UNIT 11 LINEAR EQUATIONS AND INEQUALITIES: GRAPHS AND QUADRATIC EQUATIONS

Structure

11.1 INTRODUCTION

The word equation is within the comprehension of students. Generally an equation is compared with the two pans of a weighing balance. Equations are of different types depending on the number of variables and the degree of the variables. Besides, there are many situations which are represented by inequalities. The student is familiar with the solution of the linear equations with one variable.

This unit gives methods of solving linear equations in two variables, quadratic equations, inequations and constructing their graphs.

11.2 OBJECTIVES

At the end of this unit, you should be able to:

- explain the distinction between linear equations in one variable, two variables; a system of equations in two variables and quadratic equations;
- illustrate with the help of graph the roots/solutions of equations of different types;
- develop various methods of solving different types of linear equations in one and two variables and the quadratic equations;
- describe the difference between consistent and inconsistent equations in words and also using graphs and develop a criterion for consistency;

- demonstrate the applications of system of linear equations and quadratic equations in solving various problems in every day life;
- inculcate problem-solving skills, that is;
  - i) translate word problems into mathematical models.
  - ii) apply mathematical techniques to solve word problems.
- explain the meaning of inequations in one and two variables; and
- show graphically the region where the inequations hold.

11.3 LINEAR EQUATION IN ONE VARIABLE

Main Teaching Point

Recognizing linear equation in one variable.

Teaching Learning Process

Students are familiar with the term equation, variable and constant. As a prelude to further study of equations, you may find out whether students can discriminate between an expression and an equation.

You may present them with a number of expressions and ask them to select those which are equations.

Also, ask them to find out the variables and constants in the equations. Ask the students to find out the degree of variable in each equation.

Explain

Equations in which there is only one variable and the degree of the variable is one are called equations of degree one in one variable.

They are also called linear equations in one variable.

Methodology used: Discussion with various illustrations.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

1. Which of the following expressions are equations?

i) $5x - 8 = 7$

ii) $4x - 8$

iii) $3x \neq 2x - 10$

iv) $2x + 5 > 10$

v) $5x - 3 = 2x$

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2. Which of the following are linear equations in one variable?

   i) $-2x + 7 = 0$

   ....

   ii) $8x^2 - 32 = 0$

   ....

   iii) $3x^2 = 16 - 2x^3$

   ....

   iv) $3x^2 + 5x - 10 = 3x^2$

   ....

11.4 LINEAR EQUATION IN TWO VARIABLES

An equation of degree one involving two variables is discussed in this unit. Such an equation has infinite solutions, and its graph is a straight line. The methods of solving system of equations and consistency of equations are the main points which are dealt with here.

11.4.1 Graph of Linear Equation in Two Variables

Main Teaching Point

Graph of an equation of degree one in two variables is a straight line.

Teaching Learning Process

Through examples you should bring out inductively that an equation of degree one in two variables has infinite solutions and on plotting them on a graph, they lie on a straight line.

You may then ask the students to study the following relations and represent them graphically.

Example 1: A train is moving with a uniform velocity of 60 km/hr. Draw the time-distance graph. Read the distance travelled in 2.5 hours from the graph.

The table of time and distance is as shown:

x	Time in hours	1	2	3	4	5
y	Distance in km.	60	120	180	240	300

Thus; we plot the ordered pairs: (1, 60), (2, 120), (3, 180), (4, 240), (5, 300).

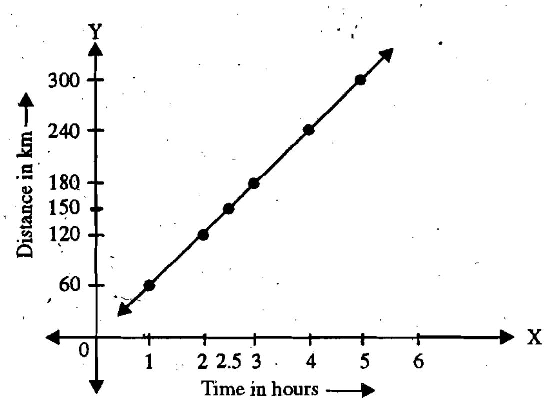

Fig. 11.1

We see that the points lie on a line. Since

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$60 = \frac{y}{x}$$

Hence, the relation is y = 60 x.

For any value of x, we can get a corresponding value of y. For x = 2.5 we find y = 60 ×2.5 = 150 km. It is also clear from the graph.

Ask the students to plot the graph of the equation x = 5.

There is only one variable x having a constant value 5. The other variable can have any value. So x = 5 is a set of points like (5, – 4); (.5, –2); (5, 3); (5, 6) .... Plot these points and join them. This is the graph of x = 5. Similarly draw graphs of x = 3 and x = 6.

Ask: What is the relation between the graphs of x = 3, x = 5 and x = –6?

Clearly, they are all straight lines parallel to the y-axis. What do 5, 3, –6 indicate in the three graphs? Ask the students to give the position of graph of x = 10, x = 4. Similarly, the graphs of y = 2, y = 7, y = –5, etc. be drawn and interpreted by the students.

ii) The table below gives measures (in degrees of two angles x and y respectively) which form a linear pair.

x	0	30	60	90	120	150	180
y	180	150	120	90	60	30	0

Plot the above values on a graph. Let angle x be represented along x-axis and angle y along y-axis. We plot the ordered pair P (0, 180); Q (30, 150); R(60,120); S(90, 90); T(120, 60); M (150, 30); N (180, 0).

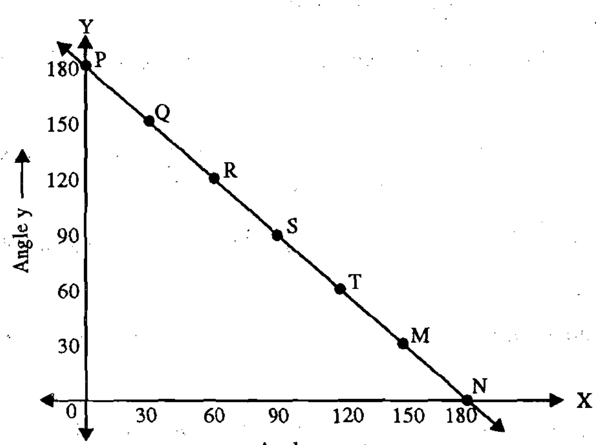

Fig. 11.2

We see that the pairs of points lie on a line. Clearly, this time the relation is:

x + y = 180 where 0 ≤ x ≤ 180 and 0 ≤ y ≤ 180.

