UNIT 4  MATRIX GROUPS

4.1  INTRODUCTION

In the previous block you have seen how actions by, and on, finite and infinite groups play a crucial role in so many areas of study. In this unit you will be studying groups that act ‘linearly’. We know that linear transformations are best studied through their ‘concrete’ cousins, matrices, which are easier to manipulate. Therefore, if both matrices and groups come together, as in this unit, this largesse is not to be taken lightly!

In the course ‘Linear Algebra’ (MMT-002), you have already studied a considerable amount about different types of matrices and some decompositions like QR and SVD. In this unit, we build on the learning you have done there. Here we focus on some special matrix groups, namely, groups of invertible matrices.

In Sec.4.2, we study groups of different types of non-singular matrices. In particular, we focus on the general linear group over \( \mathbb{R} \) or \( \mathbb{C} \), and its subgroups.

In Sec.4.3, we carry this study further. Here you will specifically study properties of the groups formed by considering all the unitary/orthogonal matrices of size \( n \).

In Sec.4.4, you will be introduced to a new concept, that of a linear representation. As you will see, this is a device to represent groups by linear groups, which are often easier to handle.

Let us now go through the specific learning objectives of this unit.

Objectives

After studying this unit, you should be able to:

• give examples, and non-examples, of linear groups;
• define symplectic, orthogonal and unitary groups;
• prove, and apply, the Gram-Schmidt orthogonalisation process;
• describe how SU(2) can be identified with the sphere \( S^3 \);
• define, and give examples of, linear representations;
In your earlier studies, you have come across several examples of sets of matrices over \( \mathbb{R} \) or \( \mathbb{C} \) which form a group under matrix multiplication. In fact, the elements of such matrices can be from any field. Before going further, recall what a field is. (You will study more about fields in Block 4.)

**Definition:** A non-empty set \( F \), along with two binary operations \( *_1 \) and \( *_2 \) defined on it, is called a field if

i) \((F, *_1)\) is an abelian group;

ii) \((F \setminus \{0\}, *_2)\) is an abelian group;

iii) \( *_2 \) is distributive over \( *_1 \).

Now, let \( F \) be a field, as for example, \( \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \). Then, recall the examples of groups of matrices you have already worked with in MMT-002, and earlier.

i) The set of all \( n \times n \) matrices with entries in \( F \) which have non-zero determinant, is a group under matrix multiplication. This is called the **general linear group over** \( F \), and is denoted by \( \text{GL}_n(F) \).

ii) The set of elements of \( \text{GL}_n(F) \) which have determinant 1 forms a normal subgroup of \( \text{GL}_n(F) \). This normal subgroup is called the **special linear group over** \( F \), and is denoted by \( \text{SL}_n(F) \).

For instance, \( \text{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \, ad - bc = 1 \right\} \).

Here are some exercises to help you re-acquaint yourselves with these groups.

**E1)** Apply the Fundamental Theorem of Homomorphism to show that \( [\text{GL}_n(\mathbb{C})/\text{SL}_n(\mathbb{C})] \cong \mathbb{C}^* \).

**E2)** Give an example of an action of \( \text{GL}_n(\mathbb{R}) \) on \( M_n(\mathbb{R}) \), with justification. Also obtain the orbits and stabilisers of the identity matrix and the zero matrix with respect to this action.

**E3)** Prove that there is a bijection between \( \text{GL}_n(F) \) and the set of all ordered \( F \)-bases of the \( n \)-dimensional vector space \( F^n \).

**E4)** Show that \( G = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R}^* \right\} \) forms a group under matrix multiplication, but it is not a subgroup of \( \text{GL}_2(\mathbb{R}) \).

**E5)** i) Find a subgroup of \( \text{GL}_2(\mathbb{R}) \) which is isomorphic to \( \mathbb{C}^* \).
ii) Show that $GL_n(\mathbb{C})$ is isomorphic to a subgroup of $GL_{2n}(\mathbb{R})$, $\forall \ n \in \mathbb{N}$.

There are several other matrix groups, which are subgroups of $GL_n(F)$. They form a subset of matrix groups that we now define.

**Definition:** A subgroup of a general linear group is called a **linear group**.

So, $SL_n(\mathbb{C})$ is a linear group and the group in E4 is **not** a linear group.

Some more examples of linear groups are:

i) **The orthogonal group**, $O(n) = \{A \in GL_n(\mathbb{R}) \mid AA^t = I\}$.

ii) **The special orthogonal group**, $SO(n) = O(n) \cap SL_n(\mathbb{R}) = \{g \in SL_n(\mathbb{R}) \mid gg^t = 1\}$.
   [This is also called the **group of rotations**.]

iii) **The unitary group**, $U(n) = \{g \in GL_n(\mathbb{C}) \mid \overline{gg^t} = I\}$.

iv) **The special unitary group**, $SU(n) = U(n) \cap SL_n(\mathbb{C}) = \{g \in SL_n(\mathbb{C}) \mid gg^t = I\}$.

Note that an orthogonal group is a unitary group with elements from $GL_n(\mathbb{R})$. You will study more about these groups later in this unit. For now, to help you get used to these groups, try the following exercises.

**E6** i) Check whether or not $\det: O(n) \to (\pm 1, \cdot)$ is a group homomorphism. If it is, use it to find $|O(n):SO(n)|$. If $\det$ is not a homomorphism, explain why it is not.

ii) Let $Q$ be the quadratic form given by $Q(v) = x_1^2 + \cdots + x_n^2$, for $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Show that $O(n) = \{T: \mathbb{R}^n \to \mathbb{R}^n : Q(T(v)) = Q(v)\}$.

**E7** Give the geometrical representation of $U(1)$.

**E8** Show that:

i) $SO(2) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a^2 + b^2 = 1, \ a, \ b \in \mathbb{R} \right\}$, and

ii) $SO(2) \supset U(1)$.

**E9** Show that:

i) $Z(GL_n(F)) = \{\lambda I \mid \lambda \in F^\times \}$, and

ii) $Z(SL_n(\mathbb{C}))$ is isomorphic to the group of nth roots of 1.
Special Groups and Semigroups

There is yet another important linear group that you shall now study.

Example 1: Consider \[ \text{SP}_{2n}(\mathbb{R}) = \left\{ g \in \text{SL}_{2n}(\mathbb{R}) \mid g^t \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \]

Check whether or not it is a linear group. (Here \(0\) and \(I\) denote the \(n \times n\) zero and identity matrices, respectively.)

Solution: Firstly, since \(I \in \text{SP}_{2n}(\mathbb{R})\), the given set is non-empty.

Next, let \[ J = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}. \] Then, for \(A \in \text{SP}_{2n}(\mathbb{R})\), \(A^tJ = J\). So, \((A^{-1})^tJ A^{-1} = J\).

Thus, \(A^{-1} \in \text{SP}_{2n}(\mathbb{R})\).

Finally, you should check that if \(A, B \in \text{SP}_{2n}(\mathbb{R})\), then so does \(AB\).

Hence \(\text{SP}_{2n}(\mathbb{R})\) is a linear group.

The linear group in Example 1 is called the symplectic group over \(\mathbb{R}\). The name ‘symplectic’ was coined by the great mathematician Hermann Weyl. Some people use the notation \(\text{Sp}(n)\) or \(\text{Sp}(2n)\) for \(\text{SP}_{2n}(\mathbb{R})\).

The elements of \(\text{SP}_{2n}(\mathbb{R})\) are called symplectic matrices.

