UNIT 3 VERTEX DEGREES

3.1 INTRODUCTION

In the last unit we have discussed about the paths, cycles and trails in a graph which are useful in many applications of graph theory; for example to find the shortest route between two particular vertices or routing of a telephone call between one subscriber and the another, etc.

In this unit, we will define the degree of a vertex in a graph $G$ which is an important parameter of a graph. Such a concept occurs, for example in a road map, where a junction has three or more roads meeting. It also arises in electrical network theory where we may be interested in the number of wires at a given terminal, or in architecture where we may be concerned with the number of rooms accessible from a given one. In chemistry the term valency is used to indicate the number of bonds connecting an atom to its neighbours.

In this unit, we discuss about counting the edges of a graph $G$ using the degrees of its vertices. We also discuss the first theorem of Graph theory also known as “Handshaking Lemma” and a theorem on degree sequence of a graph $G$ given by Havel and Hakimi.

Objectives

After studying this unit, you should be able to

- count the number of edges in a graph if the degrees of the vertices are known by applying the Handshaking Lemma;
- use Havel-Hakimi theorem to determine whether a given sequence of non-negative integers is a graphic sequence or not.

3.2 COUNTING AND BIJECTIONS

In this section, we discuss about how to count vertex degrees, edges, etc. which will be used very often to study many properties of a graph and another technique for counting a set using bijection (one-to-one and onto function from the given graph to a graph whose information is already known). This technique is useful when it is very difficult to count vertex degrees, edges and other information for the given graph. We also familiarize you with a very famous result in the graph theory known as “First theorem of graph theory” or the “Handshaking Lemma”.
Let us start with some important definitions:

**Definition 1:** The *degree* of a vertex $v$ in a graph $G$, written $d(v)$ or $d_G(v)$, is the number of edges incident to $v$, except that each loop at $v$ counts twice. The maximum degree is written as $\Delta(G)$, the minimum degree is $\delta(G)$, and the graph $G$ is called *regular* if $\Delta(G) = \delta(G)$. It is $k$-regular if the common degree is $k$, for example a complete graph $K_5$ with 5 vertices, each vertex has degree 4 and it is a regular graph of degree 4. Fig. 1 gives the examples of regular graphs with various vertices.

**Definition 2:** The *neighborhood* of $v$, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to $v$. The number of vertices in $G$ is known as *order* of the graph $G$ and is equal to $|V(G)|$ and the size of the graph can be found as $|E(G)|$, where $E(G)$ is the number of edges in $G$ (also discussed in the Unit 1).

Now one basic question arises is that If we have a graph with $n$ vertices then in how many ways we can connect the given vertices i. e. how many edges occur?

This question can be well answered using the handshaking lemma as discussed below. The name handshaking lemma arises from the fact that a graph can be used to represent a group of people shaking hands in a party. In such graph, people are represented by the vertices, and an edge is included whenever the corresponding people have shaken hands.

**Theorem 1:** *(THE HANDSHAKING LAMA)* In any graph, the sum of all the vertex-degrees is equal to twice the number of edges.

Since each edge has two ends; it must contribute exactly 2 to the sum of the degrees. The complete proof can be read from the *Text Book DW*.

**Remarks 1:** There are some important consequences of the handshaking lemma as given:

1) In any graph, the sum of all vertex-degrees is an even number.
2) In any graph, the number of vertices of odd degree is even.
3) If $G$ is a graph which has $n$ vertices and is regular of degree $r$, then $G$ has exactly $\frac{1}{2}n(r)$ edges.

Now you can start reading the following section keeping in mind the guidelines for reading the text book as given in Unit 1.

