# UNIT 10 POLYNOMIALS: BASIC CONCEPTS AND FACTORING

## Structure



## 10.1 INTRODUCTION

This unit assumes that the students have already understood the basic ideas regarding dividend, divisor, quotient and remainder in arithmetic in earlier classes. Also, they are familiar with the concepts of prime and composite numbers. They also know factorization of numbers into prime factors in arithmetic. However, a brief recapitulation by you (the teacher) can help them to revise their basic concepts before you start actual teaching of this unit. The concepts to be taught in this unit will be effectively learnt if they are related to the previous knowledge of the students. The concepts and techniques learnt in this unit would be helpful in learning algebra later.

## 10.2 OBJECTIVES

At the end of this unit, you will be able to:

- clarify to your students the basic concept of variables, constant, algebraic expressions, their numerical values and zeros of a polynomial;
- explain to your students the difference between monomials, binomials and trinomials in one and two variables along with the meanings of degree, coefficients and the constant terms;

- develop in your students, the skills of performing fundamental operations on polynomials proficiently and accurately;
- clarify to your students the meaning and purpose of factorization of polynomials in algebra;
- develop in them skills to perform factorization of polynomials by using basic algebraic identities;
- develop in your students abilities and skills to factorize quadratic polynomials by splitting the middle term;
- clarify to your students the meaning of the statement of the remainder theorem;
- enable your students to compute remainder without actual division when dividend polynomial is given and divisor is of the form (x–a);
- enable them to examine whether a given binomial of the form (x–a) is a factor of a given polynomial p(x) or not;
- enable your students to use remainder theorem to factorize polynomials of a degree higher than two, and
- enable them to apply the ideas and skills developed in this chapter to solve problems.

## 10.3 BASIC CONCEPTS OF POLYNOMIALS

### Main Teaching Points

(a) Meaning of a variable, terms and expression

(b) Value of a expression for given values of the variables.

### Teaching Learning Process

Teaching of algebra should begin with the relationship between arithmetic and algebra. While arithmetic is mainly concerned with the techniques of performing the four fundamental operations with numbers, algebra is concerned with general principles and properties of these operations. In other words, algebra generalises arithmetical concepts.

In algebra, we use alphabets to represent value, size and quantity in different situations. The fact, that a variable denoted by a letter is not tied to any fixed numerical value, but can take different values, is a great help in deriving general results which can be applied to any specific numerical values subsequently. For example, the fact that the "perimeter of a rectangle can be obtained by adding the lengths of its four sides" can be written algebraically as follows:

$$P = 2a + 2b$$

or $$P = 2(a + b)$$

Where P is the perimeter of the rectangle and 'a' and 'b' are its length and breadth respectively. This formula is general and is applicable to rectangles of all sizes.

In algebra, the letters like P, a and b, which are used to represent different values, or elements of a number system are often referred to as unknowns or variables. $x$, $x^2$, $5x^3$ $x/y$ are expressions in which a variable has been multiplied by a number (as in $2x$) or a variable has been raised to a power (as in $x^2$), or multiplied by a number (as in $5x^3$) or one variable has been divided by another as in $x/y$. All such expressions are called **terms**. A number of terms put together and connected by one or more fundamental operations is called an algebraic expression e.g.

$$2x + y,\ 3x^2 + 2x + 1,\ (x^2 + 5)(x + 2),\ x/y + 4xy - 7.$$

An algebraic expression becomes a number when all its variables are assigned definite numerical values and indicated operations are performed for those numerical values. Such a number is called the **value** of the algebraic expression. For example, when $x = 2$, $y = 3$ the value of the expression $2x + y$ is evaluated as follows:

$$2x + y = 2(2) + 3 = 4 + 3 = 7$$

If the first term in an expression or within a bracket has a positive sign, the sign is not indicated. The expression $3x^2 - 2xy + y^2$ is made of three terms $+3x^2$, $-2xy$ and $+ y^2$. For given values of the variables, the value of an algebraic expression is determined by adding the value of its terms. In the above example, when $x = 2$ and $y = 3$,

$$+3x^2 = +3\,(2)^2 = +12$$

$$-2xy = -2\,(2)\,(3) = -12$$

$$+y^2 = +(3)^2 = +9$$

Therefore, the full value of the expression $3x^2 - 2xy + y^2 = 12 - 12 + 9 = 9$.

**Methodology used:** Lecture-cum-discussion method to be used.

## 10.4 POLYNOMIALS: CONCEPTS AND DEFINITIONS

**Main Teaching Points**

(a) Definition of Polynomial

(b) Monomials, Binomials

**Teaching Learning Process**

If an algebraic expression consists of only terms in which the variables is raised to a positive integral exponent, then it is called a polynomial.

**Ask:** Consider the following algebraic expressions:

| | | |
|---|---|---|
| (i) $3x^2$, | (ii) $3x + 2$, | (iii) $x^2 - 4x + 2$, |
| (iv) $x^5 - 7$, | (v) $x^2 + 3xy + y^2$, | (vi) $x^3 - \sqrt{x} + 4$, |
| (vii) $x/y$, | (viii) $5x^2/y + z^3$. | |

How many variables are there in these algebraic expressions? There is only one variable in the first four expressions and in (vi); in (v) and (vii) there are two variables, whereas (viii) has three variables x, y and z.

What are the exponents of the variable? In (i), (ii), (iii), (iv) and (v) they are positive whole numbers.

**Explain:**

> An algebraic expression which consists of **one or more terms** such that the **variable** or variables in the terms are **raised** to a **positive integral power** is called a polynomial.

If there is only one variable, say, x, in all its terms, we call it a polynomial in x, denoting it as p(x).

Ask the students to recall the notion of a function and let them deduce that a polynomial in one variable (say x) is a function of that variable (that is, a function of x).

The standard expression of a polynomial in x is:

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

Where $a_0$, $a_1$, .... $a^n$ are constants and n is a positive integer.