Now consider an interesting relationship between the symplectic groups and the special linear groups over \(\mathbb{R}\).

Example 2: Prove that \(\text{SP}_{2}(\mathbb{R}) = \text{SL}_{2}(\mathbb{R})\), but \(\text{SP}_{4}(\mathbb{R}) \neq \text{SL}_{4}(\mathbb{R})\).

Solution: \(\text{SP}_{2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_{2}(\mathbb{R}) \mid \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \)

So \(\text{SP}_{2}(\mathbb{R}) \subseteq \text{SL}_{2}(\mathbb{R})\), by definition.

To prove the reverse inclusion, let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_{2}(\mathbb{R})\). Now you should use the fact that \(\det(A) = 1\) to check that \(A^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Thus, \(A \in \text{SP}_{2}(\mathbb{R})\). Hence, \(\text{SL}_{2}(\mathbb{R}) \subseteq \text{SP}_{2}(\mathbb{R})\).

For the second part of the problem to be solved, note that \(\text{SP}_{4}(\mathbb{R}) \subseteq \text{SL}_{4}(\mathbb{R})\).

Now take \(A = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\). Then \(A \in \text{SL}_{4}(\mathbb{R})\), since \(\det(A) = 1\).

However, \(A^t \begin{pmatrix} 0 & I \\ -1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ -3 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) So \(A \notin \text{SP}_{4}(\mathbb{R})\).

Hence the result.
Try some exercises now.

E10) Consider the action of \( \text{GL}_n(\mathbb{R}) \) on itself by \( (A, B) \mapsto A'B \). Show that \( \text{Stab}(I) = \text{O}(n) \).

E11) Consider the action of \( \text{SL}_{2n}(\mathbb{R}) \) on itself by the action given in E9. Show that \( \text{Stab}(J) = \text{SP}_{2n}(\mathbb{R}) \), where \( J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \).

E12) Prove that \( \{ A \in \text{GL}_4(\mathbb{R}) \mid A'DA = D \} \) is a group, where \( D \) is the diagonal matrix \( (1, -1, -1, -1) \). [This group, denoted by \( \text{O}(1, 3) \), is called the Lorentz group, and has several applications in physics and the other sciences.]

E13) Check whether or not the following matrices are symplectic, where the blocks are in \( \text{GL}_n(\mathbb{R}) \):

\[
\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} I & B \\ 0 & 1 \end{pmatrix}
\]

where \( B = B' \) and \( A^{-1} \) exists.

The orthogonal groups, unitary groups and symplectic groups are referred to as the classical groups, as named by Hermann Weyl. Some subgroups of classical groups are also important and we introduce them together in the following example.

**Example 3:** Prove the following:

i) The set \( \text{B}_n(\mathbb{F}) \), of upper triangular invertible \( n \times n \) matrices over a field \( \mathbb{F} \) is a group. (This is an example of a Borel subgroup of \( \text{GL}_n(\mathbb{F}) \), named after the famous Swiss mathematician, Armand Borel.)

ii) The set \( \text{T}_n(\mathbb{F}) \), of triangular matrices in \( \text{B}_n(\mathbb{F}) \) whose diagonal entries are all 1, forms a normal subgroup of \( \text{B}_n(\mathbb{F}) \).

iii) The set \( \text{D}_n(\mathbb{F}) \), of diagonal matrices in \( \text{B}_n(\mathbb{F}) \), forms an abelian subgroup of \( \text{B}_n(\mathbb{F}) \).

iv) The set of \( n \times n \) matrices with integer entries and determinant \( \pm 1 \) is a group (denoted by \( \text{GL}_n(\mathbb{Z}) \)), and is a subgroup of \( \text{GL}_n(\mathbb{R}) \).

**Solution:** Firstly, note that all four sets are non-empty.

i) To prove that \( \text{B}_n(\mathbb{F}) \) is a group, note that for \( A \in \text{B}_n(\mathbb{F}) \), \( A^{-1} \) exists, and \( A^{-1} = \frac{1}{|A|} \text{Adj}(A) \).

Now, if \( A = [a_{ij}] \), with \( a_{ij} = 0 \) for \( i > j \), \( \text{Adj}(A) = [b_{ij}] \), with \( b_{ij} = 0 \) for \( i > j \). Thus, \( A^{-1} \in \text{B}_n(\mathbb{F}) \) also.
Finally, if $A, B \in B_n(F)$, then you can check that $AB \in B_n(F)$. Thus, $B_n(F) \leq GL_n(F)$.

You can prove (ii), (iii) and (iv) in the same way.

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Here are some related exercises for you.

E14) Prove that $T_2(F)$ is abelian, for any field $F$, and $T_n(F)$ is non-abelian for $n > 2$.

E15) i) For $\sigma \in S_n$, consider the permutation matrix $P(\sigma)$, whose rows are $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$, where the $R_i$s are the rows of the identity matrix (over any field). Show that $Perm_f(n) = \{P(\sigma) | \sigma \in S_n\}$ is a subgroup of $GL_n(F)$.

ii) Show that $\det(P(\sigma)) = \text{sign}(\sigma)$ for each $\sigma \in S_n$.

iii) Show that $Perm_R(n)$ is a subgroup of $O(n)$.

Let us now focus our attention on one type of classical group, which has several applications in other areas of mathematics and in the other sciences.

4.3 ORTHOGONAL AND UNITARY GROUPS

In the previous section, you have already noted several properties of $O(n), SO(n)$ and $U(n)$, for example, while doing E5, E6 and E7. We will now discuss some more properties of elements of $O(n)$ and $U(n)$.

To begin with, let us consider what an element of $O(n)$ looks like. What do the orthogonal groups have to do with orthogonality, i.e., perpendicularity? To see this, you may recall the concept of ‘dot product’ on $\mathbb{R}^n$. The dot product is a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, defined by $v \cdot w = v_1w_1 + \cdots + v_nw_n$, for $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$. You may also recall that

i) $v \cdot v$ gives the square of the length of the vector $v$, for $v \in \mathbb{R}^n$;

ii) in $\mathbb{R}^2$, the cosine of the angle between two non-zero vectors $v$ and $w$ is $\frac{v \cdot w}{v \cdot v \cdot w \cdot w}$.

Generalising this to $\mathbb{R}^n$, you get the concept of an ‘inner product’, as you may recall from MMT-002. To remind you, consider the following.

Definitions: 1) Define $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, the **standard inner product on** $\mathbb{C}^n$, by $\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i} = v^\prime \overline{w} \cdot v$, $v, w \in \mathbb{C}^n$. 

2) Two vectors \(v, w\) in \(\mathbb{C}^n\) are called **orthogonal** (or **perpendicular**) if 
\[ \langle v, w \rangle = 0. \]

Note that:

i) \[ \langle v, w \rangle = \langle w, v \rangle, \]
ii) \[ \langle v, v \rangle \in \mathbb{R}, \]
iii) \[ \langle v, v \rangle \geq 0 \forall v \in \mathbb{C}^n, \]
iv) \[ \langle v, v \rangle = 0 \iff v = 0. \]

**In what follows, when we write \(F\) we will mean either \(\mathbb{R}\) or \(\mathbb{C}\).** We will denote by \(\langle \cdot, \cdot \rangle\) the standard inner products in the two cases.

Recall that a basis \(\{v_1, \ldots, v_n\}\) of \(F^n\) is called

i) an **orthogonal basis** if \(\langle v_i, v_j \rangle = 0\) for all \(i \neq j\).

ii) an **orthonormal basis** if \(\langle v_i, v_j \rangle = \delta_{ij}\) for all \(i, j = 1, \ldots, n\).