Given any value of x, we can find the corresponding value of y.

iii) The relationship between the number of sides (n) of a polygon and sum (8) of its interior angles in degress is given below:

No. of sides (n)	3	4	5	6	7
Sum of the angles (s)	180	360	540	720	900

Plotting the number of sides along x-axis and the sum of the angles along y-axis, we get ordered pairs A(3, 180); B(4, 360); C(5, 540); D(6, 720); E(7, 900);

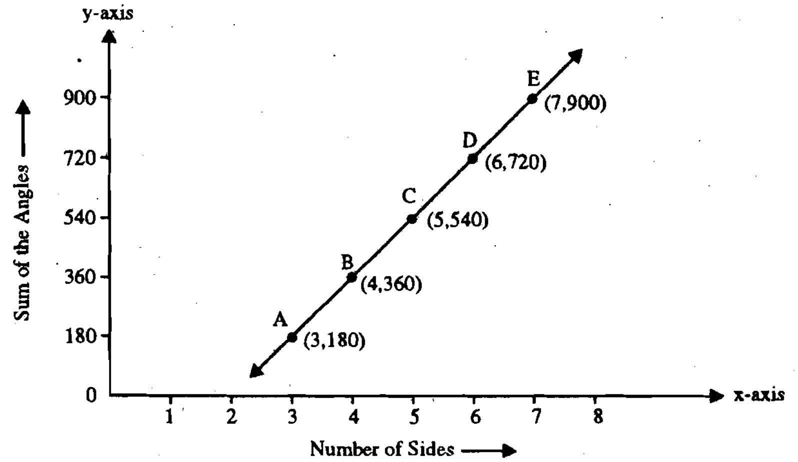

Fig. 11.3

What are the coordinates of the points A, B, C, D, E? It can be easily seen that these points lie on the same line. Let us find out relation between sides and sum of its interior angles.

The rule is S = 180, n – 360 where n is the number of sides and S is the sum of the interior angles.

Remember that n can be assigned only an integral value greater than or equal to three because at least three sides are required to make a closed polygon.

Ask the students to study the three equations i.e., $y = 60x$, $x + y = 180$; $S = 180,\ n - 360$. Each one is an equation in two variables with degree one. The general form can be expressed as:

$Ax + By + C = 0$, where A, B, C are real numbers with at least one out of A and B being non-zero. The graph of such an equation is a line. It is, therefore, called a linear equation in two variables.

Group activity: Divide the students into five groups.

Ask each group to draw an important inference in the following five cases for line $Ax + By + C = 0$.

Group I: In the equation put $A = 0$, $C = 0$ and $B \neq 0$, and draw the graph. Ask them what do they observe?

Group II: In the equation, put $B = 0$, $C = 0$ and $A \neq 0$, and draw the graph. Ask the group the special property of the graph.

Group III: In the equation, put $A = 0$, $B \neq 0$ and $C \neq 0$ and draw the conclusion regarding the graph.

Group IV: In the equation, put $A \neq 0$, $B = 0$ and $C \neq 0$, Find the value of x and draw the graph. What do you observe?

Group V: In the equation, put $A \neq 0$, $B \neq 0$ and $C = 0$, and draw the conclusion.

Methodology used: Examples are discussed and the facts are derived inductively from the different examples.

11.4.2 Graph of Linear Equation Involving Absolute Values

Main Teaching Point

To draw the graph of a linear equation involving absolute value.

The absolute value of a is written as $|a|$ and defined as

$$|a| = \begin{cases} +a, & \text{if a is positive} \\ -a, & \text{if a is negative} \end{cases}$$

Let us study the graph of a linear equation involving absolute value.

Draw the graph of $|y| = 4$.

By definition $|y| = 4 \Rightarrow y = 4$ when $y \geq 0$ and -4 when $y < 0$. The graph of the given equation is, therefore, the union of graphs of $y = -4$ and $y = 4$ as shown.

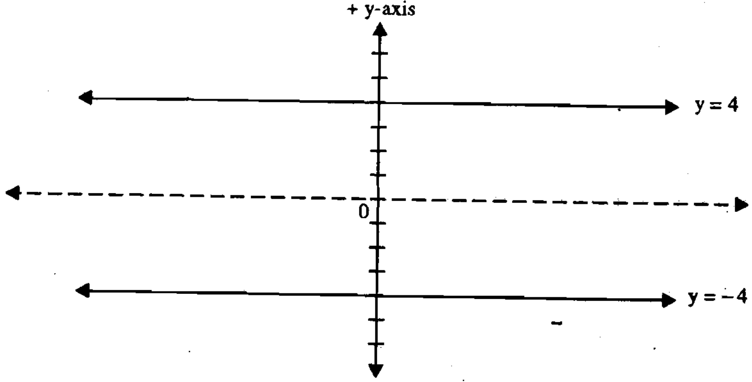

Fig. 11.4

Consider, $|2x - 5| = 4$

$$|2x-5| = \begin{cases} 2x-5, \text{ when } 2x-5 \geq 0 \\ -(2x-5), \text{ when } 2x-5 < 0 \end{cases}$$

$$\therefore \begin{cases} 2x-5 = 4, \text{ when } 2x-5 \geq 0 \\ -2x+5 = 4, \text{ when } 2x-5 < 0 \end{cases}$$

$$\text{or } \begin{cases} x = 9/2, \text{ when } x \geq 5/2 \\ x = 1/2, \text{ when } x < 5/2 \end{cases}$$

$\therefore$ The graph is as follows:

