E5) Let $A = \langle x \rangle$ and $B = \langle y \rangle$, where $o(x) = m$, $o(y) = n$.

Then $A \simeq \mathbb{Z}_m$ and $B \simeq \mathbb{Z}_n$.

If we prove that $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$, then we will have proved that $A \times B \simeq \mathbb{Z}_{mn}$, that is, $A \times B$ is cyclic of order mn.

So, let us prove that if $(m, n) = 1$, then $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.
Define $f : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n : f(r) = (r + m\mathbb{Z}, r + n\mathbb{Z})$.
Now, you can check that f is well-defined and f is a homomorphism.

$$\text{Ker } f = \{r \in \mathbb{Z} \mid r \in m\mathbb{Z} \cap n\mathbb{Z}\}$$
$$= \{r \in \mathbb{Z} \mid r \in mn\mathbb{Z}\}, \text{ since } (m, n) = 1$$
$$= mn\mathbb{Z}.$$

Finally, to show that f is surjective, take any element $(u + m\mathbb{Z},\ v + n\mathbb{Z}) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Since $(m, n) = 1$, $\exists\ s, t \in \mathbb{Z}$ such that $ms + nt = 1$. So, $u(1 - ms) + v(1 - nt) \in \mathbb{Z}$ such that $f(u(1 - ms) + v(1 - nt)) = (u + m\mathbb{Z},\ v + n\mathbb{Z})$.
Thus, f is surjective.
Now, we apply the Fundamental Theorem of Homomorphism to find that $\mathbb{Z}/\text{Ker} f \simeq \text{Im} f$, that is, $\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$, that is, $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.
$\therefore A \times B$ is cyclic of order mn.

E6) Since H_1 and H_2 are non-empty, so is $H_1 \times H_2$.
Also, for (h_1, h_2) and (h_3, h_4) in $H_1 \times H_2$, $(h_1, h_2)\ (h_3, h_4)^{-1} = (h_1 h_3^{-1},\ h_2 h_4^{-1}) \in H_1 \times H_2$. So, $H_1 \times H_2 \le G_1 \times G_2$.
Next, for $(x, y) \in G_1 \times G_2$ and $(h_1, h_2) \in H_1 \times H_2$,

$$(x, y)\ (h_1, h_2)\ (x, y)^{-1}$$
$$= (xh_1x^{-1},\ yh_2y^{-1}) \in H_1 \times H_2.$$

Thus, if $H_1 \triangleleft G_1$, $H_2 \triangleleft G_2$, then $(H_1 \times H_2) \triangleleft (G_1 \times G_2)$.

E7) Take $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$. Note that $\mathbb{Z}/2\mathbb{Z}$ has two subgroups: the trivial one and the whole group. So the subgroups of type $H_1 \times H_2$ are four in number. But $G_1 \times G_2$ is the Klein 4-group (non-cyclic of order 4) which has five subgroups: the trivial group, the whole group, and three of order 2.

E8) We know that each $x \in G$ can be expressed as hk, where $h \in H$ and $k \in K$. $\therefore G = HK$.
We need to show that $H \cap K = \{e\}$. For this, take $x \in H \cap K$.
Then $x \in H$ and $x \in K$. $\therefore xe \in HK$ and $ex \in HK$.
So, x has two representations, xe and ex, as a product of an element of H and an element of K. But we have assumed that each element must have **only one** such representation. So the two representations xe and ex must coincide, that is $x = e$. $\therefore H \cap K = \{e\}$.
$\therefore G = H \times K$.

E9) Let $P(n)$ be the statement that if $G = H_1 \times \ldots \times H_n$, then

$$o(G) = \prod_{i=1}^{n} o(H_i).$$

$P(1)$ is trivially true, as you can see.
Now, assume that $P(k)$ is true for some $k \in \mathbb{N}$.
Consider $G = H_1 \times \ldots \times H_{k+1}$ now.

Since $H_{k+1} \lhd G$, $G/H_{k+1} \simeq H_1 \times \ldots \times H_k$. Thus, $o\left(G/H_{k+1}\right) = \prod_{i=1}^{k} o(H_i)$, so that $o(G) = \prod_{i=1}^{k+1} o(H_i)$, i.e., $P(k+1)$ is true.

