UNIT 11 TWO-PERSON ZERO-SUM GAMES WITH SADDLE POINT

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11.1 INTRODUCTION

In "Game Theory", the word game is not used in the way it is commonly used in different types of sports such as hockey, cricket, football, etc. Also, it does not refer to computer games. In the usual sports or games, the main objective of the opponents is to win the game. But in the games discussed under game theory between two opposing parties with conflicting interests, winning means selecting an optimal strategy, e.g., selecting an optimal course of action as we have discussed in Units 9 and 10. Game theory deals with decision making processes of players in conflicting and competitive situations where the strategy (or action or move) of a player depends upon the move of the opponent.

Recall that in Sec. 9.5 of Unit 9, we have discussed four types of environments under which a decision maker may have to make decisions, namely:

- Decision making under certainty;
- Decision making under uncertainty;
- Decision making under risk; and
- Decision making under conflict.

We have discussed the first three types of environments in detail in Units 9 and 10. We now discuss the fourth type of environment which is present in games between two or more players. In Sec. 11.2, we introduce the key terms used in game theory. Saddle point (explained in Sec. 11.3) is one of the key concepts in game theory. On the basis of whether a saddle point exists in the game or not, games can be further classified as:

- Games with saddle point, and
- Games without saddle point.

The games with saddle point are discussed in the present unit and the games without saddle point are discussed in Unit 12. In Sec. 11.3, we discuss Maximin-minimax principle for solving two-person zero-sum games with saddle point.

In the next unit, we shall discuss the games without saddle point.

> The notion of game theory existed even before John Von Neumann (1903-1957). But it is John Von Neumann who is known as the father of game theory. He first published a paper on a mathematical treatment of game theory in 1928. In 1944 he published a book in collaboration with Oskar Morgenstern on game theory entitled "Theory of Games and Economic Behaviour".

Objectives

After studying this unit, you should be able to:

- define the key terms involved in game theory;
- solve the prisoner's dilemma game; and
- solve two-person zero-sum games using the maximin-minimax principle.

11.2 KEY TERMS USED IN GAME THEORY

In this section, we introduce the key terms and terminology used commonly in game theory. Then we explain what is meant by a game in game theory. But before doing so, let us consider the following situation:

Suppose two children X and Y agree that:

- Each one of them will simultaneously place a coin on the table.
- Each one of them will show the outcome (head or tail).
- If the faces of both coins match (i.e., either both coins show head or both show tail), child X wins and gets Rs 1 from child Y.
- If the coins do not match, child Y wins and gets Rs 1 from child X.

We can present this information in the form of a matrix as shown below. The first numeral in the four cells having entries $(1,-1), (-1, 1), (-1, 1), (1, -1)$, respectively, represents the amount won by child X and the second numeral represents the amount won by child Y. When child X wins Rs 1 from child Y, the winning amount for X is 1 and the winning amount for Y is -1 (i.e., loss of 1).

		Child Y (Player II)	
		Head	**Tail**
Child X (Player I)	**Head**	$1, -1$	$-1, 1$
	Tail	$-1, 1$	$1, -1$

In game theory, the situation discussed above is known as **Coin Matching Game** and its solution is provided in E2 in Sec. 12.8 of Unit 12.

Let us consider another situation.

Two persons, say X and Y, are arrested by the police with enough evidence for a minor crime. The police suspect that they are responsible for a murder, but do not have enough evidence. Both persons are put in separate cells so that they have no way of communicating with each other. The police starts interrogating them in separate rooms (as shown in Fig.11.1). Each person either confesses or does not confess. Also, each one knows the consequences of confession, which are given below:

- If both of them confess, both go to jail for 5 years.
- If one of them confesses and the other does not, then the one who confessed turns government's witness while the other who did not confess goes to jail for 20 years.
- If both do not confess, both go to jail for one year.

Assume that each one of them has to protect his self-interest, which means that each person tries to act in such a way that he would have to go to jail for a shorter period of time, regardless of the way the other acts. Also assume that they have no way of communicating with each other. What should X and Y do?