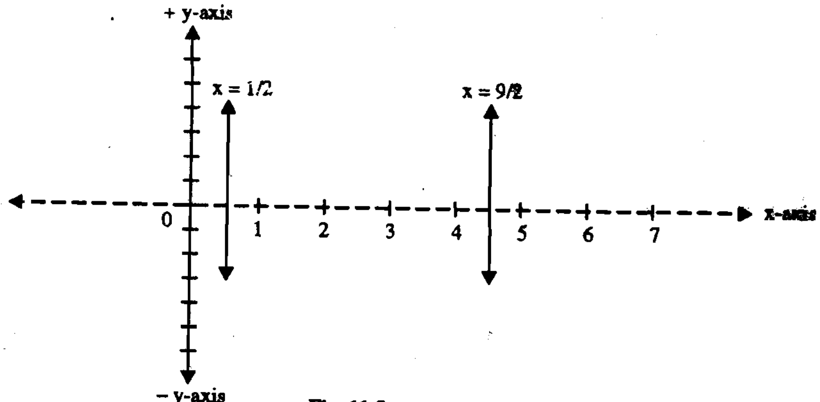

Fig. 11.5

Graph of an absolute value in two variables:

Darw the graph of $y = 2|x|$

$y = 2|x|$ can be written as

$$y = \begin{cases} 2x, x \geq 0 \\ -2x, \; x < 0 \end{cases}$$

Ask students to point out the difference in the graph of the equation $y = 2x$ and the equation $y = 2x$ for $x \geq 0$. Which part of the graph of $y = 2x$ cannot form a part of the latter graph?

The graph of linear equation $y = 2x, \forall x \geq 0$ will be part of the line in the first quadrant passing through the origin. Graph of $y = -2x, \forall x < 0$ will be part of the line in the second quadrant passing through the origin.

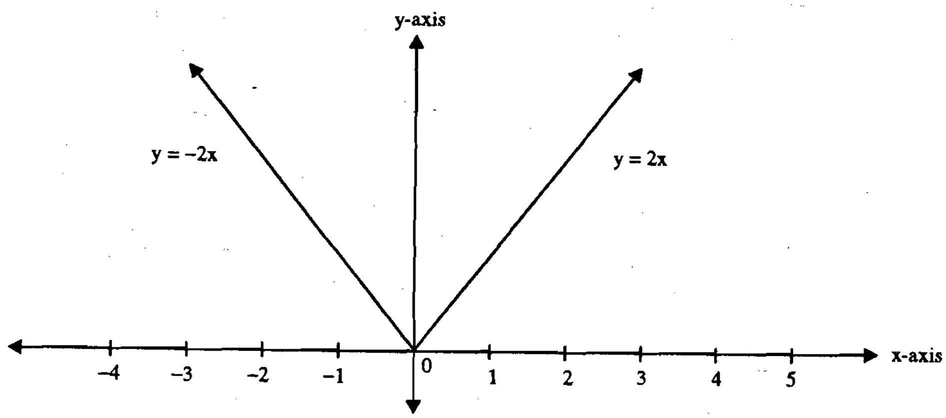

Fig. 11.6

The graph of $y = 2|x|$ is the union of two rays as shown in the figure.

Methodology used: Discussion method is used.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Draw the graphs of the following equations:

3. $2x + 3y = 5$

4. $|x-2| = 4$

11.4.3 System of Linear Equations in Two Variables

Main Teaching Point

Conditions for consistency of a system of equations.

Teaching Learning Process

Any two linear equations in two variables taken together with the help of connective 'and' are called a system of two linear equations in two variables.

Examples

i) $2x - 3y - 2$ and $x + 2y = 8$

ii) $3x - 5y + 4 = 0$ and $9x = 15y - 12$

iii) $2x + 4y = 7$ and $3x + 6y = 10$

The above three are examples of system of equations. If we draw the graph of each pair of equations we find:

i) Lines are intersecting in the first case.

ii) Lines are coincident in the second case.

iii) Lines are parallel in the third case.

A system of operations is said to be consistent if the lines representing their graphs, intersect at one point, i.e., the two equations are satisfied by exactly one ordered pair of numbers. Algebraically, the system of equation.

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

is consistent if $$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Consider the system $2x + 3y - 5 = 0$ and $6x + 9y - 15 = 0$,

Note that $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

i.e. the ratio of coefficients of x, y and constant term is the same.

If one equation in each pair is multiplied by common ratio of coefficients of x or y in the two equations, the second equation is obtained. The two equations are essentially equivalent. The graphs of such equations are coinciding.

If the graphs of the equations are coinciding, the system has infinite pairs in common. Such a system is called dependent.

Algebraically, the system of equation

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$\text{is dependent if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Ask students to find the ratio

$$\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$$

in the system of equations $2x + 3y = 8$

and $6x + 9y = 12$

Whereas $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{3}, \frac{c_1}{c_2} \neq \frac{1}{3}.$

If we multiply first equation by 3, we get $6x + 9y = 24$ which is not the same as $6x + 9y = 12$. Hence any value of (x, y) which satisfies equation (i) does not satisfy, (ii) and vice versa. The graph of such a system of equations is a pair of parallel lines. There is no common solution of the equations. Such a system of equations is called an inconsistent system of equations. Algebraically, the system of equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$\text{is inconsistent if } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Methodology used: Induction method is used. Several examples are discussed and then the condition is derived inductively.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Classify the following systems of equations as consistent, dependent or inconsistent.

5. $2x + 3y = 5$
   $4x + 6y = 10$

   ....

6. $3x + 4y + 2 = 0$
   $6x + 8y - 1 = 0$

   ....

7. $4x - y = 5$
   $2x - y = 3$

   ....

8. $8x - 6y + 2 = 0$
$-4x + 3y - 1 = 0$

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11.4.4 Methods of Solution of System of Equations

Main Teaching Point

Different methods of solving system of equations.

Teaching Learning Process

1) Method of Elimination

This method consists essentially of eliminating one of the variables from each of the two equations and thus getting a new equation in a single variable which can be solved.