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**READ DW:** Chapter 1, Section 1.3, page 34 up to 1.3.26 on page 44

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**NOTES:**

i) *Page 35, definition 1.3.7*
Hypercubes $Q_k$ is the simple graph whose vertices are the $k$-tuples with entries in \{0,1\} (binary values) and whose edges are the pairs of $k$-tuples that differ in exactly one position. For examples

\begin{align*}
Q_1 & \quad Q_2 \\
00 & \quad 00 \\
10 & \quad 01
\end{align*}

Fig. 1

Note that $Q_k$ has $2^k$ vertices and is regular of degree $k$. It can also be verified from the Fig. 1 as $Q_1$ has $2^1=2$ vertices, $Q_2$ has $2^2=4$ vertices and $Q_3$ has $2^3=8$ vertices. It follows from consequence 3 of the hand shaking lemma that $Q_k$ has $k \times 2^{k-1}$ edges which can be verified again from the Fig. 1 as $Q_1$ has $1 \times 2^0=1$ edge and $Q_3$ has $3 \times 2^2=12$ edges. The parity of a vertex in $Q_k$ is the parity of the number of 1’s in its name, even or odd. Each edge of $Q_k$ has an even vertex and an odd vertex as endpoints. So we can separate this graph into a set of odd vertices and even vertices and then we can color them in two different colors. In fact we can say that $Q_k$ is bipartite. We can also see that the graph $Q_k$ is a regular graph as we can check that $Q_1$ has $\Delta(G) = \delta(G) = 1$, $Q_2$ has $\Delta(G) = \delta(G) = 2$ and $Q_3$ has $\Delta(G) = \delta(G) = 3$ for all the vertices. Also the graph $Q_k$ has order $V(Q_k) = 2^k$ and size $E(Q_k) = k \cdot 2^{k-1}$.

ii) \quad Page 37, example 1.3.10

Let us us look at the Petersen graph $G$ as given in the Fig. 2. Is this graph bipartite?

We have $V(G) = 10$, $E(G) = 13$. The maximum degree is $\Delta(G) = 3$ and the minimum degree is $\delta(G) = 3$ which implies that this graph is 3-regular. The Petersen graph $G$ has ten 6-cycles (abcdefa, abcjgha, abigjfa and so on). Also this graph has girth 5 (smallest cycle).
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\textbf{iii) Proof of Corollary 1.3.5, page 35}

Let \( G \) be a graph and let \( u_1, u_2, \ldots, u_k \) denote the vertices of odd degrees and let \( v_1, v_2, \ldots, v_\lambda \) denote the vertices of even degree.

From Proposition 1.3.3,

\[ \sum_{v} d(v) = 2e(G). \]

Therefore \[ \sum_{i=1}^{k} d(u_i) + \sum_{i=1}^{\lambda} d(v_i) = 2e(G), \] an even number.

Further, \[ \sum_{i=1}^{\lambda} d(v_i) \] is even since each degree is even. Hence \[ \sum_{i=1}^{k} d(u_i) = 2e(G) - \sum_{i=1}^{\lambda} d(v_i) \] is also even. But \( d(u_i) \) is odd for each \( i \).

Hence \( k \), the number of vertices of odd degree must be even.

\textbf{iv) Page 36, Line 12}

\( e(Q_k) = k \cdot 2^{k-1} \).

This can easily be seen using consequence 3 of Handshaking Lemma given in remark 1.

But for the proof we use method of induction on \( k \).

Consider \( Q_1 \) in Fig.1, we have

\[ e(Q_1) = 1 = 1 \cdot 2^1 \]

Similarly \( e(Q_2) = 4 = 2 \cdot 2^1 \).

Now assume that result is true for graph \( Q_{k-1} \). Then we have

\[ e(Q_{k-1}) = k - 1 \cdot 2^{k-2} \]

In \( Q_{k-1} \) there are \( 2^{k-1} \) vertices.

Now append 0 to each vertex name in a copy of \( Q_{k-1} \) and append 1 to each vertex name in another copy of \( Q_{k-1} \). Add edges joining the vertices from the two copies whose first \( k - 1 \) coordinates are equal. The resultant graph is \( Q_k \).

Also \[ e(Q_k) = e(Q_{k-1}) + e(Q_{k-1}) + 2^{k-1} \]

\[ = (k - 1) \cdot 2^{k-2} + (k - 1) \cdot 2^{k-2} + 2^{k-1} \] by assumption

\[ = 2^{k-2}(k - 1 + k - 1 + 2) \]

\[ = 2^{k-2}(2k) = k \cdot 2^{k-1}. \]

Hence the proof.

Let's do some examples based on the above concepts.