We can present this information in matrix form as given below. The first numeral in the four cells having entries $(-5,-5), (0,-20), (-20, 0), (-1,-1)$ represents the time to be spent in prison by person X and the second numeral represents the time to be spent in prison by person Y. A negative sign is attached to the numerals because the time spent in prison is similar to a loss.

		Person Y (Player II)	
		Confesses	**Does not Confess**
Person X (Player I)	**Confesses**	$-5, -5$	$0, -20$
	Does not Confess	$-20, 0$	$-1, -1$

In game theory, the situation discussed above is known as **the Prisoner's Dilemma**. Its solution is provided in Example 4 in Sec. 11.3.

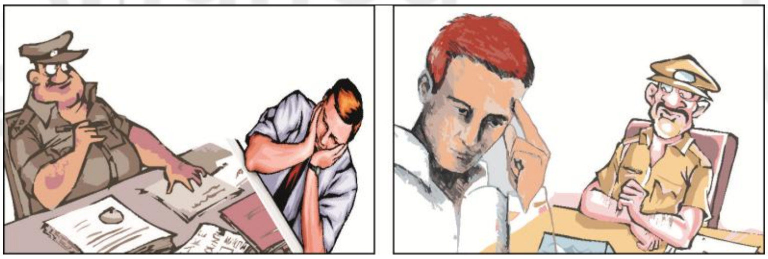

Fig. 11.1: Persons X and Y facing questions of policemen in separate rooms.

We now define some key terms used in game theory.

Player: A participant or competitor who makes decisions is known as a player. A player may be an individual or a group of individuals. For example, in coin matching game, child X and child Y are players, and in the Prisoner's Dilemma game, person X and person Y are players.

Action: The options available to the players are known as actions or courses of action or **moves**.

For example, in coin matching game, the actions are "head" and "tail"; and in the prisoner's dilemma game, "confesses" and "does not confess" are actions.

Play: A play is said to occur when each player selects a course of action.

Strategy: A predetermined rule by which a player decides his/her course(s) of action among the actions available to him/her is known as strategy for the player. A strategy may be of two types:

(i) **Pure Strategy:** A strategy is said to be a **pure strategy** if the player selects a particular course of action each time, i.e., if a player selects, say, the i^{th} course of action (A_i) each time from among n courses of actions, $A_1, A_2, ..., A_n$, available to him or her. This means that he/she assigns probability 1 to the i^{th} course of action and zero probability to each of the other courses of action $A_1, A_2, ..., A_{i-1}, A_{i+1}, ..., A_n$. So the pure strategy is denoted by $(0, 0, ..., 0, 1, 0, ..., 0)$ having values 0 at $(n-1)$ places except the i^{th} position having value 1. For example, in the prisoner's dilemma game, person X will use pure strategy (1, 0) as you will see in the solution of this game, which is provided in Example 4 in Sec. 11.3.

(ii) **Mixed Strategy:** A strategy for a player is said to be a **mixed strategy** if the player selects a combination of more than one courses of action by assigning a fixed probability to each course of action. That is, if there are n courses of action $A_i, (1 \le i \le n)$ available to the player, he/she assigns probabilities $p_1, p_2, ..., p_n$ to the courses of action $A_1, A_2, ..., A_n$, respectively, such that $p_1 + p_2 + ... p_n = 1$ and $p_i \ge 0$ for all i, $1 \le i \le n$. For example, in the coin matching game, child X will use the mixed strategy (1/2, 1/2) as you will see in the solution of E2, which is provided in Sec. 12.8 of Unit 12.

Note 1: Pure strategy is a particular case of mixed strategy because in the case of pure strategy

$$p_i = 1 \text{ for some i, and } p_j = 0 \text{ for all } j \ne i$$

Payoff Values and Payoff Matrix: You have learnt about payoff values and payoff matrices in Unit 9. Here the payoff value means money or anything else that motivates players and when these payoff values are represented in matrix form, the resulting matrix is known as payoff matrix. For example, in the coin matching game, payoff values are the money that child X gets from child Y or vice versa, while in prisoner's dilemma game, the time (in years) to be spent in prison represents payoff values.