What is the connection of such bases with the matrix groups? The following theorem tells us about this.

**Theorem 1:** An \(n \times n\) matrix \(A\) over \(\mathbb{R}\) (respectively, \(\mathbb{C}\)) is in \(O(n)\) (respectively, \(U(n)\)) if and only if \(\{Ae_1, Ae_2, \ldots, Ae_n\}\) is an orthonormal basis of \(\mathbb{R}^n\) (respectively, \(\mathbb{C}^n\)). Here \(\{e_1, \ldots, e_n\}\) is the standard basis of \(\mathbb{R}^n\) (or \(\mathbb{C}^n\)).

**Proof:** Let us first prove this for \(A \in M_n(\mathbb{R})\).

For \(v, w \in \mathbb{R}^n\), \[ \langle Av, Aw \rangle = (Av)^t(Aw) = v^t(A^tA)w. \] If \(A \in O(n)\), then \(A^tA = I\). So \(\langle Av, Aw \rangle = \langle v, w \rangle\) for all \(v, w \in \mathbb{R}^n\). Hence \(\langle Ae_i, Ae_j \rangle = \langle e_i, e_j \rangle = \delta_{ij} \forall i, j = 1, \ldots, n\). Thus, \(\{Ae_1, \ldots, Ae_n\}\) is an orthonormal basis of \(\mathbb{R}^n\).

Conversely, let \(\{Ae_1, \ldots, Ae_n\}\) be an orthonormal basis of \(\mathbb{R}^n\). Then \(\langle Ae_i, Ae_j \rangle = \delta_{ij} \forall i, j = 1, \ldots, n\). Thus, \[ e_i^t(A^tA)e_j = \delta_{ij}, \quad \ldots(1) \]
But, if \(a_{ij}\) is the \((i, j)\)-th entry of \(A^tA\), then the LHS of (1) is simply \(a_{ij}\).

Therefore, we find \(A^tA = I\). So, \(A \in O(n)\).

Now, you can prove the statement about \(U(n)\) on similar lines.

**Remark 1:** What Theorem 1 tells us is that orthogonal matrices preserve angles between vectors, and lengths of vectors, in \(\mathbb{R}^n\).

Let us see an immediate corollary to Theorem 1.
Corollary 1: For \( i, j = 1, \ldots, n \), let \( z_{ij} \in \mathbb{C} \) satisfy

1) \( |z_{ii}|^2 + |z_{i2}|^2 + \cdots + |z_{in}|^2 = 1 \) \( \forall i = 1, \ldots, n \);

2) \( z_{ii} z_{ij} + z_{i2} z_{j2} + \cdots + z_{in} z_{jn} = 0 \).

Prove that \( |z_{ij}|^2 + |z_{j2}|^2 + \cdots + |z_{nj}|^2 = 1 \) \( \forall j = 1, \ldots, n \), and
\( z_{ii} z_{jj} + z_{j2} z_{j2} + \cdots + z_{nj} z_{nj} = 0 \) \( \forall i \neq j \).

Proof: Let \( A = [z_{ij}] \). Then, the given conditions tell us that the rows of \( A \) form an orthonormal basis of \( \mathbb{C}^n \). Thus, \( A \in U(n) \). So, \( A^{-1} \in U(n) \), that is, \( A' \in U(n) \). Hence, \( \{A' e_j, \ldots, A' e_n\} \) forms an orthonormal basis of \( \mathbb{C}^n \). Hence,

\[
\sum_{a=1}^{n} |z_{aq}|^2 = 1 \quad \forall j = 1, \ldots, n \quad \text{and} \quad \sum_{a=1}^{n} z_{aq} \bar{z}_{aq} = 0 \quad \forall i \neq j.
\]

Thus, \( \sum_{a=1}^{n} |z_{aq}|^2 = 1 \quad \forall j = 1, \ldots, n \quad \text{and} \quad \sum_{a=1}^{n} z_{aq} \bar{z}_{aq} = 0 \quad \forall i \neq j. \)

We shall now consider another application of Theorem 1, to prove a result you have used again and again in MMT-002, the Gram-Schmidt process. But first, let us consider an example, to help you understand the proof.

Take a basis \( \{v_1, v_2, v_3\} = \{(1, 0, 1), (0, 1, 0), (-1, 2, 0)\} \) of \( \mathbb{R}^3 \).

On applying the Gram-Schmidt process, w.r.t. the standard inner product on \( \mathbb{R}^3 \), we get the orthonormal basis

\[
w_1 = \frac{1}{\sqrt{2}} (1, 0, 1), \quad w_2 = (0, 1, 0), \quad w_3 = \sqrt{2} \left( \frac{-1}{2}, 0, \frac{1}{2} \right).
\]

Here, \( A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \) = \( QC \)

where \( Q = [w_1, w_2, w_3] \) is in \( O(3) \), and \( C \) is in \( B_3(\mathbb{R}) \), with positive entries along the diagonal.

Let us now generalise what we have seen, and done, in this example.

Theorem 2 (Gram-Schmidt Orthogonalisation): Every matrix \( A \) in \( GL_n(\mathbb{R}) \) can be uniquely expressed as \( A = QC \), where \( Q \in O(n) \) and \( C \in B_n(\mathbb{R}) \), with the diagonal entries of \( C \) being in \( \mathbb{R}^+ \).

Proof: The proof is equivalent to the Gram-Schmidt orthogonalisation process, and the QR decomposition, you have studied earlier in Unit 6 of MMT-002. In fact, the columns of \( A \) form a basis \( \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \), as \( A \in GL_n(\mathbb{R}) \). On applying the Gram-Schmidt process to the \( v_i \)'s, we obtain a basis \( \{w_1, w_2, \ldots, w_n\} \), defined by

\[
w_1 = v_1, \quad w_2 = \frac{v_2 - \langle v_2, v_1 \rangle}{\|v_1\|} v_1, \quad w_3 = \frac{v_3 - \langle v_3, v_1 \rangle}{\|v_1\|} v_1, \quad \ldots, \quad w_{i+1} = \frac{v_{i+1} - \sum_{j=1}^{i} \langle v_{i+1}, w_j \rangle}{\|v_{i+1} - \sum_{j=1}^{i} \langle v_{i+1}, w_j \rangle \|} v_{i+1}.
\]
for \( i + 1 \leq n \).

Note that \( A = [v_1, v_2, \ldots, v_n] = [w_1, w_2, \ldots, w_n] C \), where \( C \) is an upper triangular matrix with positive entries along the diagonal (as this is the matrix which changes the basis \( \{v_i\} \) to the basis \( \{w_i\} \)). The matrix \( Q \), whose columns form the orthonormal basis \( \{w_1, \ldots, w_n\} \), is in \( O(n) \), by Corollary 1. So, we get \( A = QC \).

Next, to prove uniqueness, note that if \( Q_1C_1 = Q_2C_2 \), then
\[
Q_2^{-1}Q_1 = C_2C_1^{-1} \in O(n) \cap B_n(\mathbb{R}).
\]
Since such a matrix is in \( O(n) \), it must have its transpose (which has to be lower triangular) as its inverse. But the matrix is also in \( B_n(\mathbb{R}) \), so its inverse must be upper triangular. Hence, \( C_2C_1^{-1} \) must be diagonal. Also, since \( C_1 \) and \( C_2 \) have positive entries along the diagonal, so does \( C_2C_1^{-1} \). Further, since the eigenvalues of an element of \( O(n) \) have absolute value 1, \( C_2C_1^{-1} = I \), i.e., \( C_1 = C_2 \). So, \( Q_1 = Q_2 \).