\textbf{Example 1}: Let us find the degrees of each of the vertex of a graph given in Fig. 3
As we can see there are only two edges incident to $u$ so we can write $d(u) = 2$. Vertex $v$ has one cycle of length 1 and another cycle of length 2. So $d(v) = 5$. Similarly $d(w) = 4$ and $d(z) = 5$. Also $\Delta(G) = 5$, $\delta(G) = 2$. Since $\Delta(G) \neq \delta(G)$, this graph is not a regular graph. Also $N(u) = \{v, w\}$, $N(w) = \{u, v, z\}$ and so on.

**Example 2:** If $u, v$ are the only vertices of odd degrees in a graph $G$, then let us show that $G$ contains a path $u, v$-path.

If $u$ and $v$ are the only vertices of odd degree in a graph $G$ then $u$ and $v$ have to be in the same component of $G$ since any component of $G$ is a simple graph on its own and must contain even number of vertices of odd degree. Hence $G$ contains a $u - v$ path.

**Example 3:** In a class with nine students, each student sends valentine cards to three others. Let us determine whether it is possible that each student receive cards from the same three students to whom he or she sent cards.

Suppose it is possible. Convert the given problem into a graph theoretic problem by considering each student as a vertex and an edge between two vertices if the corresponding students send cards to each other. So, we get a graph with 9 vertices each of degree 3. This is not possible as the number of vertices of odd degree in a graph is even.

Hence it is not possible that each student receives cards from the same three students to whom he or she sent cards.

**Example 4:** Let $u$ and $v$ be adjacent vertices in a simple graph $G$. Let us prove that $uv$ belongs to at least $d(u) + d(v) - n(G)$ triangles in $G$.

Number of triangles in which $uv$ is an edge equals the number of vertices which are adjacent to both $u$ and $v$.

$$\text{No. of vertices adjacent to } u + \text{No. of vertices adjacent to } v \geq d(u) + d(v) - \text{No. of vertices of } G$$

Hence $uv$ belongs to at least $d(u) + d(v) - n(G)$ triangles in $G$. 

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**Fig. 3**
Now we want you to try some exercises.

E1) Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for loopless graphs?

E2) For each \( k \geq 3 \), determine the smallest \( n \) such that there is a simple \( k \)-regular graph with \( n \) vertices.

### 3.3 GRAPHIC SEQUENCES

In this section, we introduce you to the concept of graphic sequences. Let us now write all the vertex degrees together in a sequence and call it degree sequence \( d = (d_1, d_2, \ldots, d_n) \) where \( d_i \) is the degree of \( i \)th vertex, provided it satisfies some properties.

The degree sequence of a graph is the list of vertex degrees, usually written in nonincreasing order i.e. \( d_1 \geq d_2 \geq \ldots \geq d_n \).

What we can say about a negative degree of a vertex? Is it possible?

Suppose we are given a sequence of nonnegative numbers. Can we say that the given sequence is a degree sequence of a graph?

All these questions can be answered once we go through this section.

A degree sequence of nonnegative numbers is called a graphic sequence if it is a degree sequence of a **simple graph**. In other words, a graphic sequence is a sequence of nonnegative integers \( d = (d_1, d_2, \ldots, d_n) \), if there exists a simple graph whose degree sequence is \( d \). From the first consequence of hand shaking lemma, it is clear that a necessary condition for a sequence \( d = (d_1, d_2, \ldots, d_n) \) to be graphic is that \( \sum_{i=1}^{n} d_i \) is even and \( d_i \geq 0 \), \( 1 \leq i \leq n \).

There is very useful result which tell us how to decide whether a given sequence of integers is a graphic sequence or not and that result is known as **Havel and Hakimi theorem**. For example, for \( n > 1 \), an integer sequence, say \( d \), of size \( n \) is a graphic sequence if and only if \( d' \) is graphic, where \( d' \) is obtained from \( d \) by deleting its largest element \( \Delta \) and subtracting 1 from its next largest elements. The only 1-element graphic sequence is \( d_1 = 0 \). You will learn more about the **Havel and Hakimi theorem** from the Text Book DW.

You can start with reading the following:

READ DW: Chapter 1, Section 1.3, 1.3.27 on page 44 up to 1.3.33 on page 47.

### NOTES:

i) page 44 Proposition 1.3.28
Let us illustrate the construction in Proposition 1.3.28 with an example. Consider the degree sequence (4, 2, 3, 3).