Develop with student participation various methods of elimination. They are,

i) Comparing :

$$a_1x + b_1y = c_1 \Rightarrow x = \frac{c_1 - b_1y}{a_1}$$

$$a_2x + b_2y = c_2 \Rightarrow x = \frac{c_2 - b_2y}{a_2}$$

$$\text{hence } \frac{c_1 - b_2y}{a_1} = \frac{c_2 - b_1y}{a_2}$$

Solve it to get the value of y.

ii) Substitution:

$$a_1 x + b_1 y = c_1 \Rightarrow x = \frac{c_1 - b_1y}{a_1}$$

Substituting in $a_2x + b_2y = c_2$ we have

$$a_2 \left[\frac{c_1 - b_1y}{a_1}\right] + b_2y = c_2$$

Solve it to get the value of y.

iii) Making coefficients of either x or y equal (whichever is the most convenient).

$$a_1x + b_1y = c_1 \Rightarrow a_2 a_1x + a_2b_1y = a_2c_1 \quad \ldots(i)$$

$$a_2x + b_2y = c_2 \Rightarrow a_1a_2x + a_2b_2y = a_1c_2 \quad \ldots(ii)$$

Subtracting, II from I we get a new equation in y, i.e.,

$$(a_2 b_1 - a_1b_2)_y = a_2c_1 - a_1c_2$$

This gives the value of y.

Substituting the value of y in one of the equations, we get the value of x.

2) The Method of Cross Multiplication

Let the equations be $a_1x + b_1y + c_1 = 0 \quad \ldots(i)$

$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$

Equating the coefficients of x, i.e., multiplying (i) by a_2 and (ii) by a_1 and subtracting, we get,

$$y = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1} \qquad \ldots(a)$$

Similarly, by multiplying both sides of (i) by b_2 and (ii) by b_1 and subtracting we get

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \qquad \ldots(b)$$

From (a) and (b) we get

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - a_1 c_2} = \frac{1}{a_1 b_2 - a_2 b_1}$$

which can be exhibited as

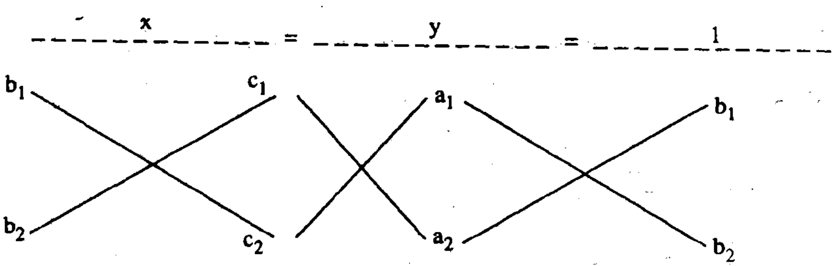


Fig. 11.7

To write this we write the detached coefficients:

a_1, b_1, c_1 and a_2, b_2, c_2 in four columns as below:

b_1	c_1	a_1,	b_1
b_2	c_2	a_2,	b_2

Below x we write the coefficients of y and the constant terms; below y we write the coefficients of x and the constant terms and below 1 we write the coefficients of x and y. We put the arrows as shown. The product of the coefficients with arrows pointing downward are taken with positive signs and the product of the coefficients pointing upward are all taken with negative sign.

Getting values of x and y by multiplying coefficients is called cross multiplication method.

In this method, we follow the steps given below:

i) Rewrite the equations in such a way that variables and constants are on the left-hand side, and the right-hand side is zero.

ii) Write variables x and y in order i.e., first x and then y.

iii) Take the detached coefficients. In the first row write the coefficients of first equation and in the second row, write the coefficients of the second equation.

iv) Below x we write coefficients of y and constant term in the two equations. Below y we write the coefficients of x in the two equations. Below 1, we write the coefficients of y in the two equations again.

v) We mark arrows as in the example:

vi) Simplify the value under x, y and 1.

vii) We get the values of x and y.

Example: Solve the system of equations:

$4y - 3x - 23 = 0$

$3y + 4x - 11 = 0$

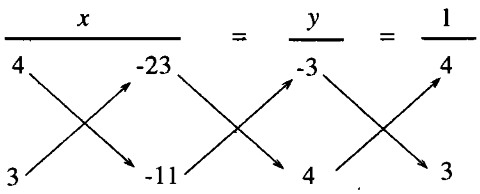
i) Writing the term x first and y next we get,

$-3x + 4y + (-23) = 0$

$4x + 3y + (-11) = 0$

Detached coefficients are:

–3	4	–23
4	3	–11



$$\frac{x}{-44+69} = \frac{y}{-92-33} = \frac{1}{-9-16}$$

$$\frac{x}{25} = \frac{y}{-125} = \frac{1}{-25} \Rightarrow x = -1, y = -5$$

Note: Cross multiplication method is applicable when the given system is consistent, i.e. it has a unique solution.

Otherwise $\frac{1}{\text{denominator}}$ becomes $\frac{1}{0}$, which is meaningless.

Methodology used: Lecture method with discussion involving students in telling different ways of elimination of one variable from two equations.

11.4.5 Solution of Word Problems

Main Teaching Point

Translating word problem into a mathematical model.

Teaching Learning Process

For solving problems on system of equations, the main emphasis should be on the formation of the equation. It is called translating word problems into mathematical models. For example, the cost of 5 chairs and 4 tables is Rs. 3200/-. This is a statement in words. To translate it into a mathematical problem, we suppose the cost of one chair as x and the cost of one table as y. The cost of 5 chairs and 4 tables is 5x + 4y. The given cost is Rs. 3200.

Hence, $5x + 4y = 3200$.

Similarly, if the cost of 3 chairs and 2 tables is Rs. 1700/- the mathematical model is

$3x + 2y = 1700$

The translation of word problem into mathematical models gives us two equations:

$5x + 4y = 3200$

and $3x + 2y = 1700$

Solving the equations, we get,

$x = 200, y = 550$

Cost of chair = Rs. 200, Cost of table = Rs. 550

Consider one more example.

Example: A fraction becomes 1/4 if 3 is subtracted from the numerator, and 1/2 if 2 is added to the denominator. Find the fraction.

Let the fraction be x/y. Translating the first condition we get $\frac{x-3}{y}=\frac{1}{4}$

It gives $4x - 12 = y \Rightarrow 4x - y = 12$.

Translating the second condition we get $\frac{x}{y+2}=\frac{1}{2}$.

or $2x = y + 2$

or $2x - y = 2$

Solving two equations,

$4x - y = 12$

and $2x - y = 2$

We get $x = 5$ and $y = 8$

The required fraction is $\frac{5}{8}$.

