Conversely, if $G_1(p_i) \simeq G_2(p_i) \forall i = 1,\ldots,k$, then
$G_1 \simeq \prod_i G_1(p_i) \simeq \prod_i G_2(p_i) \simeq G_2$.

E2) We proceed by induction on r. When $r = 1$, $|G| = p$. Then $G \simeq H_1$ and $G \simeq K_1$, so $m = n = 1$ and $|H_1| = |K_1|$.
Now suppose that the statement is true for all abelian groups of order less than p^s, for some $s \in \mathbb{N}$. For any abelian group L, $L^p = \{x^p | x \in L\}$ is a subgroup of L. Further, if L is finite then $|L/L^p| = p$. Also $G \simeq H_1 \times H_2 \times \cdots \times H_m$ implies that $G^p \simeq H_1^p \times H_2^p \times \ldots \times H_{m'}^p$, where m' is the largest integer i such that $|H_i| > p$. Similarly, $G \simeq K_1 \times K_2 \times \cdots \times K_n$ implies that $G^p \simeq K_1^p \times K_2^p \times \ldots \times K_{n'}^p$, where n' is the largest integer j such that $|K_j| > p$. Since $|G^p| < |G|$, by induction we have $m' = n'$ and $|H_i^p| = |K_i^p|$ for $i = 1,2,\ldots,m'$. Since $|H_i| = p|H_i^p|$, this proves that $|H_i| = |K_i|$ for $i = 1,2,\ldots,m'$.
All that remains to be proved is that the number of H_is of order p equals the number of K_is of order p; that is, we must prove that $n - n' = m - m'$. This follows from the fact that $|H_1||H_2|\cdots|H_{m'}|p^{m-m'} = |G| = |K_1||K_2|\cdots|K_{n'}|p^{n-n'}, m' = n'$ and $|H_i| = |K_i|$ for $1 \le i \le m'$.

E3) The elementary divisors of $G \simeq \mathbb{Z}_5 \times \mathbb{Z}_{5^2} \times \mathbb{Z}_{11^2}$ are $5, 5^2, 11^2$.
Since $\mathbb{Z}_{55} \simeq \mathbb{Z}_5 \times \mathbb{Z}_{11}, \mathbb{Z}_{55} \times \mathbb{Z}_{55} \times \mathbb{Z}_5 \simeq \mathbb{Z}_5 \times \mathbb{Z}_{11} \times \mathbb{Z}_5 \times \mathbb{Z}_{11} \times \mathbb{Z}_5$.
Its elementary divisors are $5,5,5,11,11$. These are not the same as those of G.

E4) i) $105 = 3.5.7$. Therefore, looking at the algorithm given before the exercise, $\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ is the only abelian group of order 105. Note that $\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \simeq \mathbb{Z}_{105}$.

ii) $270 = 2^1.3^3.5^1$. Therefore, looking at the partitions of 1, 3, 1, it follows that the non-isomorphic abelian groups of order 270 are $\mathbb{Z}_2 \times \mathbb{Z}_{27} \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

iii) $9801 = 3^4.11^2$. Now, the partitions of the exponents are

4	2
$1+3$	$1+1$
$1+1+2$	
$2+2$	
$1+1+1+1$	

Therefore, the required non-isomorphic abelian groups are
$\mathbb{Z}_{81} \times \mathbb{Z}_{121}$,
$\mathbb{Z}_3 \times \mathbb{Z}_{27} \times \mathbb{Z}_{121}$,

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{121}$,
$\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_{121}$,
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{121}$,

$\mathbb{Z}_{81} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$,
$\mathbb{Z}_3 \times \mathbb{Z}_{27} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$,
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$,
$\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$,
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$.

E5) i) The smallest positive integer n such that there are exactly two non-isomorphic abelian groups of order n is $n = 4$. This is because for $n = 2, 3$, the groups of order n are cyclic, and hence, unique up to isomorphism. For $n = 4$, there are two non-abelian groups of order n, namely $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

ii) The smallest positive integer n such that there are exactly three non- isomorphic abelian groups of order n is $n = 8$. This is because for $n = 2,3,5,7$, the groups of order n are cyclic, and hence, unique up to isomorphism. For $n = 4$, there are two non-abelian groups of order n, as shown in (i) above.
For $n = 6$, there is a unique abelian group of order n, i.e.,
$\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$.
For $n = 8$, there are three non-isomorphic abelian groups, namely,
$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

E6) In both the cases, there is a unique group since the exponents of the primes are 1.

