So $xy \neq yx$. Thus, $T_3(\mathbb{Q})$ is not abelian.

E15) i) Firstly, $\text{Perm}_F(n) \neq \emptyset$ as $S_n \neq \emptyset$.

Next, for $\sigma, \tau \in S_n$, $P(\sigma \circ \tau) = [R_{\sigma\tau(1)}, \ldots, R_{\sigma\tau(n)}]^t$

$= P(\sigma)[R_{\tau(1)}, \ldots, R_{\tau(n)}]^t$

$= P(\sigma)P(\tau)$

Finally, $P(I) = I$, and hence $P(\sigma^{-1}) = P(\sigma)^{-1}\ \forall \sigma \in S_n$.

Thus, $\text{Perm}_F(n) \leq GL_n F)$.

ii) If σ is a transposition, then $P(\sigma)$ is obtained from I by interchanging two rows.

$\therefore\ |P(\sigma)| = -1 = \text{sign}(\sigma)$.

Further, since $P(\sigma_1\sigma_2) = P(\sigma_1)P(\sigma_2)$, and every permutation is a product of transpositions, $P(\sigma) = \text{sign}(\sigma)\ \forall \sigma \in S_n$.

Then $C = Q^t A = \begin{bmatrix} \sqrt{a^2+b^2} & \dfrac{ac+bd}{\sqrt{a^2+b^2}} \\ 0 & \dfrac{ad-bc}{\sqrt{a^2+b^2}} \end{bmatrix}$.

So $A = QC$.

E17) Let $P = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $Q = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix}$. Write

$a = x_1 + ix_2$, $b = x_3 + ix_4$, $c = y_1 + iy_2$ and $d = y_3 + iy_4$,

where x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 are real numbers satisfying

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$.

Now $PQ = \begin{pmatrix} ac - b\bar{d} & ad + b\bar{c} \\ -\bar{b}c - \bar{a}\bar{d} & -\bar{b}d + \overline{ac} \end{pmatrix} = \begin{pmatrix} e & f \\ -\bar{f} & \bar{e} \end{pmatrix}$, say.

So

$e = (ac - b\bar{d}) = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4) + i(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3),$

$f = (ad + b\bar{c}) = (x_1y_3 + x_3y_1 - x_2y_4 + x_4y_2) + i(x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2).$

Now, PQ corresponds to the vector $(\text{Re}(e), \text{Im}(e), \text{Re}(f), \text{Im}(f))$.

You should check that

$$\text{Re}(e)^2 + \text{Im}(e)^2 + \text{Re}(f)^2 + \text{Im}(f)^2$$
$$= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2$$
$$+ (x_1y_3 + x_3y_1 - x_2y_4 + x_4y_2)^2 + (x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2)^2$$
$$= (x_1^2 + x_2^2 + x_3^2 + x_4^2)\,(y_1^2 + y_2^2 + y_3^2 + y_4^2) = 1.$$

E18) Any element of $SO(2)$ is of the form $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta \in \mathbb{R}$, and any element of $SU(2)$ is of the form $A = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$, $z, w \in \mathbb{C}^*$, $|z|^2 + |w|^2 = 1$. Also, then $A^{-1} = \begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix}$.

If $SO(2)$ is conjugate to D, then for some z, w,

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \text{ for some}$$

$\lambda_1, \lambda_2 \in \mathbb{C}^*$.

On solving this, we get

$-w(z\cos\theta + w\sin\theta) + z(-z\sin\theta + w\cos\theta) = 0$, and

$\bar{z}(-\bar{w}\cos\theta + \bar{z}\sin\theta) + \bar{w}(\bar{w}\sin\theta + \bar{z}\cos\theta) = 0$.

Simplifying, we get

$(z^2 + w^2)\sin\theta = 0 = (\bar{z}^2 + \bar{w}^2)\sin\theta$.

Let us pick z, w so that $z^2 + w^2 = 0, |z|^2 + |w|^2 = 1$.

