$\psi(A,\psi(B,C)) = A(BCB^{-1})A^{-1} = (AB)C(AB)^{-1} = \psi(AB,C)\ \forall\, A,B \in GL_n(F)$ and $C \in \mathbb{M}_n(F)$.

E2) Consider $\varphi : G \times G/H \to G/H : \varphi(g, \bar{x}) = \overline{gx}$ and $\psi : G \times G/H \to G/H : \psi(g, \bar{x}) = \overline{gxg^{-1}}$. Check that φ and ψ are G-actions. Next, they are distinct because $\varphi(g, \bar{x}) = \psi(g, \bar{x}) \Leftrightarrow g \in H$.
So $\exists\ \alpha \in G \setminus H$ s.t. $\varphi(\alpha, \bar{x}) \neq \psi(\alpha, \bar{x})$.

E3) Take $G = S_X$. Then G acts on X (see Example 1). Any proper subgroup of G will also act on X. So the group acting on X is not unique.

E4) There are several such examples. For instance, consider $\mathbb{Z} \times \mathbb{N} \to \mathbb{N} : (z,n) \mapsto n^{|z|}$.
Now $(-2,(3,3)) \mapsto (-2,3^3) \mapsto (3^3)^2 = 3^6$
And $(-2+3,3) \to 3^1 \neq 3^6$.

E5) i) $G \times G \to G : (g,h) \to gh$ is the regular action.
For any $g \in G$, we will show that $G = Gg$.
Firstly, $Gg \subseteq G$. Next, for any $h \in G$, $h = hg^{-1}g \in Gg$, so that $G \subseteq Gg$. Hence $G = Gg$.

ii) Refer to Example 4.
Then $G \cdot xH = \{gxH \forall\, g \in G\}$
Now, for any $gH \in G/H$, $gH = (gx^{-1})xH \in G \cdot xH$.
Hence the result.

iii) The H-orbit of $g \in G$ under this action would be $\{hg \mid h \in H\} = Hg$, the right coset of H containing g. Conversely, any right coset is of the form $Hx, x \in G$. Now $Hx = \{hx \mid h \in H\}$, which is the H-orbit of $x \in G$. Hence the result.

iv) Consider $S_n \times [n] \to [n] : (\sigma,i) \mapsto \sigma(i)$, where $[n] = \{1,2,\cdots,n\}$.
Now $S_n \cdot i = \{\sigma(i) \mid \sigma \in S_n\} \subseteq [n]$.
Next, let $j \in [n], j \neq i$. Then $\tau = (i\ j) \in S_n$ and $\tau(i) = j$, so that $j \in S_n \cdot i$.
Thus, $[n] = S_n \cdot i$.

E6) i) Let $H \lhd G$. Then $g^{-1}Hg \subseteq H\ \forall\, g \in G$, i.e., H is G-invariant.
Conversely, let H be G-invariant, that is, $g^{-1}Hg \subseteq H\ \forall g \in G \Rightarrow H \lhd G$.

ii) Let $S \subseteq G$. S is H-invariant iff $H \times S \mapsto S : (h,s) \mapsto hs$ is an action iff $S = \bigcup_{s \in S} Hs$.

iii) Let $D = \{(x,x) \mid x \in X\}$.
Then $(g,(x,x)) \mapsto (gx,gx) \in D$. Hence D is G-invariant.

E7) i) For $x \in G$, $Stab_G(x) = \{g \in G \mid gx = x\} = \{g \in G \mid g = e\} = \{e\}$.

ii) For $x \in G$, $Stab_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\} = Z(x)$.

iii) For $x \in G$, $Stab_G(xH) = \{g \in G \mid gxH = xH\} = \{g \in G \mid x^{-1}gx \in H\}$
$= \{g \in G \mid g \in xHx^{-1}\} = xHx^{-1}$.

E8) Let $\alpha : S_{n-1} \to S_n : \alpha(\varphi) = \tilde{\varphi}$. Then α is a 1-1 group homomorphism.
$Stab_{S_n}(n) = \{\sigma \in S_n \mid \sigma(n) = n\}$
$\sigma \in Im\,\alpha \Rightarrow \sigma(n) = n \Rightarrow \sigma \in Stab_{S_n}(n)$, so that $Im\,\alpha \subseteq Stab_{S_n}(n)$.
Conversely, let $\sigma \in Stab_{S_n}(n)$. Define $\varphi \in S_{n-1}$ by
$\varphi(i) = \sigma(i)\ \forall\, i = 1, \cdots, n-1$. Then $\sigma = \tilde{\varphi} \in Im\,\alpha$, that is, $Stab_{S_n}(n) \subseteq Im\,\alpha$.
Hence $Im\,\alpha = Stab_{S_n}(n)$.