Polynomials consisting of one term are known as monomials. Examples of monomials are 2x, $3x^2$, $7x^3$, $\frac{1}{2}x^2$ and 7. A similar expression with two terms is called a **binomial** and the one having three terms is called a **trinomial**. Examples of binomials and trinomials are given below:

| Binomials | Trinomials |
|---|---|
| $x + 1$ | $2x^2 + 3x + 1$ |
| $2x + 4$ | $3t^2 - 2t + 4$ |

$a + b$ $\qquad$ $x^2 + xy + y^2$

**Methodology used:** Inductive method using different examples is used.

### 10.4.1 Polynomials in One Variable

**Main Teaching Points**

(a) Degree of a polynomial, and

(b) Standard form of the polynomial.

**Teaching Learning Process**

The polynomials involving one variable are called **polynomials** in one variable. Since polynomials are functions, these are represented by a letter 'p'. A polynomial in x is designated as p(x). Similarly, polynomials in x, t, y may be written as

$$p(x) = 3x^2 + 2x + 1$$

$$p(t) = 2t^3 - 3t^2 + t - 5$$

$$p(y) = 2y^2 - 3y + 4$$

The value of the polynomial p(x) for x = 2 is written as p(2).

Here $\quad p(2) = 3(2)^2 + 2(2) + 1$

$\qquad\qquad = 12 + 4 + 1 = 17.$

The polynomial $p(x) = 3x^2 + 2x + 1$ has three terms $3x^2$, 2x and 1. In the first term $3x^2$, 3 is called the coefficient of $x^2$, in the second term 2x, 2 is the coefficient of x; the third term 1 is known as the constant term.

A **polynomial** consisting of **only constant** terms is known as a **constant polynomial.**

**Ask:** What is the highest value of the exponent of the variable in each of the polynomials?

$3x^2 + 4x - 7$, $3x^3$

In the first polynomial, there are three terms. $3x^2$ has the highest exponent of x. It is 2.

In the second polynomial, there is only one term, $3x^3$ and the exponent of x is 3.

**Explain**

The **exponent** (loosely described as power) of the **variable** in a **term** is called the **degree of that term.**

The **highest exponent** of the variable in any of the terms of a polynomial is called the **degree** of the polynomial. E.g. $3x^2 + 4x + 7$ is a polynomial of degree 2.

$3x^3$ is a monomial (a polynomial with one term only). Its degree is 3.

In general, a polynomial of the form $a_0 x^n + a_1 x^{n-1} + .... + a_n$ is a polynomial of degree n where $n \in I^+$.

The real numbers $a_0, a_1, a_2, .... a_n$ are called the coefficients of the terms of the polynomial. The coefficients may be integers, rational numbers or real numbers. The order of the terms in the polynomial is immaterial. For example.

$p(x) = 3x^3 + 4x^2 - 2x + 2$ is the same as

$p(x) = 2 - 2x + 4x^2 + 3x^3$.

However, generally polynomials are written in either ascending or descending powers of the variable. This form of the polynomials is called a standard form of the polynomial.

**Methodology used:** After giving various examples, inductive reasoning is used to generalise the concept.

### 10.4.2 Polynomials in Two Variables

**Ask:** Observe the polynomials: $3x^2 + 5xy + y^2$, $s^2 - sr - r^2$, $x^2 - y^2$. How are these polynomials different from the examples given earlier?

Each of these polynomials have two variables while there was only one variable in the previous examples.

**Explain**

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The **polynomials** involving **two variables** or unknowns such as x and y are known as **polynomials in two variables.**

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Such a polynomial is designed by p(x. y). The following are a few examples of polynomials in two variables.

$$p(x, y) = 3x^2 + 2xy + y^2$$

$$p(r, s) = 2r^2 - 3rs + 4s^2$$

$$p(s, t) = s^2 + 3st - 2t^2.$$

## 10.5 OPERATIONS ON POLYNOMIALS

Before teaching the operations on polynomials, the teacher should clarify to his/her students that there is a lot of similarity between basic operations on integers and operations on polynomials. Like numbers, polynomials can also be added, subtracted, divided or multiplied. Except in the case of division, the result of other operations in polynomials is also a polynomial. This is the case with integers also where except for division, the result of other operations on integers is again an integer.

### 10.5.1 Value of a Polynomial and Zeroes of a Polynomial

**Main Teaching Points**

(a) To find the value of a polynomial, and

(b) To clarify the meaning of zeroes of a polynomial.

**Teaching Learning Process**

The value of a polynomial for given values of the variables is obtained by substituting these values of the variables in the polynomial and simplifying the numerical result obtained.

**Ask:** How do you find the value of a polynomial for some assigned value of the variable?

**Explain:** It is illustrated as follows:

Let $p(x) = 2x^3 + x^2 - 3x - 2$ be a polynomial in x. Find value of p(x) for x = 2.

The value of p(x) for x = 2 is denoted by p(2) and

$$\begin{aligned} p(2) &= 2(2)^3 + 2^2 - 3(2) - 2 \\ &= 2(8) + 4 - 6 - 2 \\ &= 16 + 4 - 6 - 2 \\ &= 20 - 8 \\ &= 12. \end{aligned}$$

**Explain:** The value of a polynomial will change as the value assigned to x changes. For x = −2, the above polynomial assumes the value p (−2) given by :

$$\begin{aligned} p(-2) &= 2(-2)^3 + (-2)^2 - 3(-2) - 2 \\ &= 2(-8) + 4 + 6 - 2 \end{aligned}$$

$$= -16 + 4 + 6 - 2$$
$$= -18 + 10$$
$$= -8$$

For x = 5, it assumes the values p(5) given by:

$$\begin{aligned} p(5) &= 2(5)^3 + 5^2 - 3(5) - 2 \\ &= 2(125) + 25 - 15 - 2 \\ &= 250 + 25 - 15 - 2 \\ &= 275 - 17 \\ &= 258 \end{aligned}$$

Similarly, if x = –1, it assumes the values p(–1) given by:

$$\begin{aligned} p(-1) &= 2(-1)^3 + (-1)^2 - 3(-1) - 2 \\ &= 2(-1) + 1 + 3 - 2 \\ &= -2 + 1 + 3 - 2 \\ &= -4 + 4 \\ &= 0 \end{aligned}$$

**Explain**

Any **value** n assigned to x for which the value of **p(x)** becomes zero i.e., p(n) = 0 is called a '**zero**' of the polynomial **p(x)**.

There are as many zeros of a polynomial as the degree of the polynomial. However, these may be real numbers or complex numbers. The teacher should carefully select a few polynomials, one or more zeroes of which may be integral.

