- $G \simeq S_5 \times \mathbb{Z}/5\mathbb{Z}$ in case $srs^{-1} = r^{11}$.
- $G \simeq D_{30}$ in case $srs^{-1} = r^{14}$.

vi) Show that these cases are distinct.
[**Hint:** Find $Z(G)$ in each case, and look at their orders.]

E15) Give a presentation of all possible groups of order 10.

Now, from Unit 2, you know that a group of order p^2 is abelian, where p is a prime. What about groups of order $2p^2$, for an odd prime p? We give the result in this case, without proof.

Theorem 6: Let G be a group of order $2p^2$, for an odd prime p. Then there are 5 isomorphism classes of G:

i) G is cyclic;

ii) $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$;

iii) G is the dihedral group D_{2p^2};

iv) $G \simeq \mathbb{Z}/p\mathbb{Z} \times D_{2p}$;

v) $G = \left\langle s, r_1, r_2 \,\middle|\, s^2, r_1^p, r_2^p, r_1r_2r_1^{-1}r_2^{-1}, sr_1s^{-1}r_1, sr_2s^{-1}r_2 \right\rangle$. ■

Theorem 6 tells us, for example, that there are 3 non-abelian groups of order 50, upto isomorphism.

Why don't you try an exercise now?

E16) What are the possible presentations of groups of order 242?

With this we come to the end of our discussion on group presentations. We now summarise what you have studied in this unit.

6.5 SUMMARY

In this unit, you have studied the following points.

1) A group F is a free group on a set S if $\exists\, i : S \to F$ s.t. whenever there is a group G and a map $\alpha : S \to G$, there is a unique group homomorphism $\tilde{\alpha} : F \to G$ that extends α. Here S generates F.

2) Given any set S, a method for constructing a free group $F(S)$ on it.

3) A finite group is not free on any set.

4) Every group is isomorphic to the quotient group of a free group.

5) G has a presentation $\langle X|R\rangle$ if G is generated by the elements of X, with defining relations being the elements of R.

6) There are 5 groups of order 8, up to isomorphism.

7) There are 5 groups of order 12, up to isomorphism.

8) There are 4 groups of order 30, up to isomorphism.

9) There are 5 groups of order $2p^2$, up to isomorphism, where p is an odd prime.

6.6 SOLUTIONS / ANSWERS

E1) For $u \in W(S), u \sim u$, trivially. So ~ is reflexive.
Next, let $u, v \in W(S)$ such that $u \sim v$. Suppose v is obtained from u by deleting $x_1x_1^{-1},\ldots,x_kx_k^{-1}$ and adding $y_1y_1^{-1},\ldots,y_ry_r^{-1}$ for $x_i, y_j \in S \cup S^{-1}$.
Then u is obtained from v by adding $x_ix_i^{-1}, i=1,\ldots,k$, and deleting $y_iy_i^{-1}, i=1,\ldots,r$, in the same places they were deleted/added from.
Thus, $v \sim u$. So, ~ is symmetric.
Finally, if $u \sim v$ and $v \sim w, u, v, w \in W(S)$, then $u \sim w$ by combining both sets of additions and deletions. Hence, ~ is transitive.

E2) i) The reduced form is $a^2b^2a^3c^3b^{-2}$, and its inverse is $b^2c^{-3}a^{-3}b^{-2}a^{-2}$.

ii) $a^{-1}b^3a^4c^4a^{-1}$ and $ac^{-4}a^{-4}b^{-3}a$, respectively.

E3) You should check that this follows from the universal mapping property.