Let \( v_1, v_2, v_3, v_4 \) be the vertices of a graph having the given degree sequence. Now \( v_3 \) and \( v_4 \) have odd degree. Make an edge having these two vertices as endpoints. The remaining degree needed at \( v_3 \) and \( v_4 \) is 2. Place 1 loop at each vertex \( v_3 \) and \( v_4 \). For \( v_1 \) and \( v_2 \), the degrees are 4 and 2 respectively. Place 2, 1 loops respectively at \( v_1, v_2 \).

This construction is not possible for loopless (simple) graphs.

\[ v_1, v_2, v_3, v_4 \]

**Example 5**: Let us find out whether a sequence \( d = (7, 6, 3, 3, 2, 1, 1, 1) \) is a graphic sequence?

We can see that each term of \( d \) is a nonnegative integer and the sum of the terms is even. Indeed, if \( d \) were graphic, there must exist a simple graph \( G \) with eight-vertices whose degree sequence is \( d \). Let \( v_0 \) and \( v_1 \) be the vertices of \( G \) whose degrees are 7 and 6, respectively. Since \( G \) is simple, \( v_0 \) is adjacent to all the remaining vertices of \( G \), and \( v_1 \), beside \( v_0 \), should be adjacent to another five vertices. This means that in \( V - \{ v_0, v_1 \} \) there must be at least five vertices of degree at least 2. But this is not the case.

**Example 6**: Let us show that in any group of \( n \) persons \((n \geq 2)\), there are at least two with the same number of friends.

Denote the \( n \) persons by \( v_1, v_2, \ldots, v_n \). Let \( G \) be the simple graph with vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \) in which \( v_i \) and \( v_j \) are adjacent if, and only if, the corresponding persons are friends. Then the number of friends of \( v_i \) is just the
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degree of \( v_i \) in \( G \). Hence, to solve the problem, we must prove that there are two vertices in \( G \) with the same degree. If this were not the case, the degrees of the vertices of \( G \) must be 0, 1, 2, ..., \((n-1)\) in same order. However, a vertex of degree \((n-1)\) must be adjacent to all the other vertices of \( G \), and consequently there cannot be a vertex of degree 0 in \( G \). This contradiction shows that the degrees of the vertices of \( G \) cannot all be distinct, and hence at least two of them should have the same degree.

Example 7: Write down the degree-sequence of each of the following graphs and also verify the handshaking lemma for each graph.

![Graphs](image)

Fig. 3

The degree sequence for fig. 3(a) is \((4, 4, 2, 1, 1, 1, 1, 1)\) as it is written in non-decreasing form. There are 2 vertices whose degree is 4, one vertex with degree 2 and 6 vertices with degree just 1. For Fig. 3(b) we have \((4, 4, 4, 4)\) and fig. 3(c) has \((5, 5, 4, 4, 3, 1, 0)\). In Fig. 3(a) the sum of the degrees is 16 and the number of edges is 8; in graph 3(b) the sum of the degrees is 20 and the number of edges is 10; in graph 3(c) the sum of the degrees is 22 and the number of edges is 11. So in each case the sum of the degrees is exactly twice the number of edges.

Example 8: Let us verify consequence 3 of the handshaking lemma for each of the graphs in Fig. 3.

(a) \( n=5, r=2 \), so the number of edges is \( \frac{1}{2} (5)(2)=5 \);
(b) \( n=10, r=3 \), so the number of edges is \( \frac{1}{2} (10)(3)=15 \);
(c) \( n=12, r=5 \), so the number of edges is \( \frac{1}{2} (12)(5)=30 \).

Example 9: Let us show that the list of integers \((7, 6, 5, 4, 3, 2)\) is not graphic.

Suppose the given integer list is graphic. Let \( G \) be a simple graph which realizes the list. Then \( G \) has 6 vertices. Hence the maximum degree of any
vertex in G is 5 but the number 7 is there in the list. This is a contradiction. Hence the given list is not graphic.

Example 10: Let us show that the list (6, 6, 5, 4, 3, 3, 1) is not graphic.

Suppose the given list is graphic and let G is a realization of the given list. Then G has 7 vertices and there are 3 vertices of odd degree. This is a contradiction since the number of vertices of odd degree in a graph is even. So, the given list is not graphic.