Students should be asked to explore and find these zeroes.

**Methodology used:** Through illustrations, students should be taught to find the value of a polynomial and then inductively define zeroes of a polynomial.

## 10.5.2 Addition of Polynomials

**Main Teaching Point**

How to add two polynomials?

**Teaching Learning Process**

**Ask:** If we add $2x^2$ and $5x^2$, the sum is $7x^2$. However , if we add $2x^2$ and 5x, the sum is written differently, i.e., the sum is written as $2x^2 + 5x$. (This is often referred to as 'indicated sum'). Why are these two sums written differently?

In the first example, the terms are **similar** (i.e., except for their coefficients, the variable and its exponents are identical). Similar terms are added by adding coefficients and writing the variable part as it is. In the second example, terms are not similar, so the sum cannot be simplified.

**Explain:** In the case of polynomials also, similar terms in the ploynomials are added, other terms are shown as indicated addition.

While adding any two or more polynomials, the first step is to re-arrange the terms of the polynomials such that similar terms occur in the same column. Suppose the following polynomials.

$$p(x) = 3x^3 + x^2 - 5$$
$$p'(x) = 2x^2 + 5x + 2$$

are to be added. The terms of the polynomials may be rearranged in the following manner:

$$\begin{array}{r} 3x^3 + x^2 \qquad - 5 \\ + 2x^2 + 5x + 2 \\ \hline 3x^3 + 3x^2 + 5x - 3 \\ \hline \end{array}$$

Explain that it can be seen in the example that the term $2x^2$ of the second polynomial is written below the corresponding term $x^2$ of the first polynomial. Similarly, the constant term + 2 is placed below the constant term –5. Since the term involving $x^3$ does not exist in the other polynomial, their places have been left blank to facilitate the process of addition. After such a rearranged presentation, the monomials (or terms) in each column are added with due regard to their algebraic signs. The sum of the polynomials is the polynomial

$$3x^3 + 3x^2 + 5x - 5$$

In the same way, the sum of the polynomials

$3x - y$ $2y - 2x$ and $x + y$ may be found as follows

$$\begin{array}{r} 3x - y \\ -2x + 2y \\ x + y \\ \hline 2x + 2y \\ \hline \end{array}$$

**Explain:** This method of addition is called **column method of addition.** However, polynomials may be added in a row, combining like terms. e g.,

$$(3x - y) + (-2x + 2y) + (x + y) = (3x - 2x + x) + (-y + 2y + y) = 2x + 2y$$

Similarly in

$$(3x^3 + x^2 - 5) + (2x^2 + 5x + 2)$$
$$= 3x^3 + (x^2 + 2x^2) + 5x + (-5 + 2)$$
$$= 3x^3 + 3x^2 + 5x - 3$$

This method of addition may be described as row method of addition.

The teacher should give a few more similar examples to clarify the process of addition of polynomials to the students. In the end, a few problems may be given as exercises. Students should be asked to practice both column method of addition and row method of addition.

**Methodology used:** Lecture-cum-discussion method is used to illustrate that only similar terms (on like terms) can be added.

**Check Your Progress**

**Notes :** a) Write your answer in the space given below.
b) Compare your answer with the one given at the end of the unit.

How will you illustrate to your students the addition of the following polynomials?

1. $p(x) = 2x^2 + 3x + 5$, $q(x) = 3x^2 - 4x - 7$

2. $p(x) = x^3 - 2x - 3$, $q(x) = x^2 + 3x + 1$

3. $p(t) = 2t^2 + t - 1$, $q(t) = 3t - 5 - 3t^2$ $r(t) = 1 - 3t - 3t^2$

4. $p(k) = k^4 + k^2 + 2$, $q(k) = k^2 + 2k + 1 r(k) = k^3 - 3k + 3$

### 10.5.3 Subtraction of Polynomials

**Main Teaching Point**

To subtract one polynomial from another.

**Teaching Learning Process**

The subtraction of polynomials involves the same principles and processes as the subtraction of signed numbers (integers). You should make your students realise that only like terms or quantities can be added or subtracted. Polynomials being composed of terms, may be added or subtracted by placing like terms in the same column. If 3xy is to be subtracted from 5xy, the first step is to place them such as given below:

$$\begin{array}{r} 5xy \\ -3xy \\ \hline 2xy \\ \hline \end{array}$$

Since the two terms are similar in terms of powers of x and y, these may be subtracted as given above. The above process may also be presented as given below:

$$5xy - 3xy = (5 - 3)xy = 2xy$$

Explain the steps in subtraction with the help of an example as given below:

**Example:** Subtract $x - y$ from $2x + 3y$

| Step | Solution |
|---|---|
| 1. Write the two polynomials one below the other with like terms in the same column. | $\begin{array}{r} 2x + 3y \\ -(x - y) \end{array}$ |
| 2. Because of negative sign outside the brackets, change the sign of each term and then add. | $\begin{array}{r} 2x + 3y \\ -x + y \\ \hline x + 4y \\ \hline \end{array}$ |
| 3. The answer is $x + 4y$. | |

**Example:** Substract $3c + 7d^2$ from $4c - d^2$

**Solution**

$$\begin{array}{r} 4c - d^2 \\ -(3c + 7d^2) \end{array} \quad \text{or} \quad \begin{array}{r} 4c - d^2 \\ -3c - 7d^2 \\ \hline c - 8d^2 \\ \hline \end{array}$$

Explain that in the alternate method in the above two examples, we are adding the opposite (in respect of sign) of the subtrahend (the expression to be subtracted). Thus subtraction is the inverse operation of addition.

We can also perform the subtraction in a row.

**Example:** Subtract $2x^2 + 2y^2 - 6$ from $3x^2 - 7y^2 + 9$

**Solution**

$$3x^2 - 7y^2 - 9 - (2x^2 + 2y^2 - 6)$$
$$= 3x^2 - 7y^2 + 9 - 2x^2 - 2y^2 + 6$$
$$= (3x^2 - 2x^2) + (-7y^2 - 2y^2) + (9 + 6)$$
$$= x^2 - 9y^2 + 15$$

The above examples clarify the process of subtracting one polynomial from another polynomial.

