E17) The distinct cosets are A_5 and $A_5 \circ (1\ 2)$.
This is because any permutation in S_5 is either odd or even. If it is even, it is in A_5. If σ is odd, then $\sigma \circ (1\ 2) \in A_5$, say $\sigma \circ (1\ 2) = \alpha \in A_5$.
Then $\sigma = \alpha \circ (1\ 2)^{-1} = \alpha \circ (1\ 2) \in A_5 \circ (1\ 2)$.
So, $S_5 = A_5 \cup A_5 \circ (1\ 2)$.
This partition is not unique, as you can see from the construction. For instance, $S_5 = A_5 \cup A_5 \circ (3\ 4)$ also. Here, also note that
$A_5 \circ (1\ 2) = A_5 \circ (3\ 4)$ since $(1\ 2)(3\ 4)^{-1} \in A_5$.

E18) $o(x) = o(<x>)$, which divides $o(G)$ by Lagrange's theorem.
$\therefore o(x) \mid o(G)$.
Let $o(x) = m$. Since $m \mid n$, $n = mr$ for some $r \in \mathbb{Z}$.
Now $x^m = e \Rightarrow (x^m)^r = x^{mr} = x^n = e$.

E19) Since $o(\mathbb{Z}_{18}) = 18, o(\overline{4}) = 1, 2, 3, 6, 9$ or 18.
Since $2 \cdot \overline{4} = \overline{8} \neq \overline{0}, o(\overline{4}) \neq 2$.
Similarly, you can check that $o(\overline{4}) \neq 1, 3$ or 6.
Now $9 \cdot \overline{4} = \overline{36} = \overline{0}$. Thus, 9 is the least natural number such that $m.\overline{4} = \overline{0}$. So $o(\overline{4}) = 9$.
Similarly, check that $o(\overline{4})$ in $\mathbb{Z}_5$ is 5.

E20) Since $o(x) = n, o(x^m)$ must be finite, say r.
Then $x^{mr} = e \Rightarrow n \mid mr$.
Now let $(n, m) = d$ and $m = dm_1, n = dn_1$, where $(n_1, m_1) = 1$.
Then $dn_1 \mid dm_1 r$, that is, $n_1 \mid m_1 r$. But $(n_1, m_1) = 1$.
So $n_1 \mid r$.
Also $(x^m)^{n_1} = x^{m_1 dn_1} = x^{m_1 n} = e$. Thus, $r \mid n_1$.
Hence $r = n_1 = \dfrac{n}{(n, m)}$, that is, $o(x^m) = \dfrac{n}{(n, m)}$.

E21) Suppose $o(x^m) = r \in \mathbb{Z}$, for some $m \in \mathbb{Z}^*$.
Then $x^{mr} = e$, which means that $o(x)$ is finite, a contradiction. Thus, what we assumed was wrong, that is, x^m cannot be of finite order.

E22) Not necessarily. For example, $o(A_4) = 12$, but A_4 has no subgroup of order 6.

E23) You know that $\{e\}$ and $Z(G)$ are subgroups of G. So, now we only need to check that they are normal in G. For any $g \in G, g^{-1}eg = e \in \{e\}$.
Hence $\{e\} \lhd G$.
Next, let $g \in G$ and $z \in Z(G)$. Then
$g^{-1} z g = g^{-1} g z$, since $zg = gz$.
$= z \in Z(G)$.

Thus, $Z(G) \triangleleft G$.

E24) a) i) Let $x \in H \cap K$ and $g \in G$.
Since $H \triangleleft G$ and $x \in H$, $g^{-1}xg \in H$.
Similarly, $g^{-1}xg \in K$.
Thus, $g^{-1}xg \in H \cap K$. Hence $H \cap K \triangleleft G$.

ii) Let $hk \in HK, g \in G$. Then $g^{-1}hg \in H, g^{-1}kg \in K$.
So $g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg) \in HK$, and hence, $HK \triangleleft G$.

b) For example, $\langle(1\ 2)\rangle$ and $\langle(1\ 3)\rangle$ are not normal in S_3, and $\langle(1\ 2)\rangle\langle(1\ 3)\rangle \not< S_3$.

E25) From the elements given in Example 3, you can see that $G = HK$.
Note that $y^{-1}xy = xy^2 \notin H$. So $H \not\triangleleft G$.

E26) Let $x^{-1}y^{-1}xy \in G'$ and $g \in G$. Then
$$g^{-1}(x^{-1}y^{-1}xy)g = (g^{-1}x^{-1}g)(g^{-1}y^{-1}g)(g^{-1}xg)(g^{-1}yg)$$
$$= (g^{-1}xg)^{-1}(g^{-1}yg)^{-1}(g^{-1}xg)(g^{-1}yg) \in G'.$$
Therefore, for any $\alpha \in G'$, $g^{-1}\alpha g \in G' \forall g \in G$.
So $G' \triangleleft G$.