Optimal Strategy: The strategy for a player, which optimises his/her payoff irrespective of the strategy of his /her competitor is known as an optimal strategy.

For example, in the coin matching game, the optimal strategy for child X is (1/2, 1/2), which is explained in the solution of E2 in Sec. 12.8 of Unit 12.

Now, we are in a position to define what we mean by a game in game theory.

Definition of Game: A competitive situation is called a game if

(i) The number of competitors, called players, is finite.

(ii) The number of possible courses of action for each player is finite. However, the courses of action need not be the same for each player.

(iii) Each player selects a course of action simultaneously from among the courses of action available to him/her without directly communicating with the other player.

(iv) Every combination of courses of action results in an outcome known as payoff value, which motivates the players. Payoff value may represent loss or gain or any other thing of interest. The payoff values may be positive, negative or zero.

n-Person Game: If the number of players involved in the game is $n\,(>2)$, it is known as n-person game.

Two-Person Game: If the number of players involved in the game is 2, it is known as two-person game.

Zero-Sum Game: If the algebraic sum of the payoff values of all players after each play is zero, the game is known as a zero-sum game. Mathematically, for zero-sum game,

if $a_i, \left(1 \le i \le n\right)$ represents the payoff value of the i^{th} player, then $\sum_{1=1}^{n} a_i = 0$

Non-Zero-Sum Game: A game is said to be non-zero-sum game if there exists at least one play such that the algebraic sum of all the payoff values is not equal to zero. For example, the prisoner's dilemma game is a non-zero-sum game.

Two-Person Zero-Sum Game: If in a game, each payoff value of one player is negative of the payoff value of the other player, it is known as two-person zero-sum game. For example, coin matching game is a two-person zero-sum game.

Note 2: In two-person zero-sum game, if we call the two players as player I and player II, we see that:

each payoff value of player II is **equal in magnitude** to the payoff value of player I but **opposite in sign**.

Hence, if the payoff value of player I is known, the payoff value of player II is automatically known. So, for a two-person zero-sum game, generally, we write the payoff values of only player I instead of writing payoff values of both players. For example, payoff matrix in the case of coin matching game can simply be written as follows:

		Child Y (Player II)	
		Head	Tail
Child X (Player I)	Head	1	–1
	Tail	–1	1

Note 3: In this unit and in Unit 12 we shall limit our discussion to two-person zero-sum games. Only in Example 4, we provide the solution of prisoner's dilemma game, which is a non-zero sum game. So henceforth, in the payoff matrix we shall write only payoff values of player I as explained in Note 2.

11.3 THE MAXIMIN-MINIMAX PRINCIPLE

In Sec. 9.5 of Unit 9, you have studied the maximin-minimax criterion for a decision making situation. The basic idea of the criterion remains the same except that here we are working under the environment of conflict whereas in Unit 9 we solved problems of decision making under uncertainty. The second distinguishing feature is that here two players are making decisions simultaneously, while in Unit 9, there was only one decision maker.

With this clarification, let us define the principle:

Under this principle, first of all, a player lists the worst possible payoff values of all strategies available to him/her. Then he/she selects the strategy corresponding to the optimum payoff value from among the worst possible payoff values.

Let us consider a two-person zero-sum game to explain this principle. The payoff matrix for the game is given below:

		Player B	
		I	II
Player A	I	1	4
	II	2	3
	III	–3	5

If a_{ij} represents the payoff value when player A chooses his/her $i^{th}, (i = I, II, III)$ strategy and player B chooses his/her j^{th}, $(j = I, II)$ strategy, then