Hence $P(n)$ is true $\forall\ n \geq 1$.

E10) $|S_5| = 5 \times 3 \times 2^3$. Now, $\{e\}$ and S_5 are subgroups of S_5, of orders 1 and 120, respectively. By Sylow's first theorem, S_5 must have subgroups of order 5, 3, 2, 4 and 8. Further, since $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ for $(m, n) = 1$, S_5 must also have subgroups of order $5\times 3, 5\times 2, 3\times 2$, i.e., 15, 10 and 6. We also know that S_4, A_4 and A_5 are subgroups of S_5, of orders 24, 12 and 60, respectively.
Thus, the required values of n are
1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 60, 120.

E11) No. For example, consider S_3 and S_4. The order of a Sylow 2-subgroup in S_3 is 2, and in S_4 is 8. So they cannot be the same.

E12) Let $|G_1| = p^{e_1} m_1$ and $|G_2| = p^{e_1} m_2$, with m_1 and m_2 coprime to p. Since $p \,\|\, |G|, G$ has a Sylow p-subgroup say P, where $|P| = p^{e_1+e_2}$. Let π_1 denote the projection of G onto G_1, that is, $\pi_1 : G_1 \times G_2 \to G_1$ given by $(g_1, g_2) \mapsto g_1$. Similarly, let π_2 denote the projection of G onto G_2, that is, $\pi_2 : G_1 \times G_2 \to G_2$ given by $(g_1, g_2) \mapsto g_2$. The projections π_1 and π_2, being homomorphisms, their images $\pi_1(P)$ and $\pi_2(P)$ are p-subgroups of G_1 and G_2, respectively. Thus, $|\pi_2(P)| = p^{d_1}$ and $|\pi_2(P)| = p^{d_2}$, with $d_1 \leq e_1$ and $d_2 \leq e_2$. But, since P is contained in the subgroup $\pi_1(P) \times \pi(P_2)$, $p^{e_1+e_2} = |P| \leq |\pi_1(P)| \cdot |\pi_2(P)| = p^{d_1+d_2}$.
Hence $d_1 = e_1, d_2 = e_2$, $\pi_1(P)$ and $\pi_2(P)$ are Sylow p-subgroups of P_1 and P_2, respectively, and $P = \pi_1(P) \times \pi_2(P)$.
Conversely, if P_1 and P_2 are Sylow p-subgroups of G_1 and G_2, respectively, then $|P_1 \times P_2| = p^{e_1+e_2}$, so that $P_1 \times P_2$ is a Sylow p-subgroup of G.

E13) Let $x \in N_G(P)$. Then $xPx^{-1} = P$.
Now, for any $y \in N_G(P)$, $xyx^{-1} \in N_G(P)$.
So, $xN_G(P)x^{-1} = N_G(P)$, that is, $x \in N_G(N_G(P))$.
To prove the other inclusion, let g be an element in $N_G(N_G(P))$. Then $gN_G(P)g^{-1} = N_G(P)$ and $P \subseteq N_G(P)$. So, $gPg^{-1} \subseteq N_G(P)$. So gPg^{-1} is a Sylow p-subgroup of $N_G(P)$.
But P, being normal in $N_G(P)$, is the only Sylow p-subgroup of $N_G(P)$.

So $gPg^{-1} = P$, that is, $g \in N_G(P)$.

E14) No. Take, for instance, $p = 2, d = 1$, and G to be the Klein 4-group. There are three subgroups of order 2 of G. Since G is abelian, each of these is normal, so no two of them are conjugate.

E15) Let $|G| = 28 = 2^2 \times 7$. By Sylow's third theorem, the number of Sylow 7-subgroups is $1 (\bmod 7)$ and divides 4. Thus, it is 1. Call this unique subgroup H. Since H is normal in G, G is not simple.

Now, any element of order 7 generates a subgroup of order 7, which must be H. Also every element of H, apart from e, is of order 7. Thus, G has 6 elements of order 7.

E16) D_{12} has order $12 = 2^2 \times 3$. The number of Sylow 3-subgroups is 1 or 4. If it is 4, then D_{12} has 8 elements of order 3. However, by considering its elements, you know that it has only two elements of order 3, namely, the rotations by $2\pi/3$ and $4\pi/3$. So there is a unique Sylow 3-subgroup.