[e.g., take $z = \frac{1}{\sqrt{2}}i,\ w = \frac{1}{\sqrt{2}}$.

Then $A = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$ is such that $ABA^{-1} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in D$.]

Then, for A, B generally as above, $ABA^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{C}^*$.

Thus, $SO(2)$ is conjugate to D by $SU(2)$.

E19) For $A \in G$, $P \in SU(2)$, define $P.A = PAP^{-1}$.

Note that $P^{-1} = P^*$, and trace $(PAP^{-1}) =$ trace A.

Also PAP^{-1} is Hermitian since A is so.

So $P.A \in G$.

Also, $P.I = I$ and $(P_1.(P_2.A)) = (P_1P_2).A$, for $A \in G, P_1, P_2 \in SU(2)$.

Hence, $SU(2)$ acts by conjugation on G.

Now $\text{Stab}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \left\{P \in SU(2) \,\middle|\, P\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}P^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

$$= \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} \middle| |z|^2 + |w|^2 = 1,\ w\overline{z} + z\overline{w} = 0,\ z^2 - w^2 = 1,\ w, z \in \mathbb{C} \right\}.$$

E20) i) We have

$$\det(P - xI) = \det((P - xI)^t) = \det((P^t - xI))$$
$$= \det(P^{-1} - xI) \text{ since } P^t = P^{-1} \text{ for } P \in SO_3(\mathbb{C}).$$

So, P and P^{-1} have the same characteristic polynomials. However, in general, if λ is an eigenvalue of P, λ^{-1} is an eigenvalue of P^{-1}. So, since P and P^{-1} have the same characterstic polynomials, whenever λ is an eigenvalue of P, λ^{-1} is also an eigenvalue of P. Further, $\lambda = \lambda^{-1}$ if and only if $\lambda = \pm 1$. Now, we know that $\det(P) = 1$, $\det(P)$ is the product of the roots of the characteristic polynomial of P taken with correct multiplicity, and the degree of the characteristic polynomial of P is odd.
Use these facts and show that 1 is an eigenvalue of P.

ii) Let X_1 and X_2 be eigenvectors of P corresponding to eigenvalues λ_1 and λ_2, respectively. Then, $X_1^t P^t X_2 = (PX_1)^t X_2 = \lambda_1 X_1^t X_2$. But, $X_1^t P^t X_2 = X_1^t P^{-1} X_2 = \lambda_2^{-1} X_1^t X_2$. So, $(\lambda_1 - \lambda_2^{-1}) X_1^t X_2 = 0$. The result follows.

iii) Using (i), we see that its eigenvalues are $-1, -1, 1$. So, putting $\lambda_1 = \lambda_2 = 1$ in (ii), we get the result.

E21) i) It requires determination, but you will find that the determinant is 1!

ii) $\det \circ \rho : SU(2) \to GL_3(\mathbb{R}) : \det \circ \rho(A) = |\rho(A)| = 1.$

E22) i) $\rho : \mathbb{C}^* \to GL_2(\mathbb{R}) : \rho(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is well-defined, since $a^2 + b^2 \neq 0$.

$$\rho[(a + ib)(c + id)] = \rho[(ac - bd) + i(ad + bc)]$$
$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$
$$= \rho(a + ib)\rho(c + id).$$

Hence ρ is a group homomorphism.
Also, since $\rho(\mathbb{C}^*) \subseteq GL_2(\mathbb{R})$, it is a 2-dimensional real linear representation.

ii) Define $\theta : \mathbb{C}^* \to GL_3(\mathbb{R}) : \theta(a + ib) = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then you can check that θ is a well-defined group homomorphism, and hence is a 3-dimensional real representation of $\mathbb{C}^*$.

iii) Define $\alpha : \mathbb{C}^* \to \mathbb{R}^* : \alpha(z) = |z|$.

Then α is a well-defined group homomorphism of $\mathbb{C}^*$, and hence the required representation.