be matched to some vertex of \(S\) in the matching \(M\).

So, \(S\) must have at least \(k\) vertices i.e. \(\mid S \mid \geq \text{odd} (G-S) \forall S \subseteq V(G)\).

**Problem 2:** Check whether the following graph has a perfect matching.

![](image23)
Hint: No, a routine check will help.

Problem 3: Let four girls a, b, c and d know 3 boys w, x, y and z as given below.

<table>
<thead>
<tr>
<th>Girl</th>
<th>Boys known</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>w</td>
</tr>
<tr>
<td>b</td>
<td>y</td>
</tr>
<tr>
<td>c</td>
<td>x</td>
</tr>
<tr>
<td>d</td>
<td>x</td>
</tr>
</tbody>
</table>

Draw the bipartite graph and check Hall’s condition for this problem.

Hint: You have to check Hall’s condition for all 15 non-empty sub sets of \{a, b, c, d\} and if the condition fails for one no preferred matching is possible.

Problem 4: Check whether the bipartite graph given in the following figure has a matching which saturates all the vertices of X.

![Fig.24](image)

Problem 5: A school wants to fill six vacancies of teachers with one teacher each for Mathematics, Physics, Chemistry, Biology, Computer Science and Yoga. In order for a teacher to be selected for a particular subject, he or she must have majored or minored in that subject. The school received six applications from six candidates A, B, C, D, E and F with the following qualifications.

A: major Physics, minor Mathematics
B: major Biology, minor Chemistry
C: major Chemistry, minor Yoga
D: major Mathematics, minor Computer Science and Yoga
E: major Computer Science, minor Mathematics
F: major Physics, minor Mathematics and Chemistry

What is the maximum number of applicants the school can hire?

Hint: Draw a bipartite graph with partite set X representing the six subjects, partite set Y representing the six candidates and an edge joins a vertex x in X to a vertex y in Y if the candidate x is qualified for a post in the subject y. Then check the Hall’s condition for all non-empty subsets of X.