The Gram-Schmidt process is valid for \( \mathbb{C}^n \) also. It shows that each \( n \times n \) invertible complex matrix is uniquely a product \( k b \), where \( k \in U(n) \) and \( b \) is an \( n \times n \) upper triangular invertible complex matrix. Further, both in the real and complex cases, for the groups of matrices of determinant 1, we may deduce decompositions into corresponding determinant 1 matrices. In particular, we have the following remark.

**Remark 2:** Every matrix \( A = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \in SL_2(\mathbb{C}) \) is expressible uniquely as a product \( A = BC \), where
\[
B = \begin{bmatrix} a_0 & -\overline{c_0} \\ c_0 & \overline{a_0} \end{bmatrix} \in SU(2), \quad C = \begin{bmatrix} 1 & (b_0 + \overline{c_0})/a_0 \\ 0 & 1 \end{bmatrix} \in T_2(\mathbb{C}) \text{ if } a_0 \neq 0,
\]
and
\[
B = \begin{bmatrix} 0 & b_0 \\ -b_0^{-1} & 0 \end{bmatrix} \in SU(2), \quad C = \begin{bmatrix} 1 & -b_0d_0 \\ 0 & 1 \end{bmatrix} \in T_2(\mathbb{C}) \text{ if } a_0 = 0.
\]

Let us consider an example of the application of Theorem 2.

**Example 4:** Write \( A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) as a product of an element of \( O(3) \) and an element of \( B_3(\mathbb{R}) \).

**Solution:** Using the Gram-Schmidt process, we find an orthonormal basis \( \{w_1, w_2, w_3\} \) of \( \mathbb{R}^3 \), starting from the basis \( \{v_1, v_2, v_3\} \), where
\[
v_1 = (1, 1, 1), \quad v_2 = (1, 1, 0), \quad v_3 = (1, 0, 0).
\]

Check that \( w_1 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \quad w_2 = \sqrt{\frac{1}{2}} \left( 1, 1, 2 \right), \quad w_3 = \sqrt{2} \left( \frac{1}{2}, -\frac{1}{2}, 0 \right) \).
Special Groups and Semigroups

Now, \( Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -2 & 0 \end{bmatrix} \in \text{O}(3) \). So \( QQ' = I = Q'Q \).

Consider \( C = Q'A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2 \\ \frac{1}{\sqrt{2}} & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \)

So \( C \in B_4(\mathbb{R}) \) and \( C = Q'A \). Thus \( A = QC \).

***

Here is a related exercise for you now.

E16) Use Theorem 2 to write \( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \text{GL}_2(\mathbb{R}) \) as a product QC, with \( Q \in \text{O}(2) \) and \( C \in B_4(\mathbb{R}) \).

Let us now look at another interesting group which will turn out to be the same as SU(2) structurally. For this, consider the set of Hamilton’s real quaternions, \( \mathbb{H} = \{a + bi + cj + dk \mid i^2 = j^2 = k^2 = -1, \ ij = -ji, \ a, b, c, d \in \mathbb{R}\} \).

In fact, the given conditions on \( i, j, k \) lead us to the other similar conditions:

\[ jk = i = -kj, \ ki = j = -ik. \]

Note that \((a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2\), and that \(\psi : \mathbb{H}^* \rightarrow \mathbb{H}^* : \psi(a + bi + cj + dk) = a - bi - cj - dk\) is an anti-homomorphism, that is, \(\psi(\alpha B) = \psi(\beta)\psi(\alpha)\).

By convention, \(\psi(\alpha)\) is denoted by \(\bar{\alpha} \forall \alpha \in \mathbb{H}\).

Now consider \( N : \mathbb{H}^* \rightarrow \mathbb{R}^* : N(\alpha) = \bar{\alpha}\alpha.\) Then, you can check that \( N \) is a group homomorphism.

So, for \( \alpha = a + bi + cj + dk \in \mathbb{H}^* \), we have

\[ \alpha^{-1} = \frac{1}{N(\alpha)} \times \begin{bmatrix} a \\ -b \\ c \\ -d \end{bmatrix} \begin{bmatrix} N(\alpha) \\ N(\alpha) \\ N(\alpha) \\ N(\alpha) \end{bmatrix} = \frac{a}{N(\alpha)} + \frac{-b}{N(\alpha)}i + \frac{-c}{N(\alpha)}j + \frac{-d}{N(\alpha)}k. \]

Also, \( \text{Ker } N = \{\alpha \in \mathbb{H}^* \mid N(\alpha) = 1\} \). We denote \( \text{Ker } N \) by \( \mathbb{H}^1 \).

So \( \mathbb{H}^1 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} \subset \mathbb{H}^* \).

Thus, \( \mathbb{H}^1 \) is the group of quaternions of norm 1.
Note that $\mathbb{H}^1$ is the 3-dimensional unit sphere $S^3$.

We are now ready to prove our next result, about relationships between some of the classical groups and $\mathbb{H}^1$.

**Theorem 3:** There exists an isomorphism between SU(2) and $\mathbb{H}^1$. Thus, SU(2) can be thought of as $S^3$.

**Proof:** Recall, from Remark 2, that any element of SU(2) is of the form
\[
\begin{pmatrix}
  a + ib & c + id \\
  -c + id & a - ib
\end{pmatrix}
\]
with $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 + c^2 + d^2 = 1$. (We also say that this element of SU(2) is represented by $(a, b, c, d)$.)

Therefore, it is natural to define a map $\theta : SU(2) \to \mathbb{H}^1$ by
\[
\theta \left( \begin{pmatrix}
  a + ib & c + id \\
  -c + id & a - ib
\end{pmatrix} \right) = a + bi + cj + dk.
\]

You should check that $\theta$ is a homomorphism, it is 1-1 and onto $\mathbb{H}^1$. Therefore, $SU(2) \simeq \mathbb{H}^1$.

In the next section, you will see (in Theorem 4) that there exists a homomorphism from SU(2) to SO(3). For now, why don’t you try some exercises about SU(2)?

---

**E17** Let $P, Q$ be elements of SU(2) represented by the real vectors $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)$, respectively. Obtain the real vector representing $PQ$.

**E18** Prove that SO(2) is conjugate to the subgroup $D$ of diagonal matrices in $GL_2(\mathbb{C})$ by SU(2).

**E19** Show that SU(2) acts by conjugation on the group $(G, +)$ of trace zero Hermitian matrices. Also find $\text{Stab} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ w.r.t. this action.

**E20** Let $P \in SO_3(\mathbb{C})$.

i) Show that 1 is an eigenvalue of $P$.

ii) Let $X_1, X_2$ be eigenvectors of $P$ corresponding to its eigenvalues $\lambda_1$ and $\lambda_2$. Show that $X_1^tX_2 = 0$ unless $\lambda_1\lambda_2 = 1$.

iii) Prove that if $X$ is an eigenvector for the eigenvalue 1, and $P \neq I$, then $X^tX \neq 0$.

The groups SO(2) and SO(3) comprise the set of rotations of 2-dimensional space and 3-dimensional space, respectively. They have several applications in the sciences and engineering.
You have seen how convenient it is to do calculations with matrices. So, if we can represent abstract groups as matrix groups, will this lead to simplifying proofs? You can get a glimpse of the answer to this in the following section.