Similarly more examples can be taken to translate word problems into mathematical models.

Methodology used: Heuristic approach is more suitable for solving word problems. However, discussion should be encouraged and the teacher should guide the students towards logical thinking.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

9. A fraction becomes 1/2 on adding 1 to both the numerator and the denominator. On adding 2 to the denominator, the fraction becomes 1/4. Find the fraction.

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10. The difference between the ages of a father and his son is 30 years. Last year (one year ago) the age of the father was four times that of his son. Find their present ages.

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11. A boat travels 30 km. downstream in 1 hour and back in $1\frac{1}{2}$ hours. Find the speed of the boat in still water and the speed of the stream.

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11.5 INEQUATIONS

It is not always that we are faced with problems involving equations. There are also problems with inequations. The cost of 2 shirts and 3 pants is less than Rs. 2000/-. Taking the cost of one shirt as Rs. x, and one pair of pants as Rs. y, we get,

$$2x + 3y < 2000$$

Similarly if the cost of 3 goats and 4 sheep is greater than Rs. 15,000, the mathematical model becomes $3x + 4y > 15,000$, taking the cost of a goat as Rs. x and that of a sheep as Rs. y.

Consider the problem:

Raju went to a general store to purchase sugar. He found it was available in 250 g and 500 g packets. How could he buy the two types of packets so that the total weight always remained less than two kg?

If x and y are the number of packets weighing 250 g and 500 g respectively then,

$250 x + 500 y < 2000$. This statement in two variables is in the form of what is known as an **inequation in two variables.**

In geneıal, an inequation in two variables is of the form:

$ax + by < c$

or $ax + by > c$

or $ax + by \leq c$

or $ax + by \geq c$

where a, b, x, y are real numbers and a, b are not simultaneously equal to zero. An ordered pair which satisfies an inequation is called a solution of the inequation.

11.5.1 Graphical Representation of Inequation

Main Teaching Point

To represent an inequation on a graph.

Teaching Learning Process

Draw the graph of $2x + y - 4 < 0$ where $x, y \in R$.

Consider the equation obtained by replacing the inequality sign by equality

It is $2x + y - 4 = 0$

Draw the graph of the equation using a dotted line. Why? In how many sets are the points in the plane divided? There are three set of points:

i) Set of points which lie on the line.

ii) Set of points which lie on one side of the line, say, Region I.

iii) Set of points which lie on the second side of the line, say, Region II.

iv) The sign > or < means that the solution set is a set of points which lie in region I or II. Ask the students to take any point which does not lie on the line, say, (1, 1). Ask the students to find out whether the point (1,1) satisfies inequality $2x + y - 4 < 0$.

Since the point (1,1) satisfies the inequality and the point (1,1) lies in the region II as shown in the graph, all the points of region II lying between x-axis, y-axis (since $x > 0$, $y > 0$) and the line forms the solution set. The solution set is shaded. The sign ≥ or ≤ means the points on the line are also included in the solution set.

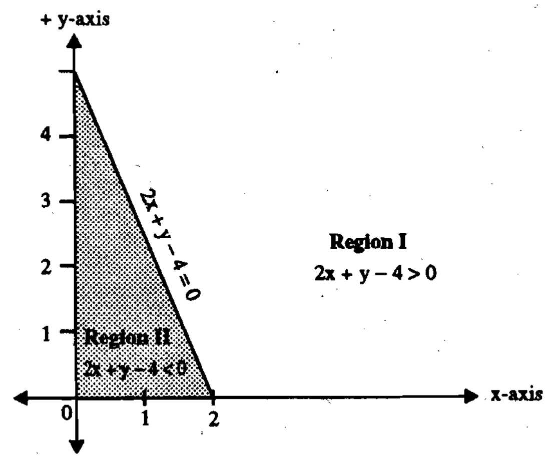


Fig. 11.8

For the graph of inequation $|x| \le 3$, we have the inequation $x \le 3$, when $x > 0$ or $x > -3$, when $x < 0$. The graph of the region lying between lines $x = 3$ and $x = -3$ represents $|x| < 3$ shown in the figure.

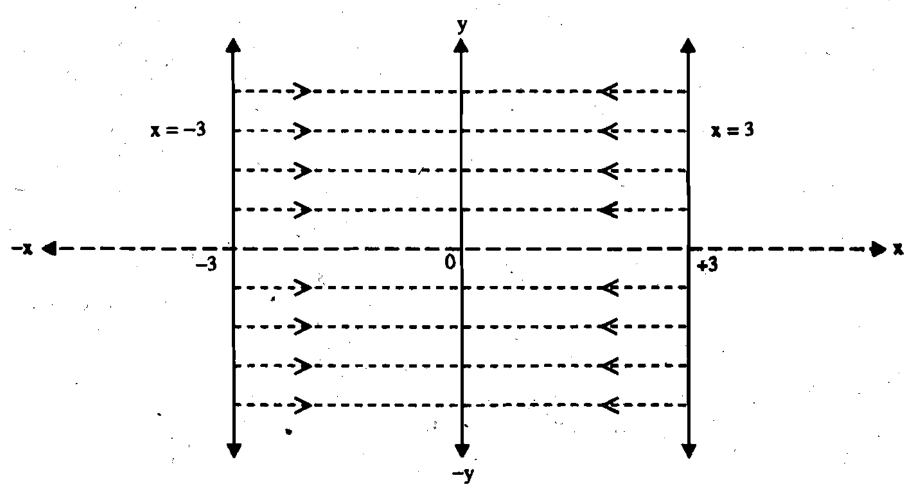


Fig. 11.9

Methodology used: Lecture method with discussion as students already know how to draw the graph of an equation.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Draw the graphs of the following linear inequations.