Thus, the process of subtraction involves the following steps:

(i) arranging the terms of the polynomial in such a way that like terms of the two polynomials occur in the same column,

(ii) changing the algebraic sign (from + to – and vice versa) of the terms of the polynomial to be subtracted, and

(iii) adding the resultant polynomials term-wise or column-wise to find the required difference.

Let $p(x) = 3x^3 - 4x + 7$

$q(x) = 2x^3 + x^2 - 5x - 3$

and $r(x) = 3x^2 - 2x + 1$

Ask the students to compute

$p(x) - q(x)$, $q(x) - p(x)$, $[\, p(x) - q(x)\,] - r(x)$ and

$p(x) - [\, q(x) - r(x)\,]$.

**Ask:** Is $p(x) - q(x)$ same as $q(x) - p(x)$?

The two answers are different.

**Explain:** $p(x) - q(x)$, $q(x) - p(x)$. In other words, subtraction is not commutative.

Also, show that $[p(x) - q(x)] - r(x)$ is not same as $p(x) - [q(x) - r(x)]$.

Explain that subtraction of polynomials is not associative.

Point out that whereas addition of polynomials is commutative and associative, subtraction is neither commutative nor associative.

**Methodology used:** Lecture-cum-discussion method is used. Sufficient practice should be provided so that the students develop the skill.

**Check Your Progress**

**Notes :** a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

5. Perform the subtractions given below:

i) $3x - y$
$-(2x - y)$

ii) $a^2 - 3b + c^2$
$-(a^2 + 5b + 3c^2)$

6. Subtract $-3x + 8y$ from $-6x - 8y$

7. Subtract $x^2 - 7y^2 + 1$ from $2x^2 + 3y^2 - 3$

8. What should be subtracted from $x^4 - 1$ to get $3x^2 - 2x + 1 + x^4$

9. What should be added to $x^3 + 3x^2 + 1$ to get $x^5 - 2x^2 - 3x$

### 10.5.4 Multiplication of Polynomials

**Main Teaching Points**

(a) To multiply polynomial by a monomial

(b) To multiply polynomial by a polynomial

**Teaching Learning Process**

Three fundamental properties of operations are most commonly used in the process of multiplication of polynomials. The associative law expresses the fact that the quantities may be associated in any order. For example the product of x, y and x may be written in any of the following forms:

$$xyz,\ (xy)z,\ x(yz)$$

The commutative law expresses the fact that the product remains the same regardless of the order in which the factors are engaged. This suggests that the following terms express the same quantity.

$$xyz,\ yxz,\ xzy,\ zyx,\ zyx,\ \text{etc.}$$

The distributive law expresses the fact that in multiplication, each term of the multiplicant is affected by the multiplier. The following identity clarifies the idea.

$$\mathbf{a\,(x + y + z) = ax + ay + az}$$

The distributive law plays a very important role in the multiplication of polynomials.

When a monomial is multiplied by another monomial, laws of exponents are also useful. For example,

$$\begin{aligned}(3x^2)(5x^3) &= (3.x.x)(5.x.x.x)\\ &= (3.5)(x.x.x.x.x)\\ &= 15\,x^5\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}(4x^3)(3x^5) &= 12x^8\ (= 12x^{3+5})\\ (x^5)(4x^7) &= 4x^{12}\ (= 4x^{5+7})\end{aligned}$$

Here, we are using

$$\boxed{\mathbf{x^m . x^n = x^{m+n}}}$$

and in general, $ax^m . bx^n = abx^{m+n}$, for all real values of a, b, m and n.

A polynomial with more than one term is multiplied by a monomial as below:

$$\begin{array}{l} 2x^2 + 3x - 5\\ \times \qquad 3x^2\\ \hline 6x^4 + 9x^3 - 15\,x^2\\ \hline \end{array}$$

This may be explained as follows:

$$\begin{aligned}&3x^2\ (2x^2 + 3x - 5)\\ &= 3x^2 . 2x^2 + 3x^2 . 3x + 3x^2 . (-5) \text{ ........ (by distributive law)}\\ &= (3 . 2) . (x^2 . x^2) + (3 . 3) . (x^2 . x) + \{3. (-5)\} . x^2\\ &= 6x^4 + 9x^3 - 15x^2\end{aligned}$$

(In actual practice, we straight-away write the first and the last steps without showing the intermediate steps).

Give several examples of multiplication of a polynomial of many terms by a monomial.

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**Explain:** To multiply a monomial by another monomial we rearrange numbers together and variables together (use commutative and associative property).

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial.

In multiplying variables with powers (exponents) we apply the laws of exponents.

Multiplication without using columns may be illustrated as below:

$$\begin{aligned}&(2x + 3y)(2x^3 - xy + 2y^2)\\ &= 2x\,(2x^3 - xy + 2y^2) + 3y\,(2x^3 - xy + 2y^2)\end{aligned}$$

$$= (4x^4 - 2x^2y + 4xy^2) + (6x^3y - 3xy^2 - 6y^3)$$
$$= 4x^4 - 2x^2y + (4xy^2 - 3xy^2) - (6x^3y - 6y^3)$$
$$= 4x^4 - 2x^2y + xy^2 + 6x^3y + 6y^3$$

**Methodology used:** The method of multiplication is illustrated through various examples and sufficient practice should be provided to develop the skill.

**Check Your Progress**

**Notes :** a) Write your answer in the space given below.
b) Compare your answer with the one given at the end of the unit.

How will you explain to find the following products?

10. $(2x + 3)(x + 4)$

11. $(x^3 - x^2 - 1)(2x^2 + x + 3)$

12. $(x + y)(x^3 - xy^2 - y^3)$

13. $(x^4 - y^4)(x^2 + xy - y^2)$

### 10.5.5 Division of Polynomials

**Main Teaching Points**

(a) To divide a polynomial by a monomial.

(b) To divide a polynomial by a polynomial.

**Teaching Learning Process**

The process of division in the case of polynomials is the same as in arithmetic. The similarity between the two processes should be made clear to the pupils by giving suitable examples. A polynomial may be divided by a monomial, a binomial or a polynomial. In these three situations, the computational procedures may be different, but the mathematical ideas are the same.

While dividing a polynomial by a monomial, each term (monomial) is divided by the divisor.

