Again, check that $\cdot$ satisfies associativity on $\mathbb{H}$, and 1 is the identity w.r.t. $\cdot$.
Finally, check that $\cdot$ distributes over $+$ in $\mathbb{H}$.

E9) i) $S \neq \emptyset$, since $\mathbf{0} \in S$.
Also check that for $A, B \in S$, $A - B \in S$ and $AB \in S$.
Hence, S is a subring of $\mathbb{M}_2(\mathbb{Z})$.

ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$. Hence S has identity, which is the same as that of $\mathbb{M}_2(\mathbb{Z})$.

E10) Since A is a subring of B, $A \neq \emptyset$ and $\forall\ x, y \in A$, $x - y \in A$ and $xy \in A$. Here the addition and multiplication are those defined on B. But these are the same as those defined on C since B is a subring of C. Thus, A is a subring of C.

E11) No, there are several counterexamples. For instance, take $R = \mathbb{Z}$, and the subset $\{1\}$. This is not a subring, since $(\{1\}, +)$ is not a group. In fact, any finite subset of $\mathbb{Z}$, apart from $\{0\}$, will not be a subring of $\mathbb{Z}$.

E12) S_1 and S_2 are subrings of R_1 and R_2, so that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.
$\therefore S_1 \times S_2 \neq \emptyset$.
Now, let (a, b) and $(a', b') \in S_1 \times S_2$. Then $a, a' \in S_1$ and $b, b' \in S_2$. As S_1 and S_2 are subrings, $a - a', a \cdot a' \in S_1$ and $b - b', b \cdot b' \in S_2$.
(We are using $+$ and $\cdot$ for both R_1 and R_2 here, for convenience.)
Hence, $(a, b) - (a', b') = (a - a', b - b') \in S_1 \times S_2$, and
$(a, b) \cdot (a', b') = (a \cdot a', b \cdot b') \in S_1 \times S_2$.
Thus, $S_1 \times S_2$ is a subring of $R_1 \times R_2$.

E13) $2\mathbb{Z} \times \mathbb{R}$, $3\mathbb{Z} \times \{0\}$ are two among infinitely many examples.

E14) Note that the two sets are $\bar{3}\mathbb{Z}_6$ and $\bar{2}\mathbb{Z}_6$. You know that they are subrings of $\mathbb{Z}_6$. Now, by elementwise multiplication you can check that $rx \in \bar{3}\mathbb{Z}_6\ \forall\, r \in \mathbb{Z}_6$ and $x \in \bar{3}\mathbb{Z}_6$.
(For instance, $\bar{5}.\bar{3} = \overline{15} = \bar{3} \in \bar{3}\mathbb{Z}_6$.)
You can similarly see that $rx \in \bar{2}\mathbb{Z}_6\ \forall\ r \in \mathbb{Z}_6$, $x \in \bar{2}\mathbb{Z}_6$.
Thus, $\bar{3}\mathbb{Z}_6$ and $\bar{2}\mathbb{Z}_6$ are ideals of $\mathbb{Z}_6$.

E15) For example, $S = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \middle|\, a, b \in \mathbb{Z} \right\}$ is a subring of $\mathbb{M}_2(\mathbb{Z})$. However, it is not an ideal of $\mathbb{M}_2(\mathbb{Z})$ since, for example, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin S$.

E16) $I_a \neq \emptyset$, since $0 \in I_a$.

$f, g \in I_a \Rightarrow (f-g)(a) = f(a) - g(a) = 0 \Rightarrow f - g \in I_a$.
$f \in I_a,\ g \in C[0,1] \Rightarrow (fg)(a) = f(a)g(a) = 0.g(a) = 0 \Rightarrow fg \in I_a$.
$\therefore I_a$ is an ideal of $C[0, 1]$.

You can check that if $f \in J_a$, then $-f \notin J_a$. Hence J_a is not even a subring of $C[0,1]$, and hence certainly not an ideal of $C[0,1]$.

E17) Ra is a subring of R (see Example 4).
Also, for $r \in R$ and $xa \in Ra$,
$r(xa) = (rx)a \in Ra$.
$\therefore Ra$ is a left ideal of R.
You can similarly show that aR is a right ideal of R.

E18) You can check that any element of the ideal generated by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is of the form $\sum_{i=1}^{n} \begin{bmatrix} a_i r_i & a_i \delta_i \\ c_i r_i & c_i \delta_i \end{bmatrix}$, $a_i, c_i, r_i, \delta_i \in \mathbb{Z}$.