4.4 LINEAR REPRESENTATIONS

In this section you will see how groups can be represented by matrix groups. The theory around this is called **representation theory**. Interestingly, some results about abstract groups can be proved only by representation theory, and there are no known purely group theoretic proofs in these cases. An important example of this is the famous theorem of Dirichlet, which states that ‘There are infinitely many primes in any arithmetic progression $an + b$, where $a$ and $b$ are coprime’. There are other examples, like that of the quantum mechanical model of the hydrogen atom, which can be explained via the representation theory of its symmetry group, SO(4). Some results, like Frobenius’s theorem, which states that ‘In any finite group $G$, the number of elements satisfying $x^n = e$ for any divisor $n$ of $o(G)$ is a multiple of $n$’, were discovered and proved using representation theory.

So, let us look at what a linear representation is.

**Definition:** Given a group $G$ and a field $K$ (think of $K = \mathbb{R}$ or $\mathbb{C}$, for concreteness), a **linear representation of dimension** $n$ is a group homomorphism from $G$ to $GL_n(K)$. If $K = \mathbb{R}$ (resp., $K = \mathbb{C}$), the representation is called a **real** (resp., a **complex**) representation.

An example of a representation of any group $G$ is the **trivial representation**, which sends every element of $G$ to the identity matrix in $GL_n(K)$.

As a **non-example** of a representation, consider the map

$$
\rho : GL_n(\mathbb{R}) \to GL_n(\mathbb{R}) : \rho(A) = A'.
$$

Since $\rho(AB) \neq \rho(A) \rho(B)$ for general $A, B, \rho$ is not a homomorphism. Hence it is not a linear representation.

**Remark 3:** If $V$ is an $n$-dimensional vector space over $K$, we can pick one ordered basis of $V$, say $\{v_1, v_2, ..., v_n\}$. Then we can identify the group $GL(V)$ of invertible linear transformations from $V$ to itself, with $GL_n(K)$. For example, if $V$ is the real vector space with basis $\{(1, 1), (-1, 1)\}$, then $GL(V) = GL_2(\mathbb{R})$.

Let us consider some non-trivial examples of representations now.

**Example 5:** Give an example each, of a 1-dimensional complex representation and a 2-dimensional real representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$.

**Solution:** For the first example, consider $\rho : \mathbb{Z}/n\mathbb{Z} \to GL_1(\mathbb{C}) = \mathbb{C}^*$, defined by $\rho(t) = e^{2\pi i / n}$.

Note that $\bar{t} = \bar{s} \Rightarrow (r - s) = nt$ for some $t \in \mathbb{Z}$. Hence $e^{2\pi i / n} = e^{2\pi i t / n}$. Thus, $\rho$ is well-defined.

Next, $\rho(t + s) = \rho(t) \cdot \rho(s) = e^{2\pi i (t+s) / n} = e^{2\pi i t / n} \cdot e^{2\pi i s / n} = \rho(t) \cdot \rho(s)$.
Hence, \( \rho \) is a group homomorphism, and so it is a 1-dimensional complex representation of \( \mathbb{Z}/n\mathbb{Z} \).

For the second example, consider
\[
\rho' : \mathbb{Z}/n\mathbb{Z} \to \text{GL}_2(\mathbb{R}) : \rho'(\bar{r}) = \begin{pmatrix}
\cos(2r \pi/n) & \sin(2r \pi/n) \\
-\sin(2r \pi/n) & \cos(2r \pi/n)
\end{pmatrix}.
\]
You should check that \( \rho' \) is well-defined, and \( \rho'(\bar{r}_1 + \bar{r}_2) = \rho'(\bar{r}_1) \rho'(\bar{r}_2) \). Thus, \( \rho' \) is a 2-dimensional real representation of \( \mathbb{Z}/n\mathbb{Z} \).

***

**Example 6:** Check whether or not the map \( \theta : S_n \to \text{GL}_n(\mathbb{R}) : \theta(\sigma) = \mathbf{P}(\sigma) \), where the \( i \)-th row of \( \mathbf{P}(\sigma) \) is the \( \sigma(i) \)-th row of the identity matrix, is a representation of \( S_n \).

**Solution:** To help you understand \( \theta \), consider an example when \( n = 3 \).
\[
\theta((1\ 2)) = \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad \theta((3\ 2\ 1)) = \begin{pmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix}
\]
and \((1,2)(3,2,1) = (2,3)\).
\[
\text{So } \theta((2,3)) = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix} = \theta(1\ 2) \theta(3\ 2\ 1).
\]
In general, you can see that \( \theta \) is well-defined, and
\[
\mathbf{P}(\sigma_1\sigma_2) = \\
\left[ \begin{array}{c} \sigma_1\sigma_2(1\ 0\ \ldots\ 0) \\
\sigma_1\sigma_2(0\ 1\ \ldots\ 0) \\
\vdots \\
\sigma_1\sigma_2(0\ 0\ \ldots\ 1) \end{array} \right] = \mathbf{P}(\sigma_1)\mathbf{P}(\sigma_2).
\]
Thus, \( \theta \) is a group homomorphism, and hence a representation of \( S_n \).

***

**Example 7:** Check whether or not \( \theta : A_n \to \mathbb{R}^* : \theta(\rho) = \rho(n) \) is a representation of \( A_n \).

**Solution:** Though \( \theta \) is well-defined, it need not be a homomorphism.
For example, if \( n = 5 \), \( \rho_1 = (1\ 2\ 3) \), \( \rho_2 = (1\ 2\ 5) \), then \( \rho_1\rho_2 = (1\ 5\ 2\ 3) \). So \( \theta(\rho_1\rho_2) = 1 \). But \( \theta(\rho_1) \theta(\rho_2) = 5 \). Thus, \( \theta \) is not a representation of \( A_n \).

***

**Example 8:** For any finite group \( G = \{g_1, \ldots, g_n\} \), consider the complex vector space \( V \) of dimension \( n \) with basis \( \{e_i\} \{i = 1, \ldots, n\} \). Show that
\[
\rho : G \to \text{GL}(V) : \rho(g) = \phi_g, \text{ where } \phi_g(e_i) = e_{gi} \forall i = 1, \ldots, n, \text{ is a representation of } G. \text{ [This is called the left regular representation of } G. \]
Solution: For each \( g \in G, \phi_g : V \rightarrow V : \phi_g \left( \sum_i z_i e_{s_i} \right) = \sum_i z_i e_{s_i}, \ z_i \in \mathbb{C} \), you should verify that \( \phi_g \) is a well-defined linear transformation.

Also, \( \phi_{gh} = \phi_g \circ \phi_h \) for \( g,h \in G \).

Hence \( \rho : G \rightarrow GL(V) : \rho(g) = \phi_g \) is an n-dimensional representation of \( G \).

***

We now use representations to prove a result, in continuation of what is done in Theorem 3.

**Theorem 4:** There is a homomorphism from \( SU(2) \) to \( SO(3) \), with kernel \( \{ \pm I \} \).

**Proof:** You know that \( \mathbb{H} \) is a real vector space of dimension 4, with basis \( \{1, i, j, k\} \). Consider its real subspace \( V \) generated by \( \{i,j,k\} \). We can think of \( SU(2) \) as acting on this space, via \( \theta \) (in Theorem 3). So, we have a homomorphism \( \rho \) from \( SU(2) \) to \( GL(V) \).

To explicitly write \( \rho : SU(2) \rightarrow GL(V) \), we use the isomorphism \( \theta \) to view \( SU(2) \) as \( \mathbb{H}^i \). Then, for any \( A = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \in SU(2) \), we write \( q = \theta(A) = a + bi + cj + dk \).