For example,

$$\left(3x^5 + 2x^2 - 3x\right) \div 3x = \frac{3x^5}{3x} + \frac{2x^2}{3x} - \frac{3x}{3x} = x^4 + \frac{2x}{3} - 1$$

The essential process involves the same process as the ordinary division of numbers. Consider

$$\frac{3x^5}{3x} = \frac{3 \cdot x \cdot x \cdot x \cdot x \cdot x}{3x} = x.\,x.\,x.\,x = x^4$$

In general we can use:

$$\frac{x^m}{x^n} = x^{m-n}$$

Also,

$$\frac{ax^m}{bx^n} = \frac{a}{b}x^{m-n}$$

Consider another example

**Example :** $\left(7x^2 - 4x\right) \div x = \frac{7x^2}{x} - \frac{4x}{x} = 7x - 4$

Similarly,

$$\frac{x^6 - 3x^4 + 2x^2}{3x^2}$$

$$= \frac{x^6}{3x^2} - \frac{3x^4}{3x^2} + \frac{2x^2}{3x^2}$$

$$= \frac{1}{3}x^4 - x^2 + \frac{2}{3}$$

The process of dividing a polynomial by another polynomial is quite similar to long division for integers. In the case of integers, we start with the division of the highest place-value digit by the divisor. In the same way we write the dividend and the divisor both in decreasing powers of the variables and then divide as illustrated in the following example:

Let us take an example to understand the steps in dividing a polynomial by a binomial.

**Example:** Divide $3x^2 + 7x - 6$ by $3x - 2$

**Steps**

1. Write the polynomials in decreasing powers of the variable.

   The polynomials are already in this form.

2. Divide the first term of the dividend by the first term of the divisor and write the monomial obtained as the first term of the quotient.

   $3x^2 \div 3x = x$

3. Multiply the divisor by the quotient of step 2, write the resulting polynomial below the dividend and subtract.

$$\begin{array}{r|l} & x \\ \hline 3x-2 & 3x^2 + 7x - 6 \\ & 3x^2 - 2x \\ & - \quad + \\ \hline & \quad 9x - 6 \end{array}$$

4. Divide the first term of the resulting polynomial with the first term of the divisor and write it as the second term of the quotient (with proper sign).

   $9x \div 3x = 3$

5. Multiply the divisor by the second term of the quotient, write the resulting polynomial below the dividend and subtract.

$$
\begin{array}{r|l}
 & x \;\; + 3 \\
\cline{2-2}
3x-2 & 3x^2 + 7x - 6 \\
 & 3x^2 - 2x \\
 & - \quad + \\
\cline{2-2}
 & \quad 9x - 6 \\
 & \quad 9x - 6 \\
 & \quad - \quad + \\
\cline{2-2}
 & \qquad 0 \\
\cline{2-2}
\end{array}
$$

6. This process continues and it stops when the remainder is either zero or a polynomial whose degree is less than that of the divisor.

The binomial x + 3 is the required quotient.

**Example:** Divide $6x^4 - 5x^3 + 2x^2 - 7$ by $2x^2 - x + 1$

$$
\begin{array}{r|l}
 & 3x^2 - x - 1 \\
\cline{2-2}
2x^2 - x + 1 & 6x^4 - 5x^3 + 2x^2 - 7 \\
 & 6x^4 - 3x^3 + 3x^2 \\
\cline{2-2}
 & \quad -2x^3 + x^3 - 7 \\
 & \quad -2x^3 + x^3 - x \\
\cline{2-2}
 & \quad -2x^2 + x - 7 \\
 & \quad -2x^2 + x - 1 \\
\cline{2-2}
 & \qquad\qquad -6 \\
\cline{2-2}
\end{array}
$$

The polynomial $3x^2 - x - 1$ is the required quotient, and the remainder is $-6$

**Example:** Divide $3p - p^2 - 1 - 3p^3 - 2p^4$ by $2p^2 - 1 + p$

In this example, as a first step, the polynomials should be written in order of descending powers of p. The problem may be set as:

$$
\begin{array}{r|l}
 & p^2 - 2p + 1 \\
\cline{2-2}
2p^2 + p - 1 & 2p^4 - 3p^3 - p^2 + 3p - 1 \\
 & 2p^4 + p^3 - p^2 \\
\cline{2-2}
 & -4p^3 + \qquad + 3p - 1 \\
 & -4p^3 - 2p^2 \quad + 2p \\
\cline{2-2}
 & \qquad 2p^2 + p - 1 \\
 & \qquad 2p^2 + p - 1 \\
\cline{2-2}
 & \qquad\qquad 0 \\
\cline{2-2}
\end{array}
$$

By giving suitable examples show that division of polynomials is neither commutative nor associative, i.e.,

$p(x) \div q(x) \neq q(x) \div p(x)$ and $[p(x) \div q(x) \div r(x) \neq r(x) \neq p(x) \div [q(x) \div r(x)]$

**Methodology used:** The method is illustrated by using different examples and a deductive approach. Sufficient practice is required to develop the skill.

**Check Your Progress**

**Notes :** a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Divide and write the quotient and remainder in each of the following:

14. $(6m^2 - 17m - 3) \div (m - 3)$

15. $(8y^2 - 6y - 5) \div (2y + 1)$

16. $(3p^2 - 5pt - 2t^2) \div (p - 2t)$

17. $(6x^2y^2 - 7xyz - 3z^2) \div (3xy + z)$

18. $(27x^3 - 1) \div (3x - 1)$

19. $(x^3 - 1) \div (x + 1)$

## 10.6 FACTORIZATION OF POLYNOMIALS

In the preceding section your students were introduced to the methods of multiplying polynomials to obtain polynomials of higher degree. Now, we shall consider the reverse procedure, i.e., techniques of writing a polynomial as a product of two or more polynomials of lower degrees.

### 10.6.1 Basic Concepts

**Main Teaching Points**

(a) What is a factor?

(b) Meaning of factorization.