$a_{11} = 1 \Rightarrow$ Player A will get Rs 1 from player B

$a_{12} = 4 \Rightarrow$ Player A will get Rs 4 from player B

$a_{21} = 2 \Rightarrow$ Player A will get Rs 2 from player B

$a_{22} = -3 \Rightarrow$ Player A will get Rs 3 from player B

$a_{31} = -3 \Rightarrow$ Player A will pay Rs 3 to player B

$a_{32} = 5 \Rightarrow$ Player A will get Rs 5 from player B

Let us first analyse this game from the point of view of player A: Here payoff values in the payoff matrix represent gains of the player A. So when player A chooses a particular strategy, player B would move in such a way that payoff to player A is minimum for that particular strategy because the interests of player A and player B are conflicting. Thus, if player A employs strategy I, he/she may gain Rs 1 or Rs 4 depending upon the strategy adopted by player B. Now, whatever strategy is adopted by player B, player A will gain at least $\min\{1, 4\} = \text{Rs } 1.$ Similarly, if player A employs strategy II, he/she will gain at least $\min\{2, 3\} = \text{Rs } 2,$ and if player A employs strategy III, then he/she will gain at least $\min\{-3, 5\} = -\text{Rs } 3.$ Obviously, player A would like to opt for the strategy, which maximises his/her minimum gains. Since $\max\{1, 2, -3\} = 2,$ player A should adopt strategy II. You may think that if player A adopts strategy III and player B adopts strategy II, the gain of player A will be Rs 5, which is greater than Rs 2. You are right but there is no guarantee that B will adopt strategy II. He/she may employ strategy I, which will result in a loss of Rs 3 to player A. So, player A should go for strategy II because, in game theory, we assume that both players are equally intelligent.

Now, we analyse this game from the point of view of player B: If player B employs strategy I, then he/she may face a loss of Rs 1, a loss of Rs 2 or a gain of Rs 3 depending upon the strategy adopted by player A. Now, whatever strategy is adopted by player A, player B cannot incur a loss of more than $\max\{1, 2, -3\} = \text{Rs } 2.$ Similarly, if player B employs strategy II, he/she cannot incur a loss of more than $\max\{4, 3, 5\} = \text{Rs } 5.$ Obviously, player B would like to opt for the strategy, which minimises his/her maximum losses. Since $\min\{2, 5\} = \text{Rs } 2,$ player B should adopt strategy I.

From the above discussion, we note that player A should opt for the strategy which corresponds to the maximum payoff value among the row minimum values, i.e., maximum among the minima. Hence, it is known as **maximin value** and is denoted by $\max_i \min_j \{a_{ij}\}.$ Player B should opt for the strategy, which corresponds to the minimum payoff value from among the column maximum values, i.e., minimum among the maxima. Hence, it is known as **minimax value** and is denoted by $\min_j \max_i \{a_{ij}\}.$

If $\text{maximin value} = \text{minimax value} = v\,(\text{say})$,

then v is known as the **value of the game** and the corresponding strategies of players A and B are known as their **optimal strategies**. Also, the position of the element corresponding to the optimal strategies of the two players is known as **saddle point**. In the above example, the value of the game is Rs 2 and strategy II of player A is the optimal strategy for player A while strategy I of player B is the optimal strategy for player B. Also the position of the saddle point is $(2,1)$, i.e., corresponding to second row and first column.

You should follow the steps given below for applying maximin-minimax principle to numerical problems:

Step 1: Identify the minimum element in each row of the payoff matrix and select the largest element among these row minima. This is the maximin value.

Step 2: Identify the maximum element in each column of the payoff matrix and select the minimum among these column maxima. This is the minimax value.

Step 3: If maximin value = minimax value $= v(\text{say})$

i.e., $\max_i \min_j \{a_{ij}\} = \min_j \max_i \{a_{ij}\} = v\,(\text{say})$ and lies at the intersection of the row of maximin value and the column of the minimax value, we say that the game is solved by **maximin-minimax principle**. The maximin (minimax) value v is called the **value of the game**. The strategies corresponding to the row of maximin value and the column of minimax value are termed the **pure optimal strategies** for player A and player B, respectively. Also, the position of the element where the row of maximin value and the column of the minimax value intersect is known as the **saddle point**.

Note 4: If maximin value $\neq$ minimax value, we say that the game does not have a saddle point. So we cannot obtain the solution of the game in terms of pure strategies by applying the maximin-minimax principle. Games without a saddle point have mixed strategies as their solutions, which are discussed in Unit 12.

Note 5: By solution of a game, we mean an optimal strategy for each player and the value of the game.

We now define a few more terms before taking up examples.