Also note that \( q^{-1} = a - bi - cj - dk \), since \( q \in \mathbb{H}^i \).

Then \( \rho(A) : V \rightarrow V : \rho(A)(v) = qvq^{-1} \).

Thus, writing \( qiq^{-1}, q j q^{-1} \) and \( qkq^{-1} \) in terms of the ordered basis \( \{i,j,k\} \), we get the following matrix:

\[
\rho(A) = \begin{pmatrix}
2(\text{bc} - \text{ad}) & 2(\text{ac} + \text{bd}) \\
2(\text{ad} + \text{bc}) & \text{a}^2 - b^2 + \text{c}^2 - \text{d}^2 \\
2(\text{bd} - \text{ac}) & 2(\text{ab} + \text{cd}) & \text{a}^2 - \text{b}^2 - \text{c}^2 + \text{d}^2 \\
\end{pmatrix}
\]

Now, since \( A^{-1} = \overline{A} \) is obtained by changing \( b,c,d \) to their negatives, you can check that \( \rho(A)^{-1} = \rho(A)^i \), that is, \( \rho(A) \in O_3(\mathbb{R}) \).

Note that if \( A \in \text{Ker} \rho \), then \( \rho(A)v = v \forall v \in V \). Therefore, \( qiq^{-1} = i \), that is, \( qi = iq \). Similarly, \( q \) commutes with \( j \) and \( k \) also. Hence, \( q \) commutes with the whole of \( \mathbb{H} \). Thus, \( q \in Z(\mathbb{H}) = \mathbb{R} \). Now, the centre of \( \mathbb{H}^i \) is \( \mathbb{R} \cap \mathbb{H}^i = \{ \pm 1 \} \).

Thus, \( q = \pm 1 \).

Thus, \( A = \pm 1 \) belongs to \( \text{Ker} \rho \).

The final assertion left is to show that the determinant of \( \rho(A) \) is always 1, which you can show by a direct (but messy) calculation (see E21 below).

Hence \( \rho : SU(2) \rightarrow SO(3) \) is a homomorphism with kernel \( \{ \pm I \} \).

Try the following exercises now.

---

E21) i) Regarding Theorem 4, by direct computation, verify that
Matrix Groups

\[
\begin{pmatrix}
    a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ac + bd) \\
    2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(cd - ab) \\
    2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2
\end{pmatrix}
\]

has determinant 1, when \(a, b, c, d\) are real numbers satisfying \(a^2 + b^2 + c^2 + d^2 = 1\).

ii) Prove that the image of \(\det \circ \rho\) is the constant function 1.

E22) i) Show that \(\mathbb{C}^* \to \text{GL}_2(\mathbb{R}) : a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\) defines a 2-dimensional real representation of \(\mathbb{C}^*\).

ii) Does \(\mathbb{C}^*\) have a 3-dimensional real representation? Why, or why not?

iii) Give a 1-dimensional real representation of \(\mathbb{C}^*\).

With this we come to the end of our discussion on linear groups and representations. Let us look at what you have studied in this unit.

### 4.5 SUMMARY

In this unit, you have studied the following points.

1) The definition, and examples, of a linear group.

2) The proof, and some applications of, the statement, ‘\(A \in O(n)\) (resp., \(U(n)\)) iff \(\{Ae_1, \ldots, Ae_n\}\) is an orthonormal basis of \(\mathbb{R}^n\) (resp., \(\mathbb{C}^n\)).’

3) \(\text{SU}(2) \cong \mathbb{H}^1\), the group of quaternions of norm 1. Hence, \(\text{SU}(2)\) can be thought of as the sphere \(S^3\).

4) The definition, and examples, of a linear representation of a group.

5) Using linear representations to prove that there is a homomorphism from \(\text{SU}(2)\) to \(\text{SO}(3)\), with kernel \(\{\pm I\}\).

### 4.6 SOLUTIONS / ANSWERS

E1) Define \(\psi : \text{GL}_n(\mathbb{C}) \to \mathbb{C}^* : \psi(A) = \det(A)\). Then you can check that \(\psi\) is a well-defined group homomorphism, which is surjective.

Further, \(A \in \text{Ker} \psi\) iff \(\det(A) = 1\) iff \(A \in \text{SL}_n(\mathbb{C})\).

Hence, using FTH, you get the result.

E2) There are several actions. Consider, for example,

\[\phi : \text{GL}_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}) : \phi(g, A) = g A\]

Then verify that \(\phi(I, A) = A\), and

\[\phi(g, \phi(h, A)) = \phi(gh, A) \quad \forall g, h \in \text{GL}_n(\mathbb{R})\] and \(A \in M_n(\mathbb{R})\).

The orbits of \(I\) and \(\emptyset\) are \(\text{GL}_n(\mathbb{R})\) and \(\{\emptyset\}\), respectively.
The stabilisers of \( I \) and \( 0 \) are \{I\} and \( \text{GL}_n(\mathbb{R}) \), respectively.

E3) \( g \in \text{GL}_n(\mathbb{F}) \) iff \( g \) is invertible iff the column rank of \( g \) is \( n \) iff the columns of \( g \) form an \( \mathbb{F} \)-basis of \( \mathbb{F}^n \).

E4) You can check that \((G, \cdot)\) is a group. However, \( G \) is not a linear group, since no element of \( G \) is in \( \text{GL}_2(\mathbb{R}) \).

E5) i) Check that 
\[
\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mapsto x + iy, \quad \text{where} \quad x, y \in \mathbb{R} \quad \text{and} \quad x^2 + y^2 \neq 0,
\]
defines an isomorphism between \( \text{GL}_2(\mathbb{R}) \) and \( \mathbb{C}^* \).

ii) Let \( z_{ij} = x_{ij} + iy_{ij} \). Check that 
\[
\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & y_{11} & \cdots & x_{1n} & y_{1n} \\ -y_{11} & x_{11} & \cdots & -y_{1n} & x_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & y_{n1} & \cdots & x_{nn} & y_{nn} \\ -y_{n1} & x_{n1} & \cdots & -y_{nn} & x_{nn} \end{pmatrix}
\]
is an injective group homomorphism from \( \text{GL}_n(\mathbb{C}) \) to \( \text{GL}_{2n}(\mathbb{R}) \).

E6) i) \( O(n) = \{g \in \text{GL}_n(\mathbb{R}) \mid g^t = g^{-1}\} \).

So \( g \in O(n) \Rightarrow gg^t = I \Rightarrow |g|^2 = 1 \Rightarrow |g| = \pm 1 \).

Thus, \( \det: O(n) \to \{\pm 1\} \) is an onto homomorphism, with kernel \( SO(n) \).

By FTH, \( \frac{O(n)}{SO(n)} \approx \{\pm 1\} \).

Thus, \( |O(n): SO(n)| = 2 \).

ii) \( Q(v) = v^t v, \quad v = (x_1, \ldots, x_n)^t \).

Now \( T \in O(n) \)
\[
\Leftrightarrow [T]^t[T] = I, \quad \text{where} \quad [T] \text{ denotes the matrix of } T \text{ w.r.t. the standard basis of } \mathbb{R}^n.
\]
\[
\Leftrightarrow [T(v)]^t[T(v)] = v^t v \forall v \in \mathbb{R}^n
\]
\[
\Leftrightarrow Q(T(v)) = Q(v), \quad \text{i.e.,} \quad T \text{ leaves } Q \text{ invariant.}
\]

E7) \( U(1) = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\} \)
\[
= \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}
\]

Thus, \( U(1) \) is the set of points on the circle of radius 1 and centre the origin, in \( \mathbb{R}^2 \).