E27) As in E17, you can show that $S_n/A_n = \{A_n, A_n(1\ 2)\}$.

E28) Since $\left|G/G\right| = \frac{|G|}{|G|} = 1$, $G/G = \{G\} = \{\bar{e}\}$.

Since $\left|G/\{e\}\right| = |G|$ and $\bar{g} = g\{e\} \in G/\{e\} \forall g \in G$, $G/\{e\} = \{\bar{g} \mid g \in G\}$.
Also note that for $g, h \in G$, $\bar{g} = \bar{h} \Leftrightarrow gh^{-1} \in \{e\} \Leftrightarrow g = h$.
So $G/\{e\} \simeq G$.

E29) i) Let $G = \langle g \rangle$ and $H \le G$. Then $H = \langle g^m \rangle$ for some $m \in \mathbb{Z}$.
Then $G/H \supseteq \{H, gH, g^2H, \cdots, g^{m-1}H\} = S$, say.
Also, any element of G/H is of the form $g^tH = (gH)^t$.
If $t \le m$, then $g^tH \in S$.
If $t > m$, then $t = mq + r$, where $r < m$.
So $g^t = (g^m)^q \cdot g^r$, hence $g^tH = g^rH \in S$.
Thus, $G/H = S$, and $S = \langle gH \rangle$, the cyclic group generated by gH.

ii) This is not true. For instance, consider the Klein 4-group. It is not cyclic, but all its non-trivial quotient groups are cyclic, being of order 2.

E30) Let $\bar{x}, \bar{y} \in G/H$, where $x, y \in G$. Then $\bar{x}\,\bar{y} = \overline{xy} = \overline{yx} = \bar{y}\,\bar{x}$. Hence the result.

E31) This is not true. For instance, $Q_8/<i>$ is abelian, since it is cyclic. But Q_8 is not abelian, as you know.

E32) π is well-defined because $g_1 = g_2 \Rightarrow g_1H = g_2H$, that is, $\pi(g_1) = \pi(g_2)$.
Further, $\pi(g_1 g_2) = \overline{g_1 g_2} = \bar{g}_1\,\bar{g}_2 = \pi(g_1)\,\pi(g_2)$.

Now, any element of G/H is $\bar{g}$ for some $g \in G$, which is $\pi(g)$. So, π is surjective.
$\text{Ker}\,\pi = \{g \in G \mid \pi(g) = H\} = \{g \in G \mid g \in H\} = H$.

E33) $i(r_1 + r_2) = r_1 + r_2 = i(r_1) + i(r_2)$.
You can show that $\text{Im}\, i = \mathbb{R}$ and $\text{Ker}\, i = \{0\}$.

E34) For $x, y \in G$, $(g \circ f)(xy) = g(f(xy)) = g(f(x)f(y))$
$= g(f(x))g(f(y)) = (g \circ f)(x)(g \circ f)(y)$.
Thus, $g \circ f$ is a group homomorphism from G_1 to G_3.

E35) To check that ϕ is well-defined, let $\bar{r} = \bar{s}$ in $\mathbb{Z}_n$.
Then $n \mid (r - s)$, say $(r - s) = nt$.
Then $\zeta^{r-s} = \zeta^{nt} = 1$, that is $\zeta^r = \zeta^s$. Hence ϕ is well-defined.
Now, $\phi(\bar{r}_1 + \bar{r}_2) = \phi(\overline{r_1 + r_2}) = \zeta^{r_1 + r_2} = \zeta^{r_1} \cdot \zeta^{r_2} = \phi(\bar{r}_1) \cdot \phi(\bar{r}_2)$, so that ϕ is a group homomorphism.
Next, $\text{Ker}\,\phi = \{\bar{r} \in \mathbb{Z}_n \mid \zeta^r = 1\} = \{\bar{r} \in \mathbb{Z}_n \mid n \mid r\} = \{\bar{0}\}$.
Finally, any element of U_n is ζ^r, where $0 \le r \le n-1$ and $\zeta^r = \phi(\bar{r})$, where $\bar{r} \in \mathbb{Z}_n$. So ϕ is onto.
Hence ϕ is a group isomomorphism.

E36) Let $f : G_1 \to G_2$ be an onto group homomorphism, where $G_1 = <g>$.
Then, any element of G_2 is of the form $f(g^m) = [f(g)]^m$ for some $m \in \mathbb{Z}$. Thus, $G_2 = <f(g)>$.
For the converse, consider $\pi : S_3 \to S_3/A_3$. Then π is an onto homomorphism and S_3/A_3 is cyclic, but S_3 is not.

E37) **Outline of proof of FTH:** Define $\psi : G/\text{Ker}\,\phi \to \text{Im}\,\phi : \psi(\bar{g}) = \phi(g)$.