E9) $\varphi(gh) = \varphi_{gh}\ \forall\, g, h \in G$.
Now $\varphi_{gh}(x) = ghx = \varphi(g) \circ \varphi(h)(x)$.
$\therefore \varphi(gh) = \varphi(g) \circ \varphi(h)\ \forall\, g, h \in G$.
$Ker\,\varphi = \{g \in G \mid \varphi_g = I\} = \{g \in G \mid gx = x\ \forall\, x \in X\}$
$= \{g \in G \mid g \in Stab(x)\ \forall\, x \in X\}$
$= \bigcap_{x \in X} Stab(x)$.
Since $Stab_G(g) = \{e\}\ \forall\, g \in G, Ker\,\varphi = \{e\}$ if $X = G$, by E7(i).
Thus $\varphi : G \to S_G$ is a 1-1 homomorphism.

E10) We may assume that the group G is finite because, using E9, we may replace G by its image in S_x under the group homomorphism $G \to S_x$ defining the action of G on X. Now, let us apply the Orbit Counting Theorem. The left hand side is 1 since the action is transitive.
Looking at the right hand side, we conclude that $\sum_{g \in G} |X^g| = o(G)$.

Suppose every element of G fixes some element of X. Then $|X^g| \geq 1$ for every $g \in G$. Moreover, for the identity element e of the group, we have $X^e = X$, so that $|X^e| \geq 2$. Hence, $\sum_{g \in G} |X^g| > |o(G)|$, a contradiction. Hence $\exists g \in G$ s.t. $|X^g| = 0$, that is, $gx \neq x\ \forall x \in X$.

E11) Consider the conjugation action of G on C. This is transitive. So, by E10, $\exists g \in G$ s.t. $g^{-1}xg \neq x\ \forall x \in C$, i.e., $gx \neq xg\ \forall\, x \in C$.

E12) Put $X = G/H$, the set of left cosets of H. By E5, you know that the action is transitive. So, by E10, there exists g in G that does not fix any left coset xH, that is, $gxH \neq xH$. This means $x^{-1}gxH \neq H$, i.e., $x^{-1}gx \notin H$, i.e., $g \notin xHx^{-1}\ \forall\, x \in G$.

E13) Consider the action of $G = GL_2(\mathbb{C})$ on $\mathbb{C}^2$. Consider the set X of lines through the origin in $\mathbb{C}^2$, i.e., $X = \left\{ \lambda \begin{bmatrix} z \\ w \end{bmatrix} \middle| z, w \in \mathbb{C}, \lambda \in \mathbb{C} \right\}$. Then X is infinite. There is an induced action of G on X. You can check that this action is transitive. However, every element of G fixes some line because every element of G has an eigenvector.

E14) Let G be an infinite group with exactly two conjugacy classes. $\{e\}$ is a conjugacy class by itself. Let C be the other conjugacy class (consisting of all elements of G other then e). Now, each element g of G fixes some element of C under the conjugacy action because if $g = e$, then of course, every element of C is fixed; if $g \neq e$, then g belongs to C, and g fixes g.

E15) **Enumeration of necklaces with 5 beads:** If the number of black beads is 0, 1, 4 or 5, then there is a single necklace. There are two necklaces with 2 black beads – one in which the two black beads are together, and another in which they are separated (by a single white bead in one side and two white beads in the other). By the same reasoning, there are two necklaces with 2 white beads, or what amounts to the same, with 3 black beads. Thus, the total number of necklaces with 5 beads, counting by ascending order of number of black beads, is 1+1+2+2+1+1=8.

Enumeration of necklaces with 6 beads: If the number of black beads is 0, 1, 5 or 6, then there is a single necklace. There are three necklaces with 2 black beads – one in which the two black beads are together, a second in which they are separated by a single white bead on one side and three white beads on the other, and a third in which they are separated by two white beads on either side. By the same reasoning, there are three necklaces with 2 white beads, or what amounts to the same, with 4 black beads. There are three necklaces with 3 black beads – one in which the black beads are together, another in which two of the black beads are together and the third is separated (from the other two by white beads in between), and a third in which no two black beads are together. Thus, the total number of necklaces with 6 beads, counting by ascending order of number of black beads, is 1+1+3+3+3+1+1=13.

Enumeration of necklaces with 7 beads: If the number of black beads is 0, 1, 6 or 7, then there is a single necklace. There are three necklaces with 2 black beads (and three more with 5 black beads, by the symmetry between black and white) – one in which the two black beads are together, a second in which they are separated by a single white bead on one side and four white beads on the other, and a third in which they are separated by two white beads on one side and three on the other. There are four necklaces with 3 black beads (and four more with 4 black beads, by the symmetry between black and white) – one in which the black beads are all together, a second in which two of them are together and are separated from the third by a single white bead on one side and three white beads on another, a third in which two black beads are together and are separated from the third by two white beads on either side, and a fourth in which the black beads are separated from each other. Thus the total number of necklaces with 7 beads, counting by ascending order of number of black beads, is 1+1+3+4+4+3+1+1=18.

E16) We follow the argument given for the case of 8 beads. Let G be the group of symmetries of the regular 12-gon: it is the dihedral group D_{24}, and has cardinality 24. Let X be the set of colourings of the vertices of the regular 12-gon by two colours, black and white.

We have $|X| = 2^{12}$.