E4) i) If $S=\{a\}$, then any reduced word is just an integral power of a. Looking at the multiplication in $F(S)$, you can see that this group is the cyclic group generated by a. Further, $[a],[a^2],\cdots$ are all distinct. Hence $F(S)$ is infinite.

ii) Consider $x, y \in S$. Then the word $x^{-1}y^{-1}xy$ is reduced and is different from the empty word. Hence $x^{-1}y^{-1}xy$ is not identity, i.e., the words x and y do not commute.

iii) Let w be a non-empty reduced word. Consider all reduced subwords t of w such that $w = tmt^{-1}$, where m is a reduced word. Let u be one such subword whose length is maximal among all such subwords t. Then $w = uvu^{-1}$ for some reduced word v (u could be e also!).
Since w is reduced, v is non-empty, and by maximality of $|u|$, there is no cancellation in the product vv, i.e., the reduced form of vv is vv. So, for any $n \geq 1, w^n = u\,v^n\,u^{-1}$ is a non-empty reduced word. Hence w can't be of order n.

E5) By the universal mapping property of free groups, the number of homomorphisms from F(S) to $\mathbb{Z}_4$ equals the number of distinct maps from $\{a, b\}$ to $\mathbb{Z}_4$, and this number is $2^4 = 16$.

E6) If $G = \{e\}$, then $F(G) = \{e\} = G$.
Let $|G| > 1$. Then $F(G)$ will be non-abelian, as you have shown in E4.
Hence $F(G) \neq G$.

E7) Let $F = F(S)$, where $S = \{a\}$. By E4(i), F is just the cyclic group $\langle a \rangle$.
Now the smallest normal subgroup of F containing a^n is $\langle a^n \rangle$. You should now check that F/N, i.e., $\langle a \rangle / \langle a^n \rangle$ is isomorphic to $\mathbb{Z}_n$.

E8) Let F be a free group generated by a set X. Since F is free, $F \simeq F(X)$. Hence, none of its generators satisfy any relations. So $F = \langle X | \emptyset \rangle$ is a presentation of F.

E9) We will show that $S_3 \simeq \langle a, b \mid a^3, b^2, b^{-1}aba \rangle$. First of all recall that S_3 is generated by $\alpha = (1, 2, 3)$ and $\beta = (1, 2)$. Moreover,
$\alpha^3 = 1, \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}$. …(1)
Let $S = \{a, b\}$ and $F = F(S)$. Then there is a homomorphism $\theta : F \to S_3$ such that $a \to \alpha$ and $b \to \beta$. θ is onto as S_3 is generated by α and β.
In view of Eqn 1, the normal subgroup N of F generated by $a^3, b^2, b^{-1}aba$ is contained in $\operatorname{Ker}\theta$. This gives a surjective homomorphism $\overline{\theta}$ from F/N to S_3 such that $aN \mapsto \alpha$ and $bN \mapsto \beta$. In particular, it tells us that F/N has at least 6 elements.
However, each element of F/N is one of N, aN, a^2N, bN, abN, a^2bN, as $a^3, b^2, b^{-1}aba$ lie in N. So, F/N has at most 6 elements.
Consequently, F/N has precisely 6 elements and $\overline{\theta}$ is an isomorphism.

You know $\mathbb{Z}^2$ is abelian, and generated by $(1, 0)$ and $(0, 1)$, with $\alpha\beta = \beta\alpha \; \forall \alpha, \beta \in \mathbb{Z}^2$. Let $S = \{a, b\}$ and $F = F(S)$. Then $\exists$ a homomorphism $\theta : F \to \mathbb{Z}^2 : \theta(a) = (1, 0), \theta(b) = (0, 1)$. As above, θ is onto. Consider the smallest normal subgroup N of $F(S)$ containing $\{aba^{-1}b^{-1}\}$. Then $N \subseteq \operatorname{Ker}\theta$. Take
$\overline{\theta} : F/N \to \mathbb{Z}^2 : \overline{\theta}(\overline{a}) = (1, 0), \overline{\theta}(\overline{b}) = (0, 1)$. Then, for any $(n, m) \in \mathbb{Z}^2$, $\overline{\theta}(\overline{a}^n \overline{b}^m) = (n, m)$. Hence $\overline{\theta}$ is onto.
Now, let $\overline{\theta}(\overline{x}) = (0, 0)$, where $x \in F$. Since $\overline{a}\overline{b} = \overline{b}\overline{a}$ in F/N, $\overline{x} = \overline{a}^n \overline{b}^m$ for some $n, m \in \mathbb{N}$.
So $\overline{\theta}(\overline{x}) = (0, 0) \Rightarrow (n, m) = (0, 0) \Rightarrow \overline{x} = \overline{e}$.
Thus, $\overline{\theta}$ is also 1-1, and hence $\overline{\theta}$ is an isomorphism. So $\mathbb{Z}^2 \simeq \langle a, b | aba^{-1}b^{-1} \rangle$.