E8) i) \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SO}(2) \iff a^2 + b^2 = 1 = c^2 + d^2, \quad ad - bc = 1.
\]
Matrix Groups

Then \((a - d)^2 + (b + c)^2 = 2 - 2 = 0\). Thus, \(a = d, b = -c\). Hence the result.

ii) Define \(\phi: \text{SO}(2) \to U(1): \phi \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + ib\).

Then \(\phi\) is well-defined.

Also, \(\phi \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = \phi \left(\begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}\right) = (ac - bd) + i(ad + bc)\)

\(= (a + ib)(c + id)\)

\(= \phi \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) \phi \left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right)\).

Thus, \(\phi\) is a group homomorphism.

Further, for any \(a + ib \in U(1), |a + ib| = 1\), i.e., \(a^2 + b^2 = 1\).

So \(a + ib = \phi \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right)\), that is, \(\phi\) is surjective.

Finally, \(\text{Ker} \phi = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a + ib = 1\right\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\).

Thus \(\phi\) is 1-1.

Hence \(\text{SO}(2) \cong U(1)\).

E9) i) \(g \in \text{Z}(\text{GL}_n(F)) \iff gA = Ag \ \forall A \in \text{GL}_n(F)\)

\(\iff gE_{ij}(l) = E_{ij}(l)g\ \forall i, j = 1, \ldots, n, i \neq j\), where \(E_{ij}(l)\) is the identity matrix with one change, that is, 1 at the \((i, j)\)th place also.

Now, \(gE_{ij}(l)\) is the matrix obtained from \(g\) by adding the \(i\)th column of \(g\) to its \(j\)th column. Also \(E_{ij}(l)g\) is the matrix obtained by adding the \(j\)th row of \(g\) to the \(i\)th row of \(g\). On running through all such \(E_{ij}(l)\), you will find that \(g = \lambda I\), for some \(\lambda \in F^*\).

ii) \(\text{Z}(\text{SL}_n(\mathbb{C})) = \{\lambda I | \lambda \in \mathbb{C}^* \text{ and } |\lambda I| = 1\}\)

\(= \{\lambda I | \lambda \in \mathbb{C}^* \text{ and } \lambda^n = 1\}\)

Now define \(\phi: \text{Z}(\text{SL}_n(\mathbb{C})) \to G: \phi(\lambda I) = \lambda\), where \(G\) is the group of \(n\)th roots of unity. Then you should check that \(\phi\) is an isomorphism.

E10) \(\text{Stab}(I) = \{A \in \text{GL}_n(\mathbb{R}) | A^t A = I\} = O(n)\).

E11) \(\text{Stab}(J) = \{A \in \text{SL}_{2n}(\mathbb{R}) | A^t JA = J\} = \text{SP}_{2n}(\mathbb{R})\).

E12) Firstly, \(O(1, 3) \neq \varnothing\), since \(I \in O(1, 3)\).

Next, you can check that if \(A \in O(1, 3)\), then so does \(A^{-1}\).

Finally, check that for \(A, B \in O(1, 3)\), \(AB \in O(1, 3)\).
E13) Let \( P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( A, B, C, D \in \text{GL}_n(\mathbb{R}) \). Then
\[
P' = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}.
\]
In the 1\(^{st}\) case, \( P = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \). Then \( P' = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \). So,
\[
P'JP = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} = J.
\]
Hence \( P \) is symplectic.

In the 2\(^{nd}\) case, \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). You can check that
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}.
\]
Hence, \( P \) is not symplectic.

In the 3\(^{rd}\) case, \( P = \begin{pmatrix} A' & 0 \\ 0 & A^{-1} \end{pmatrix} \). Then, \( P' = \begin{pmatrix} A' & 0 \\ 0 & (A^{-1})' \end{pmatrix} \). Therefore,
\[
P'JP = \begin{pmatrix} A' & 0 \\ 0 & (A^{-1})' \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & A^{-1} \end{pmatrix} = J.
\]
Hence \( P \) is symplectic.

Finally, let \( P = \begin{pmatrix} 1 & B \\ 0 & I \end{pmatrix} \). Then \( P' = \begin{pmatrix} 1 & 0 \\ B' & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \). We have
\[
P'JP = \begin{pmatrix} 1 & B \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} = J.
\]
Hence \( P \) is symplectic.

E14) Any element of \( T_2(F) \) is \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \), \( b \in F \).

Now \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \forall b, c \in F. \)

Thus, \( T_2(F) \) is abelian.

Now, consider \( x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) in \( T_2(\mathbb{Q}) \).

Then \( xy = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), and \( yx = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).
So \( xy \neq yx \). Thus, \( T_{ij}(\mathbb{Q}) \) is not abelian.

E15) i) Firstly, \( \text{Perm}_n(n) \neq \emptyset \) as \( S_n \neq \emptyset \).

Next, for \( \sigma, \tau \in S_n \), \( P(\sigma \circ \tau) = [R_{\sigma(1)}, \ldots, R_{\sigma(n)}] \)
\[ = P(\sigma)[R_{\tau(1)}, \ldots, R_{\tau(n)}] \]
\[ = P(\sigma)P(\tau) \]
Finally, \( P(I) = I \), and hence \( P(\sigma^{-1}) = P(\sigma)^{-1} \forall \sigma \in S_n \).
Thus, \( \text{Perm}_n(n) \leq \text{GL}_n(F) \).

ii) If \( \sigma \) is a transposition, then \( P(\sigma) \) is obtained from \( I \) by interchanging two rows.

\[ \therefore |P(\sigma)| = -1 = \text{sign}(\sigma). \]

Further, since \( P(\sigma_1 \sigma_2) = P(\sigma_1)P(\sigma_2) \), and every permutation is a product of transpositions, \( P(\sigma) = \text{sign}(\sigma) \forall \sigma \in S_n \).

iii) For each \( \sigma \in S_n \), the \((i,j)\)th entry of \( P(\sigma)^t \) is the \((j,i)\)th entry of \( P(\sigma) \), which is \( \delta_{\sigma(i)j(i)} \).

Also, since \( \sigma \) is a bijection, \( \delta_{\sigma(i)j(i)} = \delta_{\sigma^{-1}(j)i} \), the \((i,j)\)th entry of \( P(\sigma^{-1}) \).

\[ \therefore P(\sigma)^t = P(\sigma^{-1}) = P(\sigma)^{-1}. \]

So, \( P(\sigma) \in O(n) \).

E16) Starting with the basis \( \{v_1, v_2\} \), with \( v_1 = (a, b), v_2 = (c, d) \), you should check that the Gram-Schmidt process gives you the orthonormal basis \( \{w_1, w_2\} \).

Here \( w_1 = \left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right), w_2 = \frac{1}{\sqrt{a^2 + b^2}}(-b, a) \).

So \( Q = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \).

Then \( C = Q^tA = \begin{bmatrix} \frac{ac + bd}{\sqrt{a^2 + b^2}} & \frac{ac + bd}{\sqrt{a^2 + b^2}} \\ \frac{ad - bc}{\sqrt{a^2 + b^2}} & \frac{ad - bc}{\sqrt{a^2 + b^2}} \end{bmatrix} \).

So \( A = QC \).

E17) Let \( P = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \) and \( Q = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} \).