12. $y \geq 4$

13. $|x| \leq 2$

14. $y - x \geq 3$

11.6 QUADRATIC EQUATIONS

An equation $p(x) = 0$, where $p(x)$ is a quadratic polynomial, is called a quadratic equation. Students must be able to recognise the quadratic equations from a given set of equations as follows:

i) $3x^2 + 7x = 0$

ii) $2x^2 - 7x = 0$

iii) $x^3 + 6x^2 + 2x - 1 = 0$

iv) $2x^2 + 1/x^2 = 0$

v) $x + 3/x = x^2$

vi) $3x^2 - 5x + 8 = 3x^2$

Only (i) and (ii) are quadratic. You should discuss why others are not quadratic.

11.6.1 Solution of a Quadratic Equation

Main Teaching Points

a) Factor method

b) Completing the square method.

Teaching Learning Process

i) By factorisation

Ask question:

If $a \times b = 0$, where a and b are two numbers, then what can you say about the values of a and b? Should both of them be zero, or should only one of them need be zero?

Answer: Either $a = 0$, or $b = 0$.

Solve:

$$x^2 + 6x + 5 = 0$$

Factorizing the left hand side we get:

$$(x + 1)(x + 5) = 0$$

$$\Rightarrow x + = 1 = 0 \text{ or } x + 5 = 0$$

$$x = -1 \text{ or } x = -5$$

-1 and -5 are called roots or the solution of the quadratic equation. Why is $\{-1, -5\}$ called the solution set of the given equation?

ii) Second Method

Solve:

$$ax^2 + bx + c = 0 \text{ where } a \neq 0$$

Multiplying both sides by 4 a

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax)^2 + 2(2ax)b + 4ac = 0$$

$$(2ax)^2 + 2(2ax)b + 4ac + b^2 - b^2 = 0$$

$(2ax + b)^2 = b^2 - 4ac$. What is the sign of $b^2 - 4ae$? Positive or negative.

For real root, $b^2 - 4ac$ should be greater than or equal to zero.

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$ is called discriminant. Why?

This is because it discriminates between different types of roots:

i) if $b^2 - 4ac \geq 0$, the roots are real and unequal.

ii) if $b^2 - 4ac = 0$, the roots are real and equal.

iii) if $b^2 - 4ac < 0$, the roots are not real.

Methodology used: Deductive method is used to derive the quadratic formula.

11.6.2 Relation between Roots and Coefficients

Main Teaching Points

a) Sum of roots $= -\frac{b}{a}$

b) product of roots $= \frac{c}{a}$

Teaching Learning Process

Ask students to find the roots of the following equations.

i) $x^2 - 7x + 6 = 0$

ii) $x^2 - 3x + 2 = 0$

iii) $2x^2 - 5x - 7 = 0$

iv) $3x^2 - 11x + 10 = 0$

v) $x^2 - 3x - 10 = 0$

The roots are 6 and 1 and 2, – 1 and 7/2; 5/3 and 2; 5 and –2. Find the sum of the roots in each case. Find the relation with coefficients of x^2 and x in the equation?

i) Sum = 7 (Coefficient of x)

ii) Sum = 3 (Coefficient of x)

iii) Sum $= \frac{5}{2} = \frac{-(\text{Coefficient of x})}{\text{Coefficient of } x^2}$

iv) Sum $= \frac{11}{3}$ It is the same as above.

v) Sum = 3 (Coefficient of x)

Coefficient of x^2 in cases (i), (ii) and (v) is 1.

$$\therefore \text{ Sum of roots } = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

Again, take the product of roots and its relation with the coefficients of x^2 and constant term in the equation.

i) Product = 6 — It is the constant term.

ii) Product = 2 — It is the constant term.

iii) Production $= \frac{7}{2}$ — It is $\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

iv) Product $= \frac{10}{3}$ — It is $\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

v) Product = –10 — It is constant term.

In cases (i), (ii) and (v), Coefficient of x^2 is 1.

$$\text{Product of roots } = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

The above results can be obtained algebraically as follows:

The roots of quadratic equation $ax^2 + bx + c = 0$, are

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

and and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Adding both roots, i.e., x_1 and x_2 we get

$$x_1 + x_2 = \frac{-2b}{2a} = \frac{-b}{a} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

Multiplying we get,

$$x_1 \cdot x_2 = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Methodology used: Inductive method is used to explain the formula and then it is derived using deductive logic.

Check Your Proress

Notes: a) Write your answers in the space given below.

b) Compare your answers with the one given at the end of the unit.

15. If a and b are roots of the equation $ax^2 + bx + c = 0$, find the values of the following:

i) $\frac{1}{\alpha} = \frac{1}{\beta}$

ii) $\alpha^2 b + \alpha\beta^2$

16. For what value of p, are the roots of the equation equal?

 i) $3x^2 - 5x + p = 0$

 ii) $2px^2 - 8x + p = 0$

17. Form an equation, whose roots are:

 i) 2, 2

 ii) $3, -\sqrt{3}$,

11.6.3 Equations Reducible to Quadratic Equation

Main Teaching Point

How to transform a given equation to quadratic form in different situations.

Teaching Learning Process

There are equations which are not quadratic, but suitable transformation reduces them to quadratic equation. Such equations are called **reducible equations.** For example,

$2x^4 - 5x^2 + 3 = 0$. This is not a quadratic equation, as its degree is 4. Putting $x^2 = z$, we get

$2z^2 - 5z + 3 = 0$

This is a quadratic equation.

The equation $py + \frac{q}{y} = r$ is not quadratic. But on multiplying by y on both sides, $[y \neq 0]$, we get

$py^2 + q = ry$ or $py^2 - ry + q = 0$ which is a quadratic equation.

Consider the equation

$\sqrt{25 - x^2} = x - 1$

This is not a quadratic equation. But on squaring we get

$25 - x^2 = (x - 1)^2$

and on simplifying we get, $x^2 - x - 12 = 0$ which is a quadratic equation.

On solving it, we get:

$(x + 3)(x - 4) = 0$

$\Rightarrow x = -3$ or $x = 4$

Since $\sqrt{25 - x^2} = x - 1$

$\therefore$ solution should be such that

$25 - x^2 \geq 0$ and $x - 1 \geq 0$

$\therefore$ $x^2 \leq 25$ and $x \geq 1$

$\therefore$ $x \leq 5$ and $x \geq 1$

$\therefore$ x should be such that $1 \leq x \leq 5$. Therefore, $x = 4$ is the only root of the given equation. $x = -3$ is not a root of the given equation. It is called an extraneous root.