E10) Let $G = \{g_1, \ldots, g_n\}$ be a finite group. Take $X = G$. Let $R = \{g_i g_j g_k^{-1} \mid i, j = 1, \ldots, n, \text{ where } g_i g_j = g_k \text{ in } G\}$.

Then $\theta: F(X) \to G$, extending $I: G \to G$, is s.t. $R \subseteq \operatorname{Ker}\theta$.

So, the smallest normal subgroup N containing R is in $\operatorname{Ker}\theta$.

So there is a surjection $\bar{\theta}: F(X)/N \to G : g_i N \mapsto g_i$

$\therefore \left|F(X)/N\right| \geq |G| = n.$

Also $F(X)/N = \{g_i N \mid i = 1, \ldots, n\}$.

So $\left|F(X)/N\right| \leq n.$

Hence $\left|F(X)/N\right| = n$, and $\bar{\theta}$ is an isomorphism.

Thus, $G = \langle X \mid R \rangle$ is a presentation of G.

E11) $\langle a \mid a^4 \rangle$, $\langle a, b \mid a^4, b^2, b^{-1} \rangle$ and $\langle a, b, c \mid a^8, a^{12}, b, c \rangle$, respectively.

Explicitly, in terms of elements of $\mathbb{Z}_4$, you can see these as $\langle \bar{1} \rangle$, $\langle \bar{1}, \bar{0} \rangle$, $\langle \bar{3}, \bar{0}, \bar{0} \rangle$, respectively.

E12) i) True, as has been described in Sec.6.3.

ii) True. For example, you have seen it in E11.

iii) False. For example, $\langle a \mid e \rangle$ is a finite presentation of the infinite cyclic group $\mathbb{Z}$.

iv) False. For example, $\langle a \mid a^4 \rangle$ and $\langle a \mid a^5 \rangle$ are presentations with one generator but they do not give isomorphic groups; the former is a presentation of $\mathbb{Z}_4$ and the latter is that of $\mathbb{Z}_5$.

E13) $\left|\mathbb{Z}/2\mathbb{Z} \times S_3\right| = 12.$ From E17 of Unit 3, you know that it has 3 Sylow 2-subgroups, each isomorphic to the Klein 4-group. It has a unique Sylow 3-subgroup. So it is the case (iv) of Theorem 5.

Thus $\mathbb{Z}/2\mathbb{Z} \times S_3 \simeq D_{12}$.

E14) i) Suppose neither T nor U is normal in G. Then, by Sylow's third theorem, there are 10 Sylow 3-subgroups and 6 Sylow 5-subgroups. The intersection of any two of these subgroups is the trivial subgroup. The union of all the Sylow 3-subgroups has cardinality 21 (each of the groups contains, in addition to the identity, two elements, neither of which is contained in any other subgroup). By a similar reasoning, the union of all Sylow 5-subgroups is 25. The only thing that is common to these two unions is the identity element. Therefore, the union of the Sylow 3- and Sylow 5-subgroups has cardinality 45. This is not possible because

G has order only 30. Thus, we reach a contradiction. So at least one of T or U is normal in G.

ii) Since at least one of T and U is normal, and since $T \cap U$ is trivial, the result follows from Proposition 2, Unit 3.

iii) See Example 13, Unit 3.

iv) Suppose that $srs^{-1} = r^a$.
Then $(srs^{-1})^a = (r^a)^a = r^{a^2}$.
Also $sr^a s^{-1} = s(srs^{-1})\, s^{-1} = s^2 r s^{-2} = r$ (since $s^2 = e$).
$\therefore r^{a^2} = r$, that is, $a^2 \equiv 1 (\mathrm{mod}\, 15)$.
You should check that there are four solutions modulo 15 to this, namely, 1, 4, 11, and 14.