E7) $120 = 2^3.3.5$. Therefore, the groups of order 120 are
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.
The first group has only one element of order 2, namely, $(\bar{4}, \bar{0}, \bar{0})$.
The second group has precisely three elements of order 2, namely,
$(\bar{1}, \bar{0}, \bar{0}, \bar{0}), (\bar{1}, \bar{2}, \bar{0}, \bar{0}), (\bar{0}, \bar{2}, \bar{0}, \bar{0})$.
The third group has 7 elements of order 2, namely, $(\bar{x}, \bar{y}, \bar{z}, \bar{0}, \bar{0})$, where $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(\bar{0}, \bar{0}, \bar{0})\}$. Therefore, the required group is $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Therefore, along the same lines as in Example 4, you can check that if G has precisely 3 elements of order 2, then
$G \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

E8) By Theorem 4, you can assume that $G = \mathbb{Z}_{q_1^{e_1}} \times \mathbb{Z}_{q_2^{e_2}} \times \ldots \times \mathbb{Z}_{q_s^{e_s}}$, where $q_1, q_2, \cdots, q_s$ are primes (not necessarily distinct) and $e_i \geq 1$ for all i.
Since $|G| = q_1^{e_1} q_2^{e_2} \ldots q_s^{e_s}$, and m divides $|G|$, we must have
$m = q_1^{f_1} \; q_2^{f_2} \; \ldots \; q_s^{f_s}$, where $0 \leq f_i \leq e_i$ for all i.
Now if $\langle a_i \rangle$ is a subgroup of $\mathbb{Z}_{q_i^{e_i}}$ of order $q_i^{f_i}$ (this is possible since the

converse of Lagrange's theorem is true for cyclic groups), then $\langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$ is a subgroup of G of order m.

E9) There are 6 types of abelian groups of order $108 = 2^2.3^3$, namely,

i) $\mathbb{Z}_4 \times \mathbb{Z}_{27}$

ii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}$

iii) $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$

iv) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$

v) $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

vi) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

So, the number of elements of order 3 in the groups $(i)-(vi)$ is given as in the following table:

Group	**Number of elements of order 3**
(i)	2 [i.e., $(\overline{0},\overline{9})$, $(\overline{0},\overline{18})$]
(ii)	2 [i.e., $(\overline{0},\overline{0},\overline{9})$, $(\overline{0},\overline{0},\overline{18})$]
(iii)	8 [i.e., $(\overline{0},\overline{x},\overline{y}), \overline{x} \in \mathbb{Z}_3, \overline{y} \in 3\mathbb{Z}_9$, both $\overline{x}$ and $\overline{y}$ not zero simultaneously.]
(iv)	8
(v)	26 [i.e., $(\overline{0},\overline{x},\overline{y},\overline{z}), \overline{x},\overline{y},\overline{z} \in \mathbb{Z}_3$, not all zero simultaneously.]
(vi)	26

Now recall that if H is a subgroup of order 3 in a group G, then H is cyclic and is generated by an element of order 3. Furthermore, H has precisely 2 elements of order 3 and both of them generate it. Consequently, groups (i) and (ii) have unique subgroups of order 3; groups (iii) and (iv) have four subgroups of order 3; and the groups (v) and (vi) have 13 subgroups of order 3.

E10) Abelian groups of order 56 have been constructed in Example 2.

Elementary Divisor Decomposition	Invariant Factor Decomposition
$\mathbb{Z}_8 \times \mathbb{Z}_7$	$\mathbb{Z}_{56}$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_7$	$\mathbb{Z}_2 \times \mathbb{Z}_{28}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$

E11) We have the following:

G	Elementary divisor form	Invariant factor form
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	already in this form	$\mathbb{Z}_{10} \times \mathbb{Z}_{10}$
$\mathbb{Z}_9 \times \mathbb{Z}_{25} \times \mathbb{Z}_{35} \times \mathbb{Z}_4$	$\mathbb{Z}_9 \times \mathbb{Z}_{25} \times \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_4$	$\mathbb{Z}_{6300} \times \mathbb{Z}_5$
$\mathbb{Z}_5 \times \mathbb{Z}_{10} \times \mathbb{Z}_{50}$	$\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	already in this form

E12) i) Let H be the subgroup of $\mathbb{Z}_{12}$ generated by $\{\overline{2}, \overline{3}\}$. Then $\overline{1} = \overline{3} - \overline{2} \in H$. Therefore, $\langle \overline{1} \rangle \subseteq H$. But $\langle \overline{1} \rangle = \mathbb{Z}_{12}$. Therefore, $H = \mathbb{Z}_{12}$.

ii) $\overline{2} = \overline{6} - \overline{4} \in < \overline{4}, \overline{6} > \subseteq < \overline{2} >$.
$\therefore < \overline{4}, \overline{6} > = < \overline{2} >$.

iii) Let K be the subgroup of $\mathbb{Z}_{12}$ generated by $\{\overline{6}, \overline{8}, \overline{10}\}$. Then $\overline{2} = \overline{8} - \overline{6} \in K$. Therefore, $< \overline{2} > \subseteq K$. But $\{\overline{6}, \overline{8}, \overline{10}\} \subseteq < \overline{2} >$. Therefore, $K = < \overline{2} > = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$.