Similarly, other equations of the type

i) $\sqrt{3x+10} + \sqrt{6-x} = 6$

ii) $3\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 26 = 0$

can be solved after transforming them to quadratic form.

Methodology used: Deductive method is used for teaching the transformation in different situations.

Check Your Progress

Notes: a) Write your answer in the space given below.

b) Compare your answer with the one given at the end of the unit.

Solve the following equations.

18. $\sqrt{x-2} + \sqrt{x+1} = 63$

19. $\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 2 = 0$

11.6.4 Solution of Word Problems

Main Teaching Point

To translate the given word problem into a mathematical model.

Teaching Learning Process

There are a few problems which when translated into a mathematical model form a quadratic equation. In such a situation the main emphasis should be on translation of the language of problems into an appropriate quadratic equation.

For example: Find two natural numbers whose difference is 3 and sum of their squares is 117.

Let the first natural number = x

Second number = x + 3

Translating the language of the problem into a mathematical model.

$(x)^2 + (x+3)^2 = 117$

Simplification leads to a quadratic equation.

$2x^2 + 6x - 108 = 0 \Rightarrow x = -9; x = 6.$

x = – 9 is not applicable as the numbers are natural.

Thus, the first number = 6 and the second number = 9

Example II: The sum of a number and its reciprocal is $\frac{41}{20}$,

Find the number.

Let the number be x

Its reciprocal = $\frac{1}{x}$

Translating the problem into a mathematical model,

$x + \frac{1}{x} = \frac{41}{20} \Rightarrow 20x^2 - 41x + 20 = 0$

Solving it, we get

$x = \frac{5}{4}$ or $\frac{4}{5}$

Number = $\frac{5}{4}$ or $\frac{4}{5}$

Methodology used: Heuristic method is used to solve word problems. You, the teacher, should guide the students towards logical thinking.

11.7 LET US SUM UP

The unit provides an opportunity to the teacher

i) To distinguish between equations in one degree and two degrees.

ii) Graphical representation of linear equation in one variable and two variables.

iii) To understand different types of systems of equations, i.e., consistent, dependent or inconsistent.

iv) Solution of quadratic equation.

v) Solution of questions connected with roots of equations.

vi) Solution of inequations with the help of graph.

11.8 UNIT-END ACTIVITIES

1. Draw the graph of $|y| = 5$

2. Draw the graph of $y = |x - 3|$

3. Solve the following pairs of equations:

   a) $x + \frac{y}{2} = 4; \qquad \frac{x}{3} + 2y = 4$

   b) $\frac{x}{2} + \frac{y}{4} = 3; \qquad 2x - y = 4$

   c) $c_1x + d_1y = p_1$. Apply cross multiplication method

   $c_2x + d_2y = p_2$

4. For what value of k, has the system of equations a unique solution?

   $2x + ky - 1 = 0$

   $3x - y - 7 = 0$

5. Prove that the system of equations

   $ax + by = c$

   $lx + my = n$

   has a unique solution if $am - bl = 0$

6. The age of a father is 4 times that of his son. Five years hence, the age of father will be three times the age of his son. Find the present ages of the father and son.

7. If one is subtracted from both numerator and denominator of a fraction, the fraction becomes 1/3. If 1 is added to both numerator and the denominator, the fraction becomes 1/2. Find the fraction.

8. Solve the following equations.

   a) $6x^2 + 7x - 3 = 0$

   b) $6\left(y^2 + \frac{1}{y^2}\right) - 25\left(y - \frac{1}{y}\right) + 12 = 0$

   c) $\left(\frac{2x+1}{x-1}\right)^4 - 10\left(\frac{2x+1}{x-1}\right)^2 + 9 = 0$

   d) $\sqrt{2x^2 - 2x + 1} - 2x + 3 = 0$

9. The difference of two positive numbers is 7 and their multiplication is 170. Find the numbers.

10. Divide 16 into two numbers such that twice the square of greater number is 164 more than the square of the smaller number.

11. Find the quadratic equation whose one root is 2 and the sum of the roots is – 4.

12. Find the value of k such that the sum and product of roots of the quadratic equation

$3x^2 + (2k + 1)x - k - 5 = 0$ are equal.

11.9 ANSWERS TO CHECK YOUR PROGRESS

1. i) and (v) are equations.