Lower and Upper Values of the Game: If a_{ij} denotes the payoff value when player I chooses his/her i^{th}, $(i = 1, 2, ..., m)$ strategy and player II chooses his/her j^{th}, $(j = 1, 2, ..., n)$ strategy, then the maximin value $= \max_i \min_j \{a_{ij}\}$ is known as the **lower value** of the game and is generally denoted by $\underline{v}$. The minimax value $= \min_j \max_i \{a_{ij}\}$ is known as the **upper value** of the game and is generally denoted by $\overline{v}$.

Strictly Determinable Game: A game is said to be **strictly determinable** if $\underline{v} = v = \overline{v}$, where $\underline{v} = \max_i \min_j \{a_{ij}\}$, $\overline{v} = \min_j \max_i \{a_{ij}\}$, $v =$ value the game

Fair Game: A game is said to be **fair** if $\underline{v} = v = \overline{v} = 0$.

Let us now apply the maximin-minimax principle to a few games.

Example 1: You are given a game having the following payoff matrix:

		Player B		
		B_1	**B_2**	**B_3**
Player A	**A_1**	5	7	4
	A_2	4	2	0
	A_3	6	1	3

Obtain the

(i) Optimal strategy for player A,

(ii) Optimal strategy for player B,

(iii) Value of the game, and

(iv) Saddle point.

Also answer the questions: Is the game strictly determinable? Is the game fair?

Solution: We apply the maximin-minimax principle to the game. The following three steps are involved in applying this principle:

Step 1: We identify the minimum element of each row and then select the maximum among these minimum elements. This is called the maximin value as indicated in Table 11.1.

Step 2: Then we identify the maximum element of each column and select the minimum among these maximum elements. It is called the minimax value as indicated in Table 11.1.

Table 11.1: Calculation of Maximin-Minimax Principle

		Player B			Step 1	
		B_1	**B_2**	**B_3**	**Row Minima**	**Maximin Value**
Player A	**A_1**	5	7	4	4	$\max\{4, 0, 1\} = 4$
	A_2	4	2	0	0	
	A_3	6	1	3	1	
Step 2	**Column Maxima**	6	7	4		
	Minimax Value	$\min\{6, 7, 4\} = 4$				

Step 3: We check whether the maximin and minimax values are equal or not. In this case, maximin value = minimax value = 4. So, the game is solved using the maximin-minimax principle and we obtain the following results:

i) Optimal strategy for player A is A_1 as it corresponds to the maximin value,

ii) Optimal strategy for player B is B_3 as it corresponds to the minimax value,

iii) The value of the game is $4 [\because \text{maximin value} = \text{minimax value} = 4]$

Now,

$\underline{v} = \text{the lower value of the game} = \text{maximin value} = 4$

$\overline{v} = \text{the upper value of the game} = \text{minimax value} = 4$

$v = \text{the value of the game} = 4$

$\because \underline{v} = v(= 4) = \overline{v}$, the game is strictly determinable.

But $v = 4 \neq 0$. So the game is not fair.

Note 6: In the above example, player A always has to employ strategy A_1 to maximise his/her minimum gains. So he should associate probability 1 to A_1 and a zero probability to each of A_2 and A_3. Hence, the optimal strategy of player A can also be written as a pure strategy $(1, 0, 0)$. Similarly, optimal strategy of player B can also be written as pure strategy $(0, 0, 1)$.

Example 2: Find the maximin and minimax values for the game having the payoff matrix given below. Does the game have a saddle point? If the game has a saddle point, solve it.

		Player B		
		B_1	B_2	B_3
Player A	A_1	2	4	5
	A_2	6	3	7
	A_3	4	1	–1

Solution: Calculations for the maximin and the minimax values are shown in Table 11.2 given below:

Table 11.2: Calculation of Maximin-Minimax Values

		Player B			Step 1	
		B_1	B_2	B_3	Row Minima	Maximin Value
Player A	A_1	2	4	5	2	$\max\{2, 3, -1\} = 3$
	A_2	6	3	7	3	
	A_3	4	1	–1	–1	
Step 2	Column Maxima	6	4	7		
	Minimax Value	$\min\{6, 4, 7\} = 4$				

In the above table, we see that

Maximin value = 3 and minimax value = 4.