As to the Sylow 2-subgroups, there are either 1 or 3 of them. The number cannot be 1 since D_{12} is not abelian. So the number is 3. The rotation by an angle π belongs to the centre of the group. It is contained in all 3 Sylow 2-subgroups. Each of these subgroups consists of e, the rotation by angle π, and a pair of reflections. To describe such a pair, consider the line through a pair of opposite vertices. The reflection in this line, along with that in the line perpendicular to it, forms a pair of reflections. Note that the perpendicular line passes through the midpoints of the two edges that are not incident on the chosen pair of opposite vertices. There being three such pairs (of opposite vertices and therefore of reflections), the Sylow 2-subgroups are all described.

E17) This group has order $2 \times 6 = 2^2 \times 3$. This has Sylow 2- and Sylow 3-subgroups of orders 4 and 3, respectively. Along the lines of E12, you can show that the Sylow 3-subgroups are of the form $\{e\} \times P_1$ where P_1 is a Sylow 3-subgroup of S_3. Since P_1 is unique, $\mathbb{Z}_2 \times S_3$ has a unique Sylow 3-subgroup.
Using E12, there are three Sylow 2-subgroups, each of which is isomorphic to the Klein 4-group.

E18) i) Note that if a term in the numerator is divisible by p, it is of the form $p^r m - p^s n$, where $(p, n) = 1$. Then in the denominator there is a corresponding term $(p^r - p^s n)$. So p^s divides both terms and p^{s+1} does not divide either. So the powers of p in the numerator and denominator are the same and cancel out. Hence the result.

ii) $G \times \wp(G) \to \wp(G) : (g, S) \mapsto gS$ is an action.
If $X = \{S \subseteq G \,\big|\, |S| = p^r\}$, then $G \times X \subseteq X$.

So X is a G-invariant subset of $\wp(G)$.

iii) The number of distinct subsets in X is $\binom{p^r m}{p^r}$.

Now $p \nmid |X|$, by (i) above. Also X is a union of its G-orbits. So there is at least one G-orbit in X, say Y, such that $p \nmid |Y|$.

iv) $H = \text{Stab}_G(S)$ for $S \in Y$, where $p \nmid |Y|$ and $|S| = p^r$.
Now, by E6 (ii) of Unit 2, as S is H-invariant, it is the union of right cosets of H.
So $|H| \,\big|\, |S| = p^r$.

v) $Y \simeq G/H$ as G-sets. Since $p \nmid |Y|$, $p^r \,\big|\, |H|$.

vi) From (iv) and (v), $|H| = p^r$, so that H is a Sylow p-subgroup of G.

E19) i) Let X denote the set of cosets K/Q, with the natural action of K on it. Restrict to G the action of K on X, and consider the G-orbits in X. Since Q is a Sylow p-subgroup of K, it follows that $|X|$ is coprime to p. So there exists a G-orbit Y of X such that $|Y|$ is coprime to p. Let y be a point of Y and consider the stabiliser, $\text{Stab}_G(y)$. Call it H.
Now, if y represents the coset kQ of Q in K, then the stabiliser of y in K is kQk^{-1}, so $H = G \cap kQk^{-1}$ is a p-group (being a subgroup of the p-group kQk^{-1}). And, on the order hand, $Y \simeq G/H$, as G-sets. So the index of H in G (being equal to $|Y|$) is coprime to p. So H is a Sylow p-subgroup of G.

ii) Let P be a p-subgroup of G and let X be as in (i) above. Restrict to P the action of K on X. Since P is a p-group, we may invoke E17 of Unit 2 to conclude that $|X^P| \neq 0$. Letting kQ be the coset corresponding to a fixed point of P in X, we obtain $P \subseteq kQk^{-1}$.

iii) Let P be a Sylow p-subgroup of G. By (ii) above, there exists a conjugate Q' of Q in K such that $P \subseteq Q'$, so that $P \subseteq G \cap Q'$. But $G \cap Q'$ is a p-subgroup of G (since Q' is a p-group), so its cardinality cannot exceed that of P. Hence, $P = G \cap Q'$.