We first enumerate the elements of G. Let r denote the rotation by an angle of $30°$ in the counter-clockwise direction. Let s denote the reflection in the line joining a pair of opposite vertices. Then the 24 elements of D_{24} are as follows:

$e, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11},$

$s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7, sr^8, sr^9, sr^{10}, sr^{11}.$

Now, we have the following table:

Elements	m(g)	$\lvert X^g \rvert$
e	12	2^{12}
r, r^5, r^7, r^{11}	1	2^1
r^2, r^{10}	2	2^2
r^3, r^9	3	2^3
r^4, r^8	4	2^4
r^6	6	2^6
$s, sr^2, sr^4, sr^6, sr^8, sr^{10}$	7	2^7
$sr, sr^3, sr^5, sr^7, sr^9, sr^{11}$	6	2^6

Plugging in the values of $\lvert X^g \rvert$ from this table into the right hand side of (3), we obtain:

The number of G-orbits in X

$$= \frac{1}{24}(1\cdot 2^{12} + 4\cdot 2^1 + 2\cdot 2^2 + 2\cdot 2^3 + 2\cdot 2^4 + 1\cdot 2^6 + 6\cdot 2^7 + 6\cdot 2^6) = 224.$$

Thus, the answer is 224 for the case of 12 beads.

For p in place of 12, we follow the same procedure. There are $2p$ elements in the group D_{2p} of symmetries of the regular p-gon: let r denote the rotation by an angle of $(360/p)°$ (say counter-clockwise). Let s denote the reflection in the line joining a vertex to the midpoint of the side opposite to it. Then the elements of D_{2p} are as follows:

$e, r, r^2, \ldots, r^{p-1}, s, sr, sr^2, \ldots, sr^{p-1}.$

We have the following table:

Elements	m(g)	$\lvert X^g \rvert$
e	p	2^p
$r, \ldots, r^{p-1}$	1	2^1
$s, \ldots, sr^{p-1}$	$(p+1)/2$	$2^{(p+1)/2}$

Plugging values of $\lvert X^g \rvert$ from this table into the right hand side of (3), we obtain:

The number of G-orbits in X

$$= \frac{1}{2p}(1\cdot 2^p + (p-1)\cdot 2^1 + p\cdot 2^{(p+1)/2}) = \frac{2^{p-1}-1}{p} + 2^{(p-1)/2} + 1$$

Thus, we have a formula for the number of distinct necklaces with p beads.

E17) $|X| = |X^G| + \sum_{x\in X}\{|Gx| \mid |Gx| > 1\}$

Now $p \mid |Gx| \;\forall x \in X$ for which $|Gx| > 1$.

So $p \mid (|X| - |X^G|)$. Thus, $|X| \equiv |X^G| \pmod{p}$.

Next, suppose $(|X|, p) = 1$. Then $|X| \not\equiv 0 \pmod{p}$. So $|X^G| \not\equiv 0 \pmod{p}$.

So $|X^G| \neq 0$.

So $\exists x \in X$ s.t. $gx = x \;\forall g \in G$.

E18) i) $G^G = \{x \in G \mid g^{-1}xg = x \;\forall g \in G\}$
$= Z(x)$.

ii) $G^G = \{x \in G \mid gx = x \;\forall g \in G\}$
$= \emptyset$, unless $G = \{e\}$.

E19) i) Since $N \triangleleft G, g^{-1}xg \in N \;\forall g \in G$ and $x \in N$.

Thus, $C_x \subseteq N \;\; \forall x \in N$.

So $\bigcup_{x\in N} C_x \subseteq N$.

Also, for $y \in N, y \in C_y \subseteq \bigcup_{x\in N} C_x$.

Hence $N = \bigcup_{x\in N} C_x$.

ii) This follows immediately from (i) above.

E20) i) Here $o(Z(G)) = 3$, but $o(G) = 10$ and $o(Z(G)) \nmid o(G)$.

Hence this partition cannot form a Class Equation.

ii) Here $o(Z(G)) = 1$ and there are three conjugacy classes, of sizes 2, 2 and 5. This is the Class Equation of

$D_{10} = < \{x, y \mid x^2 = e, y^5 = e, xy = y^{-1}x\} >$.

Here $C_e = \{e\}, C_x = \{x, xy, xy^2, xy^3, xy^4\}, C_y = \{y, y^4\}, C_{y^2} = \{y^2, y^3\}$.

iii) This is not possible since this shows a conjugacy class of order 3, and $3 \nmid 10$.

iv) Here $o(Z(G)) = 10$. Hence $G = Z(G)$, i.e., G is abelian. Check that $G \simeq \mathbb{Z}_{10}$.

E21) By Corollary 2 (of Theorem 3), the required number is

$o(GL_2(\mathbb{Z}_5)) / |Z(A)|$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Now, you can check that $Z(A) = \{X \in GL_2(\mathbb{Z}_5) \mid AX = XA\}$

$$= \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{Z}_5, a, b \neq 0 \right\}$$

So $|Z(A)| = |\mathbb{Z}_5^* \times \mathbb{Z}_5^*| = 16.$

Thus, the required cardinality is $480/16 = 30.$

E22) Any group of order 9 must be abelian. Hence each conjugacy class will have only one element. Therefore, this equation is not a valid Class Equation.