**Teaching Learning Process**

As your students have learnt in arithmetic, factors are two or more numbers or expressions whose product is equal to the number or expression being factorized. In algebra also, the concept of a factor is the same. For example, we know that

1. $(2x)\cdot(3x^2) = 6x^3$
2. $2xy(x+y)=2x^3y+2xy^2$
3. $(2x+3)\cdot(x-2)=2x^2-x-6$

In these examples, factors and products are shown as follows:

| | Factor | Factor | Product |
|---|---|---|---|
| 1. | $2x$ | $3x^2$ | $6x^3$ |
| 2. | $2xy$ | $x + y$ | $2x^3y + 2xy^2$ |
| 3. | $2x + 3$ | $x - 2$ | $2x^2 - x - 6$ |

**Ask:** In the above examples, is it possible to factorize the indicated factors further?

Yes, $x - 2. x$ and $3x^2 = 3. x. x$

Similarly, $2xy = 2 . x . y$

$(x + y)$, $(2x + 3)$ and $(x - 2)$ cannot be factorized further.

Thus

$6x^3 = 2 . 3 . x . x . x$ (5 factors)

$2x^3y + 2xy^2 = 2. x. y (x + y)$ ( 4 factors)

$2x^2 - x - 6 = (2x + 3) ( x - 2)$ (2 factors)

**Explain**

A factor that cannot be factorized further is called a **prime factor.**

In the first example given above, $2x$ and $3x^2$ are factors of $6x^3$, but its prime factors are 2, 3 and x, x, x. Similarly, the prime factors of the expression $2x^2 + 2xy^2$ are 2, x, y and $(x + y)$.

A **polynomial** that is a **product** of **two-constant polynomials** is said to be factorizable or reducible.

A **polynomial** that **cannot be factorised** is known as **irreducible**.

An **irreducible polynomial** having unity as a leading coefficient is called a prime polynomial.

For example, $x + 1$, $x^2 + 2$ are prime polynomials.

Factorization, therefore, is a process of breaking down or reducing a polynomial in terms of two or more prime polynomials.

If there are three polynomials p(x), g(x) and h(x) such that g(x) h(x) = p (x).

We say that g(x) and h(x) are factors of p(x). The factorization is a process of finding g(x) and h(x) when p(x), their product, is given.

**Methodology used:** Discussion with inductive reasoning is used to clarify the concepts.

## 10.6.2 Factoring a Quadratic Polynomial

**Main Teaching Point**

To factorize using algebraic indentities.

**Teaching Learning Process**

Your students have learnt in lower classes that

$$(x+a)^2 = x^2+2ax+a^2$$
$$(x-a)^2 = x^2-2ax+a^2$$
$$(x-a)(x+a) = x^2-a^2$$

A polynomial of the form $p(x) = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$, is known as a quadratic polynomial. You may help the pupils relate this form of a polynomial to the products given in the above equations. The students may be led to discover that the right hand side expressions given in the above equations are quadratic polynomials and the left-hand side expressions are their factors, since.

$$(x + a)^2 = (x+a)(x+a)$$
$$(x - a)^2 = (x-a)(x-a)$$

If a given polynomial can be so written that it is similar to the expression on the right-hand side of any of the above quadratic equations, the factors may be easily and readily written. The following examples may be given to clarify the method:

**Example 1**

Factorize $x^2 + 10x + 25$

This quadratic polynomial may be written as

$x^2 + 2 . 5 . x + (5)^2$

which takes the form of a perfect square. Therefore, the factors may be written as:

$$x^2 + 10x + 25 = x^2 + 2 . 5. x + (5)^2$$
$$= (x + 5)^2$$

**Example 2**

Factorize $9x^2 + 12 x + 4$

$$9x^2 + 12x + 4 = (3x)^2 + 2 . 2 . 3 . x + (2)^2$$
$$= (3x + 2)^2$$

**Example 3**

Factorize $4x^2 - 12 x + 9$

$$4x^2 - 12x + 9 = (2x)^2 - 2 . 3 . 2 . x + (3)^2$$
$$= (2x - 3)^2$$

**Example 4**

Factorize $81x^6 - 16y^6$

$$81x^6 - 16y^6 = (9x^3)^2 - (4y^3)^2$$
$$= (9x^3 + 4y^3)(9x^3 - 4y^3)$$

**Example 5**

Factorize $25x^2 - 10x + 1 - 36y^2$

$$[25x^2 - 10x + 1 - 36y^2]$$
$$= (5x)^2 - 2.1.5.x + 1 - (6y)^2$$
$$= (5x - 1)^2 - (6y)^2$$
$$= [(5x - 1) + 6y][(5x - 1) - 6y]$$
$$= (5x - 1 + 6y)(5x - 1 - 6y)$$

After giving a few more similar examples, the teacher may give some exercises for practice.

**Methodology used:** Method is illustrated using various examples and drill should be provided to develop the skill.

## 10.6.3 Method of Splitting the Middle Term

**Main Teaching Point**

To factorize a quadratic of the type $x^2 + bx + c$ by splitting the middle term.

**Teaching Learning Process**

The teacher may ask the students to multiply the binomials $(x + p)$ and $(x + q)$ as follows:

**Ask:** Write the product of $(x + p)$ and $(x + q)$ in decreasing powers of x.

$$\begin{array}{l} x + p \\ x + q \\ x^2 + px \\ \hline \quad + qx + pq \\ \hline x^2 + px + qx + pq \end{array}$$

or $\quad (x + p)(x + q) = x^2 + (p + q)x + pq$

**Explain:** The product is of the form $x^2 + bx + c$. We may write

$$x^2 + bx + c = x^2 + (p + q)x + pq$$

so that $\quad b = p + q, \quad c = pq$

Therefore, to find the factors $x + p$ and $x + q$, we have to find numbers p and q such that $p + q = b$ and $pq = c$ by hit and trial. Then we can easily factorize the given quadratic. This should be illustrated through examples as follows:

**Example:** Factorize $x^2 + 7x + 12$

In order to factorize this expression, the students should be asked to find two numbers p and q such that

$$p + q = 7 \text{ and } pq = 12$$

The pupils should be made to think in the following way:

Since pq = 12, the values of p and q may be the following:

| p | q | p + q |
|---|---|---|
| 1 | 12 | 12 |
| 2 | 6 | 18 |
| 3 | 4 | 7 |

A little examination shows that it is only the last combination, i.e., 3 and 4 which satisfies the condition that p + q = 7.

Now, $x^2+7x+12=x^2+4x+3x+12$

$$=x(x+4)+3(x+4)$$

$$=(x+4)(x+3)$$

which are the required factors.