However, any element of $\mathbb{M}_2(\mathbb{Z})\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is of the form $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$, for $a, c \in \mathbb{Z}$.

So, for example, $\begin{bmatrix} 6 & 10 \\ -3 & -5 \end{bmatrix}$ is in the ideal, but not in the left ideal, generated by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Thus, the two sets are not equal.

E19) $\bar{3}\mathbb{Z}_{10} = \{\bar{3}x \mid x \in \mathbb{Z}_{10}\} = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}\}$
$= \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \bar{2}, \bar{5}, \bar{8}, \bar{1}, \bar{4}, \bar{7}\}$
$= \mathbb{Z}_{10}$.
$\bar{5}\mathbb{Z}_{10} = \{\bar{0}, \bar{5}\}$.

E20) i) Since $\bar{6}^2 = \overline{36} = \bar{0}$ in $\mathbb{Z}_9$, $\bar{6}$ is nilpotent in $\mathbb{Z}_9$.

ii) Let $N = \{a \in R \mid a^n = 0$ for some positive integer $n\}$. Then $0 \in N$.
Also, if $a, b \in N$, then $a^n = 0$ and $b^m = 0$ for some positive integers m and n.
Now, $(a-b)^{m+n} = \sum_{r=0}^{m+n} {}^{m+n}C_r a^r (-b)^{m+n-r}$ (since R is commutative).
For each $r = 0, 1, \ldots, m+n$, either $r \geq n$ or $m+n-r \geq m$, and hence, either $a^r = 0$ or $b^{m+n-r} = 0$. Thus, each term $a^r b^{m+n-r} = 0$. So $(a-b)^{m+n} = 0$.
Thus, $a - b \in N$ whenever $a, b \in N$.
Finally, if $a \in N, a^n = 0$ for some positive integer n, and hence, for any $r \in R, (ar)^n = a^n r^n = 0$, i.e., $ar \in N$.

So, N is an ideal of R.

E21) $\forall\, x \in \mathbb{Z},\ x^n = 0 \Rightarrow x = 0$. Hence, the nil radical of $\mathbb{Z} = \{0\}$. Similarly, the nil radical of $\mathbb{R}$ is $\{0\}$.

Let the nil radical of $\mathbb{Z}_8$ be N. Then $\overline{0} \in N$.

$\overline{1} \notin N$ since $\overline{1}^n = \overline{1} \neq \overline{0}$ for all n.

$\overline{2}^3 = \overline{0} \Rightarrow \overline{2} \in N$.

$\overline{3}^n \neq \overline{0}\ \forall\, n$. $\therefore\ \overline{3} \notin N$.

Similarly, you can check that $\overline{4},\ \overline{6} \in N$ and $\overline{5},\ \overline{7} \notin N$.

$\therefore\ N = \{\overline{0},\ \overline{2},\ \overline{4},\ \overline{6}\} = \langle \overline{2} \rangle$.

For any $A \in \wp(X)$, $A^n = A \cap A \cap \ldots \cap A = A\ \forall\, n$.

Thus, $A^n = \emptyset$ iff $A = \emptyset$. Thus, the nil radical of $\wp(X)$ is $\{\emptyset\}$.

E22) i) For any $a \in I$ and $b \in J$, $ab \in I$ and $ab \in J$.
Thus, $ab \in I \cap J$. Since $I \cap J$ is an ideal, any finite sum of such elements will also be in $I \cap J$. Thus, $IJ \subseteq I \cap J$.
Next, by definition $I \cap J \subseteq I$ and $I \cap J \subseteq J$.
Also, $I \subseteq I + J$, $J \subseteq I + J$.

ii) Let A be an ideal of R containing I as well as J. Then certainly $I + J \subseteq A$. Thus, (ii) is proved.

iii) Let B be an ideal of R such that $B \subseteq I$ and $B \subseteq J$. Then certainly, $B \subseteq I \cap J$. Thus, (iii) is proved.

iv) We want to show that $I \cap J \subseteq IJ$.
Let $x \in I \cap J$. Then $x \in I$ and $x \in J$.
Since $1 \in R = I + J$, $1 = i + j$, for some $i \in I$ and $j \in J$.
$\therefore x = x.1 = xi + xj = ix + xj \in IJ$.
Thus, $I \cap J \subseteq IJ$.