Since maximin value ≠ minimax value, the game has no saddle point.

Hence, the solution of the game cannot be obtained by using the maximin-minimax principle.

Note 7: Solutions of the games without saddle points are discussed in Unit 12.

Example 3: Determine the range of values of λ and μ that will make the position $(2, 2)$ a saddle point for the game having the payoff matrix given below:

		Player B		
		B_1	**B_2**	**B_3**
Player A	**A_1**	1	3	5
	A_2	8	4	λ
	A_3	2	μ	9

Solution: Since it is given that the position $(2, 2)$ is the saddle point, the maximin value = minimax value $= 4$.

Now, maximin value $= 4 \Rightarrow 4$ is the minimum element of the second row.

$\therefore \quad \lambda \geq 4$

Also minimax value $= 4 \Rightarrow 4$ is the maximun element of the second column.

$\therefore \quad \mu \leq 4$

Hence, the range of λ is $\lambda \geq 4$ and the range of μ is $\mu \leq 4$.

Example 4: What are the optimal strategies for person X and person Y in the Prisoner's Dilemma game?

Solution: We have described the Prisoner's Dilemma game in Sec. 11.2 and obtained the payoff matrix for it. Let us rewrite the payoff matrix given there.

		Person Y (player II)	
		Confesses	**Does not Confess**
Person X (Player I)	**Confesses**	–5, –5	0, –20
	Does not Confess	–20, 0	–1, –1

It is not a zero sum game. Obviously, if players have the facility to communicate with each other, then not to confess is the optimal strategy for both players because in this case both will get prison for only one year. But it is given that they cannot communicate with each other and both persons have to protect their self-interest. What is the optimal strategy for both players? Let us first find the optimal strategy for player X. If Y confesses, then X will go to jail for 5 years if he confesses, and 20 years if he does not confess. So, if Y confesses, it is better for X to "confess" rather than to "not confess". If Y does not confess, then X will be free (0 year in jail) if he confesses, and get 1 year jail term if he does not confess. Also if Y does not confess, it is better for X to "confess" rather than to "not confess".

Thus, for X it is better to confess rather than to not confess, irrespective of whether Y confesses or not. Therefore, the optimal strategy for X is to confess.

Let us now now find the optimal strategy for Y. If X confesses, then Y will go to jail for 5 years if he confesses, and 20 years if he does not confess. So, if X confesses, then it is better for Y to "confess" rather than to "not confess". If X does not confess, then Y will be free (0 year jail) if he confesses, and get 1 year jail term if he does not confess. Also if X does not confess, it is better for Y to "confess" rather than to "not confess".

Hence, under the given situation the optimal strategy for both X and Y is to confess. So, the optimal pure strategy for both persons is (1, 0).

You may like to try the following exercises to check your understanding of the key terms and concepts explained in Secs. 11.2 and 11.3.

In the exercises E1) to E5), choose the correct option.

E1) Game theory is the study of:

(A) Computer games (B) Usual sports

(C) Identifying optimal strategies (D) Two-person zero-sum games

E2) If $\underline{v}$ and $\overline{v}$ denote the lower and upper values of the game, then the game is said to be fair if

(A) $\underline{v} < \overline{v}$ (B) $\underline{v} = \overline{v} \neq 0$ (C) $\underline{v} = \overline{v} = 0$ (D) $\underline{v} < 0 < \overline{v}$

E3) A saddle point exists in a game when

(A) Maximax Value = Minimin Value

(B) Maximin value = Minimax Value

(C) Maximax value = Maximin value

(D) Maximin value = Minimin value

E4) Three strategies are available to a player in a game between two players. There is a pure optimal strategy in the game for him/her. Which of the following cannot be his/her pure strategy?

(A) $(1/3, 1/3, 1/3)$ (B) $(1, 0, 0)$ (C) $(0, 1, 0)$ (D) $(0, 0, 1)$

E5) If a player A associates probabilities $p_1, p_2, ..., p_m$ with m strategies available to him, which one of the following is the case of pure strategy?