Example 11: Let us check whether the integer list d = (4, 4, 4, 2, 2, 2) is graphic and if so find a realization of d.

Let d = (4, 4, 4, 2, 2, 2). Using Havel-Hakimi theorem, we get

d’ = (3, 3, 1, 1, 2) = (3, 3, 2, 1, 1) in non increasing order

d” = (2, 1, 0, 1) = (2, 1, 1, 0) in non increasing order.

We expect you to try some exercises.

E3) Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.
   (i) (5,5,4,3,2,2,1)  (ii) (5,5,4,4,2,2,1,1)  (iii) (5,5,3,2,2,1,1)
   (iv) (5, 5, 5, 4, 2, 1, 1, 1)

E4) Show that every cubic graph (regular graph of degree 3) has an even number of vertices.

E5) Let G be a graph with n vertices and e edges all of whose vertices have degree k or k + 1. If G has t > 0 vertices of degree k, then show that t = n(k + 1) − 2e.

3.4 SUMMARY

In this unit, we have covered the following:

1. The sum of the degrees of the vertices in a graph is an even number and equals twice the number of edges of G.
2. The number of vertices of odd degree in a graph is even.
3. The k-dimensional hypercube Q_k has 2^k vertices and k2^{k-1} edges.
4. Q_k is k-regular.
5. Any k-regular bipartite graph (k>0) has equal number of vertices in both partitions.
6. The Petersen graph has ten b-cycles.
7. When loops and multiple edges are allowed, the non-negative integers \( d_1, d_2, \ldots, d_n \) are the vertex degrees of some graph if and only if \( \sum d_i \) is even.

8. We use Havel-Hakimi theorem to find whether a given sequence of non-negative integers is a graphic sequence or not.

9. If \( G \) and \( H \) are two simple graphs with the same vertex set \( V \), then \( d_G(v) = d_H(v) \forall v \in V \) if and only if there is a sequence of 2-switches that transform \( G \) into \( H \).

### 3.5 HINTS/SOLUTIONS

**E1)** Let \( G \) be a graph with at least two vertices. To prove that \( G \) has two vertices of equal degree.

Let us assume the contrary. That is, let \( G \) have \( n \) vertices and let the degrees all be different. The implies that degrees of the vertices are \( 0, 1, 2, \ldots, n - 1 \) (since 0 and \( n - 1 \) are the minimum and maximum possible degrees. This is a contradiction since a vertex of degree \( n - 1 \) is adjacent to all the remaining vertices and so there can not exist a vertex of degree 0.

Hence we conclude that there are two vertices of equal degree in \( G \).

**E2)** There exists a \( k \)-regular graph with \( k + 1 \) vertices which is a complete graph on \( k + 1 \) vertices. Also \( k + 1 \) is the smallest \( n \) such that there is a simple \( k \)-regular graph with \( n \) vertices.

**E3**) (i) Let \( d = (5, 5, 4, 3, 2, 2, 2, 1) \)

\[
\begin{align*}
\text{Let } d' &= (4, 3, 2, 1, 1, 2) \\
&= (4, 3, 2, 2, 1, 1, 1) \text{ in non increasing order} \\
\text{Then } d'' &= (2, 1, 1, 0, 1, 1) \\
&= (2, 1, 1, 1, 1, 0) \text{ in non increasing order.} \\
\text{Finally } d''' &= (0, 0, 1, 1, 0) \\
&= (1, 1, 0, 0, 0)
\end{align*}
\]

Using Havel-Hakimi theorem, \( d''' \) is realizable \( \Rightarrow d'' \) is realizable \( \Rightarrow d' \) is realizable \( \Rightarrow d \) is realizable. Hence the given sequence is a graphic sequence.

Similarly you can try (ii), (iii) and (iv).

**E4)** **Hint:** Use Proposition 1.3.3.

**E5)** Since \( G \) has \( t \) vertices of degree \( k \), the remaining \( (n - t) \) vertices are of degree \( k + 1 \).

\[
\sum_{v} \deg v = 2e = tk + (n - t)(k + 1)
\]

\[
= tk + nk + tk - t
\]

This implies that \( t = n(k + 1) - 2e \).