**Example:** Factorize $x^2 + 7x - 30$

To factorize this quadratic, we need p and q such that p + q = 7 and pq = –30, the factors of – 30 may be written as

| p | | q | | pq | p + q |
|---|---|---|---|---|---|
| 1 | × | –30 | = | –30 | –29 |
| –1 | × | 30 | = | –30 | 29 |
| 2 | × | –15 | = | –30 | –13 |
| –2 | × | 15 | = | –30 | 13 |
| 3 | × | –10 | = | –30 | –7 |
| –3 | × | 10 | = | –30 | 7 |
| 5 | × | –6 | = | –30 | –1 |
| –5 | × | 6 | = | –30 | 1 |

We have to select that pair of factors which satisfies the condition p + q = 7. It is obvious that p = –3 and q = 10 satisfy both the conditions because

$p + q = -3 + 10 =$ and $pq -3 \times 10 = -30$

Now, $x^2 + 7x - 30 = x^2 - 3x + 10x - 30$

$$= x(x-3) + 10(x-3)$$

$$= (x-3)(x+10)$$

**Methodology used:** The method is illustrated using several examples and then practice should be provided to develop the skill.

## 10.6.4 Method of Splitting the Middle Term (contd.)

**Main Teaching Point**

To factorise a quadratic of the type $ax^2 + bx + c$ by splitting the middle term.

**Teaching Learning Process**

When a quadratic is of the form $ax^2 + bx + c$, such as $2x^2 + 9x + 4$, $3x^2 - 14x + 8$ and $6x^2 - 11x - 2$, the method of factorization may be modified. Such expressions cannot be factorized by using the identity.

$$(x + p)(x + q) = x^2 + (p + q)x + pq,$$

because, here the coefficient of $x^2$ is not unity. As a means to understand the method, ask the students to:

Multiply the binomials $(px + q)$ and $(rx + s)$ and arrange the product in decreasing power of x.

$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$

**Explain:** If we compare the right-hand side with the standard form

$$ax^2 + bx + c = prx^2 + (ps + qr)x + qs$$

It is observed that

$$a = pr,\ b = (ps + qr),\ c = qs$$

It is also seen that ac = (pr) (qs) (ps) (qr). A minute inspection may reveal that the coefficient of x, i.e., b = ps + qr, is the sum of ps and qr whose product is ac. This leads to the following conclusion.

1. Break the coefficient of x in the quadratic $ax^2 + bx + c$ into two additive components whose product is the same as the product of the coefficients of $x^2$ and the constant term c.
2. Let the two factors of ac be $b_1$ and $b_2$. Write
   $ax^2 + bx + c = ax^2 + (b_1 + b_2)x + c =$
   $ax^2 + b_1x + b_2x + c$
3. Factorise the first two terms and the last two terms separately. A common factor will emerge.
4. Rewrite the expression as the product of the two factors.

Consider the following example:

**Example:** Factorize $2x^2 + 9x + 4$

Product of coefficients of $x^2$ and the constant term $= 2 \times 4 = 8$.

Coefficient of $x = +9$

Now we should look for two numbers whose product is +8 and sum is +9. Obviously the numbers are +8 and +1. Then, $2x^2 + 9x + 4 = 2x^2 + 8x + x + 4$

$$= 2x(x + 4) + (x + 4)$$

$$= (x + 4)(2x + 1)$$

**Example:** Factorize $3x^2 - 14x + 8$

Let us look for the two numbers whose sum is $-14$ and product is $3 \times 8 = 24$. Such a pair of number is $-12$ and $-2$. The given expression, then, may be written as

$$3x^2 - 14x + 8 = 3x^2 - 12x - 2x + 8$$

$$= 3x(x - 4) - 2(x - 4)$$

$$= 3x(x - 4) - 2(x - 4)$$

$$= (3x - 2)(3x - 2)$$

Which are the required factors.

**Example:** Factorize $6x^2 - 11x - 2$

Let us again look for the numbers. This time their sum should be $-11$ and product, $-2 \times 6 = 12$. The numbers are $+1$ and $-12$.

$$6x^2 - 11x - 2 = 6x^2 - 12x + x - 2$$

$$= 6x(x - 2) + (x - 2)$$

$$= (x - 2)(6x + 1)$$

Which are the required factors.

**Methodology used:** The method is illustrated by various examples and sufficient practices is required to develop the skill.

**Check Your Progress**

**Notes :** a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Factorise the following expressions.

20. $x^2 + 8x + 15$

21. $x^2 - x - 6$

22. $5x^2 + 40x + 60$

23. $54x^2 - 15x - 4$

## 10.7 REMAINDER THEOREM

**Main Teaching Points**

(a) Statement of remainder theorem.

(b) Use of remainder theorem to find remainder.

**Teaching Learning Process**

You have already taught your students the process of dividing a polynomial by another polynomial. As a special case, if a polynomial p(x) is divided by another polynomial q(x) and polynomials r(x) and s(x) are the quotient and the remainder respectively, then the students should be led to understand the relation.

$$p(x) = q(x)\, r(x) + s(x)$$

As a special case when $q(x) = x - a$ which is a binomial of the degree 1, the above relation reduces to

$p(x) = (x - a)\, r(x) + s(x)$ where $r(x)$ is the quotient and $s(x)$ is the remainder. Since the degree of $s(x)$ is less than the degree of $x - a$, it is a constant.

Let $\quad p(x) = Ax^3 + Bx^2 + Cx + D$ and $q(x) = (x - a)$

**Ask:** Divide $p(x)$ by $q(x)$

The process goes on as follows:

$$\begin{array}{r|l}
 & Ax^2 + (B+aA)x + [a(B+aA)+C] \\ \hline
x - a & Ax^3 + Bx^2 + Cx + D \\
 & Ax^3 - aAx^2 \\ \hline
 & (B+aA)x^2 + Cx + D \\
 & (B+aA)x^2 - a(B+aA)x \\ \hline
 & [a(B+aA) + c]x + D \\
 & [a(B+aA) + c]x - a[a(B+aA)+C] \\ \hline
 & a[a(B+aA)+C] + D \\
 & = Aa^3 + Ba^2 + Ca + D
\end{array}$$

**Explain:** The remainder is

$$Aa^3 + Ba^2 + Ca + D = p(a)$$

This leads to an important theorem known as remainder theorem, which may be stated as

---

"If $p(x)$ is a polynomial of degree $\geq 1$, and it is divided by a binomial $(x - a)$ where a is a real number, then the remainder will be $p(a)$."