E23) Check that $1 + I$ is the identity of R/I, where 1 is the identity of R.

E24) Since $1 \in I$, you know that $I = R$.
$\therefore\ R/I = \{\overline{0}\}$.

E25) Let $x + N \in R/N$ be a nilpotent element.

Then $(x + N)^n = N$ for some positive integer n.

$\Rightarrow x^n \in N$ for some positive integer n.

$\Rightarrow (x^n)^m = 0$ for some positive integer m.

$\Rightarrow x^{nm} = 0$ for some positive integer nm.

$\Rightarrow x \in N$

$\Rightarrow x + N = 0 + N$, the zero element of R/N.

Thus, R/N has no non-zero nilpotent elements.

E26) For $x, y \in S$,

$i(x+y) = x+y = i(x)+i(y)$, and
$i(xy) = xy = i(x)i(y)$
$\therefore i$ is a homomorphism.
$\text{Ker } i = \{x \in S \mid i(x) = 0\} = \{0\}$.
$\text{Im } i = \{i(x) \mid x \in S\} = S$.

E27) $f : \mathbb{Z} \to 2\mathbb{Z} : f(x) = 2x$ is a group homomorphism.
However, $f(2.3) = f(6) = 12$, and $f(2).f(3) = 4.6 = 24$.
Thus, $f(2.3) \neq f(2)f(3)$.
$\therefore f$ is not a ring homomorphism.

E28) For any $x, y \in R_1, f(x+y) = 0 = 0+0 = f(x)+f(y)$, and
$f(xy) = 0 = 0.0 = f(x).f(y)$. $\therefore f$ is a homomorphism.
$\text{Ker } f = \{x \in R_1 \mid f(x) = 0\} = R_1$.
$\text{Im } f = \{0\}$.

E29) Since $\text{Im } f \neq \mathbb{M}_2(\mathbb{Z})$ (e.g., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \text{Im } f$), f is not surjective, and hence not an isomorphism.

E30) For any $(a,b),(c,d) \in A \times B$,
$\pi((a,b)+(c,d)) = \pi(a+c, b+d) = a+c = \pi(a,b)+\pi(c,d)$,
$\pi((a,b)(c,d)) = \pi(ac, bd) = ac = \pi(a,b)\pi(c,d)$.
Thus, π is a ring homomorphism.
$\text{Ker } \pi = \{(a,b) \in A \times B \mid a = 0\} = \{0\} \times B$.
$\text{Im } \pi = \{\pi(a,b) \mid (a,b) \in A \times B\} = \{a \mid (a,b) \in A \times B\} = A$.

E31) For $f, g \in C[0,1]$,
$$\begin{aligned}\phi(f+g) &= ((f+g)(0), (f+g)(1)) \\ &= (f(0), f(1)) + (g(0), g(1)) \\ &= \phi(f) + \phi(g), \text{ and}\end{aligned}$$
$$\begin{aligned}\phi(fg) &= (fg(0), fg(1) = (f(0), g(0))(f(1), g(1)) \\ &= \phi(f)\phi(g).\end{aligned}$$
$\therefore \phi$ is a homomorphism.
ϕ is not an isomorphism, because, for example,
$f : [0, 1] \to \mathbb{R} : f(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}$ is in $\text{Ker } \phi$ and $f \neq \mathbf{0}$.

E32) For any $r \in R_2 \, \exists \, s \in R_1$ such that $r = f(s)$. Now,
$r = f(s) = f(s \cdot 1) = f(s) \cdot f(1) = rf(1)$. Similarly, $r = f(1) \cdot r \;\forall\; r \in R_2$.
Hence $f(1)$ is the identity of R_2.

E33) We will prove (ii). The proof of (i) is similar.
Firstly, since $T \neq \emptyset, f^{-1}(T) \neq \emptyset$.
Next, $a, b \in f^{-1}(T)$.
$\Rightarrow f(a), f(b) \in T$

$\Rightarrow f(a)-f(b)\in T$ and $f(a)f(b)\in T$
$\Rightarrow f(a-b)\in T$ and $f(ab)\in T$
$\Rightarrow a-b\in f^{-1}(T)$ and $ab\in f^{-1}(T)$
$\Rightarrow f^{-1}(T)$ is a subring of R_1.