Write \( a = x_1 + ix_2, b = x_3 + ix_4, c = y_1 + iy_2 \) and \( d = y_3 + iy_4 \).

where \( x_1, x_2, x_3, x_4 \) and \( y_1, y_2, y_3, y_4 \) are real numbers satisfying \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \) and \( y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1 \).

Now \( PQ = \begin{pmatrix} ac - \bar{bd} & ad + bc \\ -bc - \bar{ad} & -bd + \bar{ac} \end{pmatrix} = \begin{pmatrix} e & f \\ -\bar{f} & \bar{e} \end{pmatrix} \), say.

So
Special Groups and Semigroups

\[ e = (ac - b\bar{c}) = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4) + i(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3), \]
\[ f = (ad + b\bar{c}) = (x_1y_3 + x_2y_1 - x_3y_4 + x_4y_2) + i(x_1y_4 + x_4y_1 + x_3y_3 - x_3y_2). \]

Now, PQ corresponds to the vector \((\text{Re}(e), \text{Im}(e), \text{Re}(f), \text{Im}(f))\).

You should check that
\[
\text{Re}(e)^2 + \text{Im}(e)^2 + \text{Re}(f)^2 + \text{Im}(f)^2 = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2 + (x_1y_3 + x_2y_1 - x_3y_4 + x_4y_2)^2 + (x_1y_4 + x_4y_1 + x_3y_3 - x_3y_2)^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2) (y_1^2 + y_2^2 + y_3^2 + y_4^2) = 1.
\]

E18) Any element of SO(2) is of the form
\[
B = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}, \quad \theta \in \mathbb{R}, \quad \text{and}
\]
any element of SU(2) is of the form
\[
A = \begin{bmatrix}
z & w \\
-\bar{w} & \bar{z}
\end{bmatrix}, \quad z, w \in \mathbb{C}^*.
\]

\[ |z|^2 + |w|^2 = 1. \]

Also, then
\[
A^{-1} = \begin{bmatrix}
\bar{z} & -w \\
-\bar{w} & \bar{z}
\end{bmatrix}.
\]

If SO(2) is conjugate to D, then for some \(z, w,\)
\[
\begin{bmatrix}
z & w \\
-\bar{w} & \bar{z}
\end{bmatrix} \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
z & -w \\
\bar{w} & \bar{z}
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}, \quad \text{for some} \lambda_1, \lambda_2 \in \mathbb{C}^*.
\]

On solving this, we get
\[
-w(z \cos \theta + w \sin \theta) + z(-z \sin \theta + w \cos \theta) = 0, \quad \text{and}
\]
\[
-z(-\bar{w} \cos \theta + \bar{z} \sin \theta) + \bar{w}(\bar{w} \sin \theta + \bar{z} \cos \theta) = 0.
\]

Simplifying, we get
\[
(z^2 + w^2) \sin \theta = 0 = (\bar{z}^2 + \bar{w}^2) \sin \theta.
\]

Let us pick \(z, w\) so that \(z^2 + w^2 = 0, |z|^2 + |w|^2 = 1.\)
[e.g., take \(z = \frac{1}{\sqrt{2}} i, w = \frac{1}{\sqrt{2}} i.\)]

Then
\[
A = \begin{bmatrix}
i & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & -i
\end{bmatrix}
\]
is such that
\[
ABA^{-1} = \begin{bmatrix}
e^{-i\theta} & 0 \\
0 & e^{i\theta}
\end{bmatrix} \in D.\]

Then, for \(A, B\) generally as above,
\[
ABA^{-1} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{C}^*.
\]

Thus, SO(2) is conjugate to D by SU(2).

E19) For \(A \in G, P \in SU(2),\) define \(PAP^{-1}.\)

Note that \(P^{-1} = P^*,\) and trace \((PAP^{-1}) = \text{trace } A.\)

Also \(PAP^{-1}\) is Hermitian since \(A\) is so.

So \(PAP^{-1} \in G.\)

Also, \(P.1 = 1\) and \((P_1, P_2, A) = (PP_1)_A,\) for \(A \in G, P_1, P_2 \in SU(2).\)

Hence, SU(2) acts by conjugation on \(G.\)

Now
\[
\text{Stab} \left( \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \right) = \left\{ P \in SU(2) \middle| \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} P = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \right\}
\]
\[
\begin{bmatrix}
\begin{array}{cc}
\varepsilon & \bar{w} \\
-w & z
\end{array}
\end{bmatrix}
\begin{bmatrix}
\bar{z} \\
\bar{w}
\end{bmatrix}
= 1, \ w\bar{z} + z\bar{w} = 0, \ z^2 - w^2 = 1, \ w, z \in \mathbb{C}
\text{.}
\]

E20) i) We have
\[
\det(P - xI) = \det((P - xI)^t) = \det((P^t - xI))
= \det(P^t - xI) \text{ since } P^t = P^{-1} \text{ for } P \in SO_3(\mathbb{C})
\]
So, P and P^{-1} have the same characteristic polynomials. However, in general, if \( \lambda \) is an eigenvalue of P, \( \lambda^{-1} \) is an eigenvalue of P^{-1}. So, since P and P^{-1} have the same characteristic polynomials, whenever \( \lambda \) is an eigenvalue of P, \( \lambda^{-1} \) is also an eigenvalue of P. Further, \( \lambda = \lambda^{-1} \) if and only if \( \lambda = \pm 1 \). Now, we know that \( \det(P) = 1 \), \( \det(P) \) is the product of the roots of the characteristic polynomial of P taken with correct multiplicity, and the degree of the characteristic polynomial of P is odd. Use these facts and show that 1 is an eigenvalue of P.

ii) Let \( X_1 \) and \( X_2 \) be eigenvectors of P corresponding to eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. Then, \( X_1^t P X_2 = (PX_1)^t X_2 = \lambda_1 X_1^t X_2 \).

But, \( X_1^t P X_2 = X_1^t P^{-1} X_2 = \lambda_2^{-1} X_1^t X_2 \). So, \( (\lambda_1 - \lambda_2^2) X_1^t X_2 = 0 \). The result follows.

iii) Using (i), we see that its eigenvalues are -1, -1, 1. So, putting \( \lambda_1 = \lambda_2 = 1 \) in (ii), we get the result.

E21) i) It requires determination, but you will find that the determinant is 1!

ii) \( \det : SU(2) \to GL_3(\mathbb{R}) : \det(A) = |\det(A)| = 1 \).

E22) i) \( \rho : \mathbb{C}^* \to GL_2(\mathbb{R}) : \rho(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) is well-defined, since

\[
a^2 + b^2 \neq 0.
\]

\[
\rho[(a + ib)(c + id)] = \rho[(ac - bd) + i(ad + bc)]
= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}
= \rho(a + ib) \rho(c + id).
\]

Hence \( \rho \) is a group homomorphism.

Also, since \( \rho(\mathbb{C}^*) \subseteq GL_2(\mathbb{R}) \), it is a 2-dimensional real linear representation.

ii) Define \( \theta : \mathbb{C}^* \to GL_3(\mathbb{R}) : \theta(a + ib) = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

Then you can check that \( \theta \) is a well-defined group homomorphism, and hence is a 3-dimensional real representation of \( \mathbb{C}^* \).
iii) Define \( \alpha : \mathbb{C}^* \rightarrow \mathbb{R}^+ : \alpha(z) = |z| \).

Then \( \alpha \) is a well-defined group homomorphism of \( \mathbb{C}^* \), and hence the required representation.