---

**Example:** Find the remainder when $3x^2 + 2x + 1$ is divided by $(x - 3)$.

Here, $p(x) = \mathbf{3x^2} + 2x + 1$

Comparing $x - 3$ with $x - a$, we get $a = 3$

Remainder $= p(3) = 3\,(3)^2 + 2\,(3) + 1$

$= 27 + 6 + 1$

$= 34$

**Example:** Find the remainder when $x^3 - 2x^2 + x - 5$ is divided by $(x + 2)$.

Here $\quad p(x) = x^3 - 2x^2 + x - 5$ and $a = -2$

Remainder $= p(-2) = (-2)^3 - 2\,(-2)^2 + (-2) - 5$

$= -8 - 8 - 2 - 5$

$= -23$

**Methodology used:** Deductive method is used to explain the meaning of the statement of the remainder theorem. Then various illustrations are given to find the remainder.

### 10.7.1 Use of Remainder Theorem in Factorization

**Main Teaching Point**

To find if $x-a$ is a factor of given polynomial.

**Teaching Learning Process**

**Ask:** When $p(x)$ is divided by $(x - a)$, the remainder is $p(a)$. If $p(x)$ turns out to be equal to zero, what do you conclude?

**Explain** If the remainder is zero, $p(x)$ is divisible by $(x - a)$. Conversely, 'if $p(a)$, the value of a polynomial $p(x)$ for some 'a' is zero, then $(x - a)$ is a factor of $p(x)$'.

Or, $p(x) = (x - a)\, q(x)$ where $q(x)$ is the quotient. This statement is called **factor theorem** and follows directly from the remainder theorem.

**Example:** Examine if $(x - 3)$ is a factor of $x^3 - 3x^2 + 4x - 12$

Putting $x = 3$

$p(3) = 3^3 - 3(3)^2 + 4(3) - 12$

$= 27 - 27 + 12 - 12 = 0$

Hence $(x - 3)$ is a factor of $x^3 - 3x^2 + 4x - 12$.

We have $x^3 - 3x^2 + 4x - 12 = x^2 (x - 3) + 4 (x - 3)$

$= (x - 3) (x^2 + 4)$

$x^2 + 4$ is irreducible hence cannot be factorized further.

**Example:** For what value of k is $x^3 - kx^2 + 4x - 12$ divisible by $x - 3$.

Comparing $x - 3$ with $x - a$, we get $a = 3$

$p(3) = 3^3 - k(3)^2 + 4(3) - 12$

$= 27 - 9k + 12 - 12$

$= 27 - 9k$

For divisibility, $p(3)$ must be equal to 0.

Hence $27 - 9k = 0$ or $k = 3$

**Methodology used:** Inductive logic is used to illustrate that $x - a$ is a factor of $p(x)$ if $p(a) = 0$

**Check Your Progress**

**Notes:** a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

24. Find whether $p(x)$ is divisible by $q(x)$ for

$p(x) = x^3 - 2x^2 + x + 3$

$q(x) = x - 4$

25 For what values of k, q(x) will be a factor of p(x) in the following?

$p(x) = x^3 + 2x^2 + kx - 2$

$q(x) = x + 2$

## 10.8 LET US SUM UP

In this unit you have learnt some strategies and approaches to clarify certain important ideas such as constant, variables, algebraic expression, values of an algebraic expression, polynomial and its subforms, monomial, binomial, etc. You have also learnt how to teach effectively the concepts of basic operations involving polynomials and impart skills to perform these operations. You have also studied the technique of factorising quadratic polynomial by using several methods such as the method of splitting the middle term. The use of remainder theorem in algebra to find out remainder without performing division was also presented. It was also shown that the remainder theorem may be used to factorise polynomials of higher degree. In short, this unit provides you a foundation for further development in teaching algebra.

## 10.9 UNIT-END ACTIVITIES

At the end of this unit you may

1. give your students some more exercises on basic operations with polynomials and factorization;
2. evaluate the learning of your students after this unit has been taught;
3. prepare lesson plans or develop teaching aids for teaching factorization and multiplication of polynomials; and
4. visit your nearby schools and observe the teaching of polynomials to see whether the ideas and techniques learnt in this unit find a place in classroom teaching.

## 10.10 POINTS FOR DISCUSSION

1. Can we correlate the teaching of basic ideas involved in this unit with corresponding concepts in arithmetic?
2. Can we correlate the concepts of products and factors involved in this unit to the concepts of area and volume in mensuration and develop some novel teaching aids?
3. Can remainder theorem be used to develop a general technique of factorization?

## 10.11 ANSWERS TO CHECK YOUR PROGRESS

1. $5x^2 - x - 2$
2. $x^3 + x^2 + x - 2$
3. $(-4t^2 + t - 5)$
4. $K^4 + k^3 + 2k^2 - k + 6$
5. (i) $x$

   (ii) $-8b - 2c^2$
6. $-3x - 16y$
7. $x^3 + 10y^2 - 4$
8. $-3x^2 + 2x - 2$
9. $x^5 - x^3 + x^2 - 3x - 1$
10. $2x^2 + 11x + 12$
11. $6x^5 - x^4 + 2x^3 - 5x^2 - x - 3$
12. $x^4 + x^3y - x^2y^2 - 2xy^3 - y^4$
13. $x^6 + x^5y - x^4y^2 - x^2y^4 - xy^5 - y^6$
14. Quotient = $6m + 1$; Remainder = 0
15. Quotient = $4y - 5$; Remainder = 0
16. Quotient = $3p - t$; Remainder = 0
17. Quotient = $2xy - 3z$; Remainder = 0
18. Quotient = $9x^2 + 3x + 1$; Remainder = 0
19. Quotient = $x^2 - x + 1$; Remainder = $-2$
20. $(x + 3)(x + 5)$
21. $(x + 2)(x - 3)$
22. $5(x + 6)(x + 2)$
23. $(6x + 1)(9x - 4)$
24. No
25. $K = -1$

## 10.12 SUGGESTED READINGS

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