Check that ψ is well-defined, ψ is a group homomorphism, ψ is surjective, ψ is $1-1$. ■

To prove Theorem 8, define $\lambda : H \to HK/K : \lambda(h) = \bar{h}$. Check that λ is well-defined, a group homomorphism, and surjective. Now
$\text{Ker}\,\lambda = \{h \in H \mid \bar{h} = \bar{1}\} = \{h \in H \mid h \in K\} = H \cap K$.
Now apply Theorem 7 to get the result. ■

To prove Theorem 9, define $\mu : G/K \to G/H : \mu(gK) = gH$.
Show that μ is well-defined, a group homomorphism, surjective and $\operatorname{Ker}\mu = H/K$. Now apply Theorem 7 to get the result. ■

E38) By Theorem 8, $HK/H \simeq K/H \cap K$.
Thus $\frac{o(HK)}{o(H)} = \frac{o(K)}{o(H \cap K)}$. Hence the result.

E39) i) Firstly, since $G \neq \emptyset$, the identity map, $I \in \text{Aut } G$. So $\text{Aut } G \neq \emptyset$.
Next, if $\psi_1 : G \to G$ and $\psi_2 : G \to G$ are in Aut G, then $\psi_1 \circ \psi_2 : G \to G$ is also an isomorphism. Thus, $\circ$ is closed on Aut G.
Thirdly, $\circ$ is associative, since it is associative in general.
Fourthly, $I : G \to G : I(g) = g$ is the identity w.r.t. $\circ$.
Finally, $\psi^{-1} : G \to G$ is the inverse of $\psi : G \to G$.
Thus, $(\text{Aut } G, \circ)$ is a group.

ii) You can check that f_g is 1-1, onto and a group homomorphism, for each $g \in G$.

iii) Firstly, since $G \neq \emptyset$, Inn $G \neq \emptyset$. (Why?)
Next, for $g_1, g_2 \in G$, show that $f_{g_1} \circ f_{g_2} = f_{g_2 g_1} \in \text{Inn } G$.
Also, for $g_1 \in G, (f_{g_1})^{-1} = f_{g_1^{-1}} \in \text{Inn } G$.
So $\text{Inn } G \leq \text{Aut } G$.
Now, for $\psi \in \text{Aut } G$ and $g \in G$, show that
$\psi^{-1} \circ f_g \circ \psi = f_{\psi^{-1}(g)} \in \text{Inn } G$. Hence $\text{Inn } G \lhd \text{Aut } G$.

iv) Define $f : G \to \text{Inn } G : f(g) = f_{g^{-1}}$.
Check that f is well-defined.
Now $f(g_1 g_2) = f_{(g_1 g_2)^{-1}} = f_{g_2^{-1} g_1^{-1}} = f_{g_1^{-1}} \circ f_{g_2^{-1}} = f(g_1) f(g_2)$.
Thus f is a group homomorphism.
Also, check that f is onto.
Finally, $\operatorname{Ker} f = \{g \in G \mid f_{g^{-1}} = f_e\} = \{g \in G \mid gxg^{-1} = x \, \forall \, x \in G\}$
$= \{g \in G \mid gx = xg \, \forall \, x \in G\}$
$= Z(G)$.
Now use FTH to get the result.

E40) Let $G = \langle g \rangle$ be a cyclic group.
Define $\psi : \mathbb{Z} \to G : \psi(m) = g^m$.
Now ψ is a well-defined group homomorphism, which is surjective.
If G is an infinite cyclic group, then $\psi(m) = \psi(n) \Rightarrow m = n$, that is, $\operatorname{Ker}\psi = \{0\}$.
And hence, by FTH, $\mathbb{Z} \simeq \langle g \rangle$.
If G is finite, of order n, then $g^n = e$ and if $g^m = e$ for $m \in \mathbb{Z}$, then $n \mid m$. So, in this case, $\operatorname{Ker}\psi = \{m \in \mathbb{Z} \mid g^m = e\} = n\mathbb{Z}$.

Thus, by FTH, $\mathbb{Z}/_{n\mathbb{Z}} \simeq \langle g \rangle$.

Since $\mathbb{Z}/_{n\mathbb{Z}} \simeq \mathbb{Z}_n, \langle g \rangle \simeq \mathbb{Z}_n$.

E41) Define $\psi : \mathbb{Z}_5 \to S_5 : \psi(\overline{1}) = (1\ 2\ 3\ 4\ 5)$ and $\psi(\overline{0}) = I$, and extend ψ to be a homomorphism, that is, $\psi(\overline{m}) = m\psi(\overline{1})\ \forall\ m = 2, 3, 4, 5$.
Then, the way ψ is defined, it is a group homomorphism which is 1-1, and $\text{Im}\,\psi = \langle (1\ 2\ 3\ 4\ 5) \rangle \leq S_5$.

E42) Yes, it would. This is because, by Cayley's theorem, the finite group is isomorphic to a finite permutation group; and being isomorphic they must satisfy the same properties.