E13) No. Because if H is the subgroup of $\mathbb{Z}_8$ generated by $\{\overline{4}, \overline{6}\}$, then it can be shown, as in E12, that $H = < \overline{2} > = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, and so $H \neq \mathbb{Z}_8$.

E14) The generating set need not be unique. For example, both $\{\overline{4}, \overline{6}\}$ and $\{\overline{6}, \overline{8}, \overline{10}\}$ generate $< \overline{2} >$ in $\mathbb{Z}_{12}$.
Similarly, both $\{1\}$ and $\{-1\}$ generate $\mathbb{Z}$.

E15) Let $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$ generate G_1 and G_2, respectively.
Any element of $G_1 \times G_2$ is of the form (g_1, g_2), where $g_1 \in G_1, g_2 \in G_2$.
Since $g_1 \in G_1$, $g_1 = a_1^{n_1} \cdots a_r^{n_r}$, $n_i \geq 0$. Similarly, $g_2 = b_1^{m_1} \cdots b_s^{m_s}$, $m_i \geq 0$.
Therefore, $(g_1, g_2) = \prod \{(a_i^{n_i}, b_j^{m_j}), i = 1, \ldots, r, j = 1, \ldots, s\}$.
Also. $(a_i^{n_i}, b_j^{m_j}) = \prod [(a_i, e_2)^{n_i} (e_1, b_j)^{m_j}]$, where e_1, e_2 are the identities of G_1 and G_2, respectively.
Thus, $G_1 \times G_2$ has a finite set generating it, viz.,
$\{(a_i, e_2), (e_1, b_j) \mid i = 1, \cdots, r, j = 1, \ldots, s\}$.

E16) Suppose $(\mathbb{Q}, +)$ is finitely generated, say generated by $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$.

Then any rational number is expressed as $k_1 \frac{a_1}{b_1} + k_2 \frac{a_2}{b_2} + \ldots + k_n \frac{a_n}{b_n}$ for some integers $k_1, k_2, \ldots, k_n$. Let $\ell = \text{l.c.m}\ (b_1, b_2, \ldots, b_n)$.
Consider $\frac{1}{\ell + 1} \in \mathbb{Q}$. Suppose $\frac{1}{\ell+1} = k_1 \frac{a_1}{b_1} + \cdots + k_n \frac{a_n}{b_n}$. Then
$b_1 \cdots b_n = (\ell + 1)(k_1 a_1 + \cdots k_n a_n)$, which is not possible. Hence our assumption that $(\mathbb{Q}, +)$ is finitely generated is wrong.

E17) If G is free abelian with basis X and G' is free abelian with basis X', then you can show that $G \times G'$ is free abelian with basis
$\{(x, 0), (0, x') \mid x \in X, x' \in X'\}$.

E18) Suppose that G is free abelian with basis $\{x_1, \ldots, x_n\}$. Define
$\theta : \mathbb{Z}^n \to G : \theta(a_1, \ldots, a_n) = a_1 x_1 + \cdots + a_n x_n$. You should check that θ is a well-defined group homomorphism, which is 1-1 and onto.

E19) Let G be an abelian group of order $n, n \geq 2$. Suppose G is free abelian and let X be a basis of G. Consider $x \in X$. Since $x \in G$, $nx = 0$. This is not possible if X is a basis, unless $n = 0$. We reach a contradiction. Hence G is not free abelian.

E20) Suppose $\{(a, b)\}$ generates $\mathbb{Z}\times\mathbb{Z}$. Then each element of $\mathbb{Z}\times\mathbb{Z}$ is of the form (na, nb), where $n \in \mathbb{Z}$. Now consider $(a+1, b) \in \mathbb{Z}\times\mathbb{Z}$. Then $na = a+1$, $nb = b$. Thus, $n = 1$ and $a = a+1$, which is absurd. Hence, we reach a contradiction. Thus, a singleton cannot generate $\mathbb{Z}\times\mathbb{Z}$.