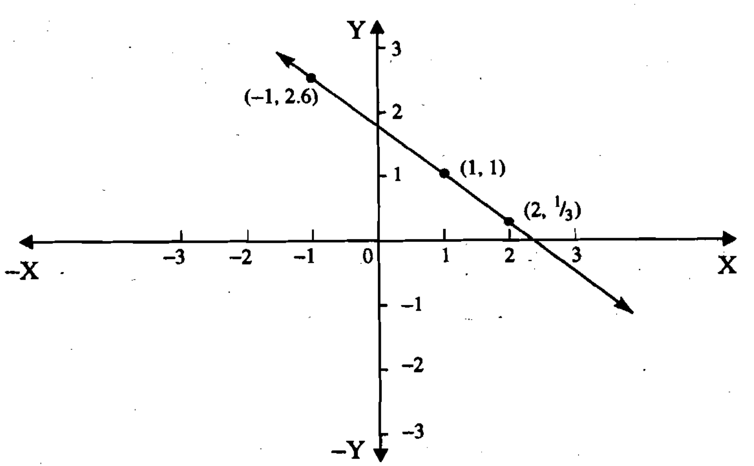
2. i) Linear

 ii) Quadratic

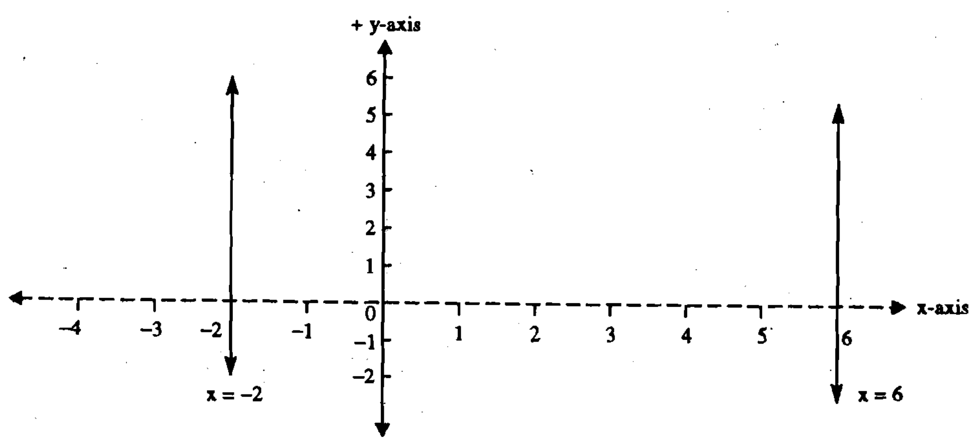
 iii) Cubic

 iv) Linear because it is equivalent to $5x - 10 = 0$

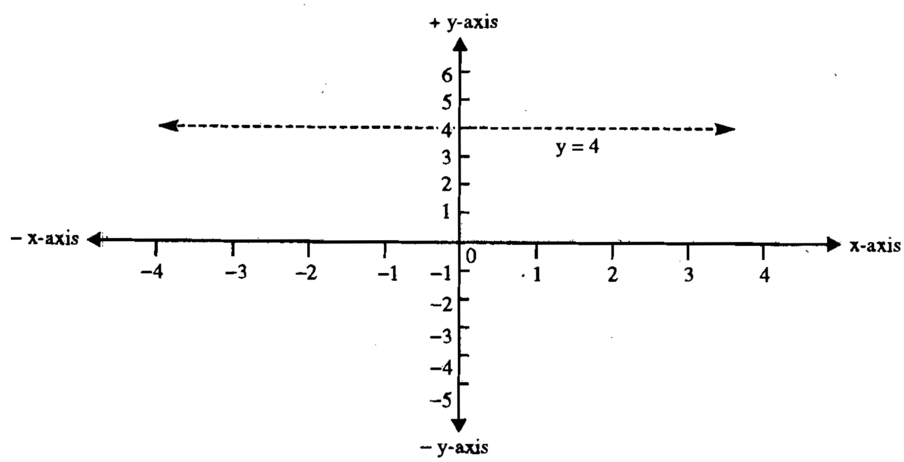
3.



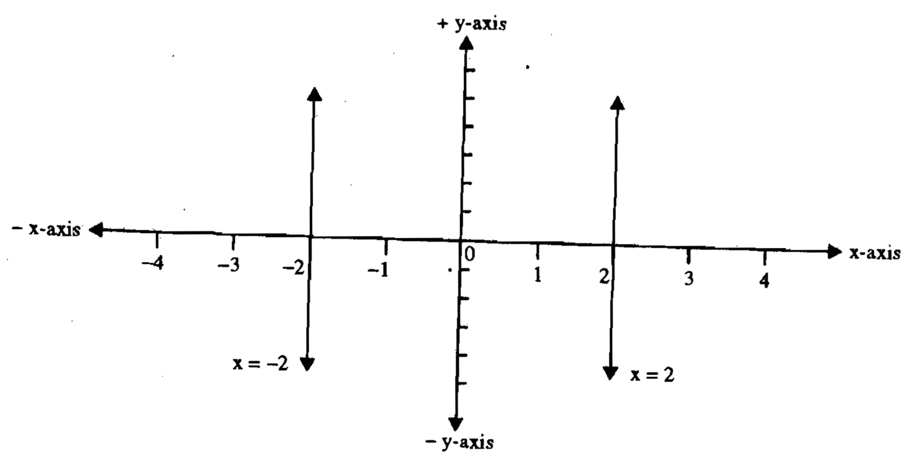
4.



5. Dependent
6. Inconsistent
7. Consistent
8. Dependent
9. 3/7
10. 41 and 11 years
11. 25 km/hr., 5 km/hr.
12.



13.



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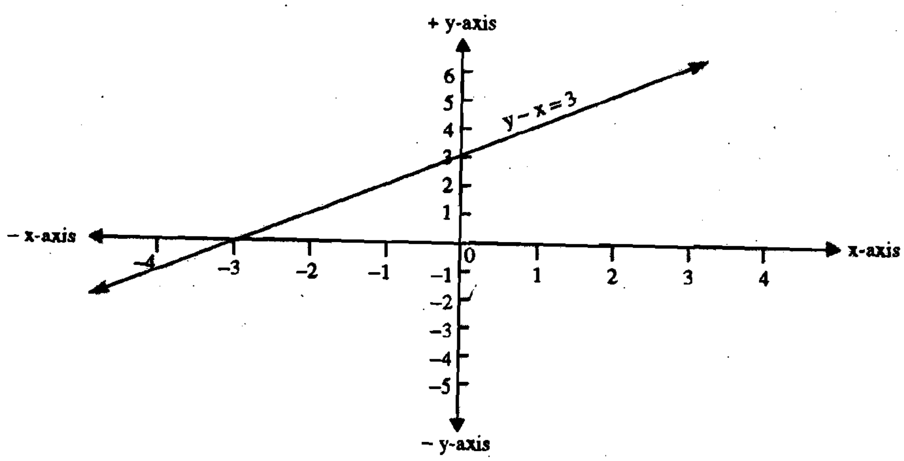

15. i) $-b/c$

    ii) $\frac{-bc}{a^2}$

16. i) $\frac{25}{12}$

    ii) $\pm 2\sqrt{2}$

17. i) $x^2 - 4x + 4 = 0$

    ii) $x^2 + \left(-3 + \sqrt{3}\right)x - 3\sqrt{3} = 0$

18. $x = 3$

19. $x = 1$

11.10 SUGGESTED READINGS

Mathematics: A Text Book for Class IX, (1993): NCERT, New Delhi.

Johnson R.E. et. al, (1961) *Modern Algebra*, First Course; Addison-Wesley Publishing Company Inc., USA.

Russel, Donald S. (1961): *Elementary Algebra*, Allyn and Bacon Inc., Boston.

Surjeet Singh and Quazi Zamiruddin, *Modern Algebra*.