E20) We apply Proposition 2, with $P_1 = H$ and $P_2 = N$. For reasons of order, $P_1 \cap P_2$ is trivial. So $|G| = |P_1P_2|$, so that $G = P_1P_2$. Thus, G is isomorphic to $P_1 \times P_2$.

For the general case, let $P_1, P_2, \ldots, P_k$ be Sylow subgroups of G, corresponding one each to the various prime divisors of $|G|$. Suppose that they are all normal. The argument of the previous paragraph shows that P_1P_2 is $P_1 \times P_2$.. It follows, from Proposition 2(iv), that P_1P_2 is normal in G. We now apply Proposition 2 again, this time with $H = P_1P_2$ and $N = P_3$, to conclude that $P_1P_2P_3$ is normal and isomorphic to $P_1P_2 \times P_3 \simeq P_1 \times P_2 \times P_3$. Proceeding thus, we conclude that G is isomorphic to the direct product $P_1 \times \cdots \times P_k$.

E21) The order of S_5 is $5! = 2^3 \cdot 3 \cdot 5$. The number of Sylow 5-subgroups can be 1 or 6. The number of Sylow 3-subgroups can be 1, 4 or 10.
Now, the order of a Sylow 5-subgroup is 5. Every subgroup of order 5 consists of four 5-cycles in addition to the identity. No two distinct subgroups of order 5 intersect non-trivially – the identity is the only element common to them both. The number of 5-cycles in S_5 being 24, we conclude that there are 6 Sylow 5-subgroups in S_5.
The analysis for Sylow 3-subgroups is similar. The 3-cycles in S_5 are $\frac{5\times4\times3}{3} = 20$ in number. They are distributed over 10 Sylow 3-subgroups.

E22) Let G be a group of order pq. Assume $p < q$. (The same argument works if $p > q$.) Then G has a Sylow q-subgroup and the number of such subgroups is $1 \pmod q$ and divides p. Hence the number is 1. So the Sylow q-subgroup has order q and is normal in G. Hence G is not simple.

Now consider a group G of order p^2q. If $p > q$, then, as above, G will not be simple.
Now assume $q > p$. The number of Sylow q-subgroups can be 1, p or p^2.
If the number is 1, then G is not simple.
Since $q > p$, the number cannot be p.
If the number of Sylow q-subgroups is p^2, then $q \mid (p^2 - 1)$, that is $q \mid (p-1)(p+1)$. Since $q \nmid (p-1), q \mid (p+1)$.
This is only possible if $p = 2$. Then $q = 3$, so that $|G| = 12$ and G has 4 Sylow 3-subgroups. Let the Sylow 3-subgroups be T_1, T_2, T_3, T_4.
Then $\bigcup_{i=1}^{4} T_i$ consists of the identity and 8 elements of order 3. Any Sylow 2-subgroup of G intersects this union trivially because any such group intersects each T_i trivially. So a Sylow 2-subgroup must contain the 3 remaining elements in $G \setminus \left(\bigcup_i T_i\right)$ and the identity. Thus, the Sylow 2-subgroup is normal, and G is not simple.

Hence, in all the cases, G is not simple.

E23) i) Firstly, check that $HN \neq \emptyset$.
Next, use the fact that H normalises N to show that if $h_1n_1, h_2n_2 \in HN$, then $(h_1n_1)(h_2n_2)^{-1} \in HN$. Hence $HN \leq G$.

ii) You have seen this in Unit 1.

iii) Since $H \cap N = \{e\}$, HN is a direct product.

iv) From (i) you know that $HN \leq G$.
Further, for any $g \in G$, $h \in H$, $n \in N$, $ghng^{-1} = ghg^{-1}gng^{-1} \in HN$.
Hence $HN \lhd G$.

E24) If $n = 1$, then $G = \{e\}$.

- If $n = 2, 3, 5, 7$, then $G \simeq \mathbb{Z}/n\mathbb{Z}$.
- If $n = 4$, then G is cyclic or the Klein 4-group (see Example 5).
- If $n = 6$ or 10, then by Theorem 8, G is either cyclic or isomorphic to the dihedral group.
- If $n = 9$, then G is abelian.
  If G has an element of order 9, it is cyclic.
  If not, consider its subgroup $H = \langle x \rangle$, where $x^3 = e$.
  As $G/H \simeq \mathbb{Z}/3\mathbb{Z}$, $G \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.