E21) i) True. For any $n \geq 1$, $\mathbb{Z}^n$ is free abelian of rank n.

ii) False. $\{1\}$ is a basis of $\mathbb{Z}$. Also $\{1\} \subseteq \{1, 2\} \subseteq \mathbb{Z}$, but $\{1, 2\}$ is not a basis because, for example, $0 = 0.1 + 0.2 = 2.1 + (-1).2$ are two different ways of writing 0 as a $\mathbb{Z}$-linear combination of 1 and 2.

iii) True. Follows from Theorem 9.

iv) False. $2\mathbb{Z}$ is a subgroup of the free abelian group $\mathbb{Z}$, but $\mathbb{Z}/2\mathbb{Z}$, being finite, is not free abelian.

v) True. If G and G' are free abelian of rank r, then both are isomorphic to $\mathbb{Z}^r$, and hence isomorphic to each other.

E22) i) Firstly, $\text{Tor}(G) \neq \emptyset$ since $0 \in \text{Tor}(G)$. Next, for $g, h \in \text{Tor}(G)$, let the order of g and h be n and m, respectively. Then $nm.(g-h) = 0$ implies that $g - h \in \text{Tor}(G)$ (we have written the binary operation in G additively). Hence $\text{Tor} G) \leq G$.

ii) It need not be a subgroup.
For example, let $G = GL_2(\mathbb{R})$.
Take $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.
Then $A^4 = I, B^3 = I$, so that $A, B \in \text{Tor}(G)$.
However, $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
You can check that $(AB)^n = \begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix}$, so that $AB \notin \text{Tor}(G)$.
Hence, $\text{Tor}(G) \not\leq G$.

E23) Let F be a free abelian group. Suppose $a \in F$ is an element of finite order. By Theorem 9, $\langle a \rangle$ is free abelian. Also, since $a \in \text{Tor}(F), \langle a \rangle$ is finite. By E19, this is possible only if $\langle a \rangle$ is trivial, i.e., $a = \{0\}$.

E24) $(x, y) \in \text{Tor}(G_1 \times G_2)$
$\Leftrightarrow (x, y)^n = (e_1, e_2)$ for some $n \in \mathbb{N}$
$\Leftrightarrow x^n = e_1, y^n = e_2$ for some $n \in \mathbb{N}$
$\Leftrightarrow x \in \text{Tor}(G_1), y \in \text{Tor}(G_2)$.
Hence the two sets are equal.

E25) $G \times F$ is abelian because
$(a, b).(a', b') = (a.a', b.b') = (a'.a, b'.b) = (a', b').(a, b)$ for all
$(a, b), (a', b') \in G \times F$.
Also, if F is generated by the finite set X, then $G \times F$ is generated by $G \times X$, which is a finite set. Hence, $G \times F$ is also finitely generated.
Next, $\text{Tor}(G \times F) = \text{Tor}(G) \times \text{Tor}(F) = G \times \{e\}$, by E23 and E24.
$\simeq G$.

E26) By E25, the torsion subgroup of $\mathbb{Z}_4 \times \mathbb{Z} \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$, which has order 12.
The torsion subgroup of $\mathbb{Z}_{12} \times \mathbb{Z} \times \mathbb{Z}_{12}$ has order 144.

E27) We have

$$\mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{12} \times \mathbb{Z}_{10} \simeq \mathbb{Z}^3 \times \mathbb{Z}_6 \times \mathbb{Z}_{12} \times \mathbb{Z}_{10}$$
$$\simeq \mathbb{Z}^3 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2$$
$$\simeq \mathbb{Z}^3 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

This is the elementary divisor decomposition of the given group.
Using the method explained in the Structure Theorem,
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{60}$, so that the invariant factors of the given group are 2, 6, 60. Furthermore, its Betti number is 3.

E28) The Betti number of a finite abelian group is 0.

E29) No. For example, $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}_3$ have the same Betti number but are not isomorphic. This is because, for example, $\mathbb{Z} \times \mathbb{Z}_2$ has an element of order 2 but no element of order 3; similarly, $\mathbb{Z} \times \mathbb{Z}_3$ has an element of order 3 but no element of order 2.
The necessary and sufficient condition for two finitely generated abelian groups to be isomorphic is that the two groups should have the same Betti number **and** the same invariant factors; or, equivalently, the two groups should have the same Betti number **and** the same elementary divisors. This has been proved in the two Structure Theorems in this section.

E30) i) The group G/H is finitely generated. In fact, if $e_1, e_2, \ldots, e_n$ generate G, then $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$ generate G/H over $\mathbb{Z}$.

ii) No, the group G/H need not be free. For example, $\mathbb{Z}$ is a free group over $\mathbb{Z}$ generated by 1. But, the quotient $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is not a free abelian group.