E34) Consider the inclusion $i:\mathbb{Z}\to\mathbb{R}: i(x)=x$. You know that $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. But is $i(2\mathbb{Z})$ (i.e., $2\mathbb{Z}$) an ideal of $\mathbb{R}$? No. For example,
$2\in 2\mathbb{Z}$, $\frac{1}{4}\in\mathbb{R}$, but $(2)\left(\frac{1}{4}\right)=\frac{1}{2}\notin 2\mathbb{Z}$.

E35) Here we will prove (i), and leave (ii) to you.
Firstly, since I is a subring of R_1, $f(I)$ is a subring of R_2.
Secondly, take any $f(x)\in f(I)$ and $r\in R_2$. Since f is surjective, $\exists s\in R_1$ such that $f(s)=r$.
Then $rf(x)=f(s)f(x)=f(sx)\in f(I)$, since $sx\in I$.
Similarly, $f(x)r\in f(I)$.
Thus, $f(I)$ is an ideal of R_2.

E36) Example 11: $\mathbb{Z}_6/\{\bar{0},\bar{3}\}\simeq\mathbb{Z}_3$, that is, $\mathbb{Z}_6/\langle\bar{3}\rangle\simeq\mathbb{Z}_3$.

Example 12: Ker $\phi=\{f\in C[0,1] \mid f\left(\frac{1}{2}\right)=0\}$. Also, Im $\phi=\mathbb{R}$.

So $C[0,1]/\text{Ker }\phi\simeq\mathbb{R}$.

Example 13: Ker $f=\{0\}$, Im $f=\{nI \mid n\in\mathbb{Z}\}$. So, FTH says that
$\mathbb{Z}\simeq\{nI \mid n\in\mathbb{Z}\}$.

Example 14: $\wp(X)/\text{Ker } f\simeq\wp(Y)$.

E37) Define $\pi:\mathbb{C}^5\to\mathbb{C}^2:\pi(z_1, z_2, z_3, z_4, z_5)=(z_1, z_2)$.
Then π is well-defined.
Also $\pi[(z_1, z_2, z_3, z_4, z_5)+(w_1, w_2, w_3, w_4, w_5)]$
$=\pi[(z_1+w_1), (z_2+w_2), (z_3+w_3), (z_4+w_4), (z_5+w_5)]$
$=(z_1+w_1, z_2+w_2)=(z_1, z_2)+(w_1, w_2)$
$=\pi(z_1, z_2, z_3, z_4, z_5)+\pi(w_1, w_2, w_3\ w_4, w_5)$.
Similarly, check that $\pi(\mathbf{z}.\mathbf{w})=\pi(\mathbf{z}).\pi(\mathbf{w})$, where $\mathbf{z}, \mathbf{w}\in\mathbb{C}^5$.
For any $(z_1, z_2)\in\mathbb{C}^2$, $\pi(z_1, z_2, 0, 0, 0)=(z_1, z_2)$. Hence Im $\pi=\mathbb{C}^2$.
Ker $\pi=\{(z_1, z_2, z_3, z_4, z_5)\in\mathbb{C}^5 \mid z_1=0=z_2\}=\{(0, 0, z_3, z_4, z_5)\in\mathbb{C}^5\}$.
So, by FTH, $\mathbb{C}^5/\text{Ker }\pi\simeq\mathbb{C}^2$ …(1)
Now define $f:\text{Ker }\pi\to\mathbb{C}^3: f(0, 0, z_3, z_4, z_5)=(z_3, z_4, z_5)$.
Then f is an isomorphism. Hence Ker $\pi\simeq\mathbb{C}^3$. …(2)
Thus, by (1) and (2) we get
$\mathbb{C}^5/\mathbb{C}^3\simeq\mathbb{C}^2$.

E38) Since I is an ideal of R and $I\subseteq S+I$, it is an ideal of $S+I$.

Thus, $(S+I)/I$ is a well-defined ring.

Define $f : S \to (S+I)/I : f(x) = x + I$.

Then, you can check that f is well-defined, $f(x+y) = f(x) + f(y)$, and $f(xy) = f(x)f(y) \forall x, y \in S$.

Further, check that f is surjective and $\text{Ker } f = S \cap I$.

Thus, the FTH tells us that $S/(S \cap I) \simeq (S+I)/I$.

E39) Define $f : R/J \to R/I : f(r+J) = r + I$.

You should check that f is well-defined, f is a ring homomorphism, f is surjective and $\text{Ker } f = I/J$.

Thus, I/J is an ideal of R/J, and by the FTH, we see that $(R/J)/(I/J) \simeq R/I$.