v)
- In case $a = 1$, then $sr = rs$, so G is abelian.
- In case $a = 4$, $sr^3 s^{-1} = r^{12} = r^{-3}$, and $sr^5 s^{-1} = r^5$. The elements s and r^3 together generate a normal subgroup isomorphic to the dihedral group D_{10}. This subgroup intersects trivially the subgroup $\{1, r^5, r^{10}\}$ which is central. By Proposition 2 of Unit 3, we conclude that G is isomorphic to the direct product $D_{10} \times \mathbb{Z}/3\mathbb{Z}$.
- The analysis in case $a = 11$ is similar to that in the case of $a = 4$. The elements s and r^5 together generate a normal subgroup isomorphic to the symmetric group S_3. This subgroup trivially intersects the subgroup $\{1, r^3, r^6, r^9, r^{12}\}$, which is central. We conclude that $G \simeq S_3 \times \mathbb{Z}/5\mathbb{Z}$.
- In case $a = 14$, G is generated by r and s, subject to the relations $s^2 = r^{15} = e$ and $srs^{-1} = r^{-1}$. Thus G is the dihedral group D_{30}.

vi) In case $a = 1$, $G = Z(G)$.
In case $a = 4$, $Z(G) \simeq Z(D_{10}) \times Z\left(\mathbb{Z}/3\mathbb{Z}\right) = \{e\} \times \mathbb{Z}/3\mathbb{Z}$.
In case $a = 11$, $Z(G) \simeq Z(S_3) \times Z\left(\mathbb{Z}/5\mathbb{Z}\right) = \{e\} \times \mathbb{Z}/5\mathbb{Z}$.
In case $a = 14$, $Z(G) = Z(D_{30}) = \{e\}$.
Thus, the orders of $Z(G)$ in these cases are 30, 3, 5, 1, respectively. Hence these cases are all distinct.

E15) If G is of order 10, then by Theorem 8 of Unit 3, you know that $G \simeq \mathbb{Z}_{10}$, or $G \simeq D_{10}$.
If $G \simeq \mathbb{Z}_{10}$, then $\langle a \,|\, a^{10} \rangle$ is a presentation of G.
If $G \simeq D_{10}$, let $G = \{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$, where $x^5 = e = y^2$ and $yxy^{-1} = x^4 = x^{-1}$. We will show that

$G \simeq \langle a, b \mid a^5, b^2, bab^{-1}a^{-1} \rangle$.

For this, let F be the free group on $\{a, b\}$, and N be the smallest normal subgroup of F containing $a^5, b^2, bab^{-1}a^{-1}$. The surjective homomorphism $\theta: F \to G$ which maps $a \mapsto x$ and $b \mapsto y$, carries N to identity, because $x^5 = e, y^2 = e$ and $yxy^{-1}x^{-1} = e$. So F/N has at least 10 elements. But every element of F/N is one of $a^i b^j N, 0 \le i \le 4$ and $j = 0,1$. Thus, F/N has precisely 10 elements, and $N = \text{Ker}\,\theta$. Consequently $F/N \simeq G$, the desired result.

E16) $242 = 2(11)^2$. Thus, by Theorem 6, we know there are 5 possible structures for a group of order 242. Accordingly, their presentations are

i) $\langle a \mid a^{242} \rangle$,

ii) $\langle a, b, c \mid aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}, a^2, b^{11}, c^{11} \rangle$

iii) $\langle a, b \mid a^{121}, b^2, bab^{-1}a^{-120} \rangle$

iv) $\langle a, b, c \mid a^{11}, b^{11}, c^2, cbc^{-1}b^{-10} \rangle$

v) $\langle s, r_1, r_2 \mid s^2, r_1^{11}, r_2^{11}, r_1r_2r_1^{-1}r_2^{-1}, sr_1s^{-1}r_1, sr_2s^{-1}r_2 \rangle$.