(A) $p_1 = p_2 = ... = p_m = \frac{1}{m}$

(B) $p_1 = \frac{1}{2}, \ p_2 = \frac{1}{2}, \ p_3 = p_4 = ... = p_m = 0$

(C) $p_1 = p_2 = ... = p_{m-2} = 0, \ p_{m-1} = \frac{1}{2}, \ p_m = \frac{1}{2}$

(D) $p_i = 1$ for some i and $p_j = 0$ for all $j \neq i$

E6) Solve the game for which the payoff matrix is given by

		Player B		
		B_1	**B_2**	**B_3**
Player A	**A_1**	2	4	3
	A_2	1	–2	–3
	A_3	0	6	1

Let us now summarise the main points that we have covered in this unit.

11.4 SUMMARY

1) **Player:** A participant or competitor who makes decisions is known as a player. A player may be an individual or a group of individuals. For example, in coin matching game, child X and child Y are players, and in the Prisoner's Dilemma game, person X and person Y are players.

2) **Action:** The options available to the players are known as actions or courses of action or **moves**.

3) **Play:** A play is said to occur when each player selects a course of action.

4) **Strategy:** A predetermined rule by which a player decides his/her course(s) of action among the actions available to him/her is known as strategy for the player. A strategy may be of two types:

   - **Pure Strategy:** A strategy is said to be a **pure strategy** if the player selects a particular course of action each time.
   - **Mixed Strategy:** A strategy for a player is said to be a **mixed strategy** if the player selects a combination of more than one courses of action by assigning a fixed probability to each course of action.

5) **Optimal Strategy:** The strategy for a player which optimises his/her payoff irrespective of the strategy of his /her competitor is known as an optimal strategy.

6) **Definition of Game:** A competitive situation is called a game if

   (i) The number of competitors, called players, is finite.

   (ii) The number of possible courses of action for each player is finite. However, the courses of action need not be the same for each player.

   (iii) Each player selects a course of action simultaneously from among the courses of action available to him/her without directly communicating with the other player.

   (iv) Every combination of courses of action results in an outcome known as payoff value, which motivates the players. Payoff value may represent loss, gain or any other thing of interest. The payoff values may be positive, negative or zero.

7) **n-Person Game:** If the number of players involved in the game is $n(>2)$, it is known as n-person game.

8) **Two-Person Game:** If the number of players involved in the game is 2, it is known as two-person game.

9) **Zero-Sum Game:** If the algebraic sum of the payoff values of all players after each play is zero, the game is known as a zero-sum game.

10) **Non-Zero-Sum Game:** A game is said to be non-zero-sum game if there exists at least one play such that the algebraic sum of all the payoff values is not equal to zero. The prisoner's dilemma game is a non-zero-sum game.

11) **Two-Person Zero-Sum Game:** If in a game, each payoff value of one player is negative of the payoff value of the other player, it is known as two-person zero-sum game. The coin matching game is a two-person zero-sum game.

12) **Lower and Upper Values of the Game:** If a_{ij} denotes the payoff value when player I chooses his/her i^{th} $(i=1,2, ..., m)$ strategy and player II chooses his/her j^{th} $(j=1, 2, ..., n)$ strategy, then the maximin value $= \max_i \min_j \{a_{ij}\}$ is known as the **lower value** of the game and is generally denoted by $\underline{v}$. The minimax value $= \min_j \max_i \{a_{ij}\}$ is known as the **upper value** of the game and is generally denoted by $\overline{v}$.

13) If maximin value $=$ minimax value $= v$(say), then v is known as the **value of the game** and the corresponding strategies of players A and B are known as their **optimal strategies**. Also, the position of the element corresponding to the optimal strategies of the two players is known as **saddle point**.

14) **Strictly Determinable Game:** A game is said to be **strictly determinable** if $\underline{v} = v = \overline{v}$.

15) **Fair Game:** A game is said to be fair if $\underline{v} = v = \overline{v} = 0$.

11.5 SOLUTIONS/ANSWERS

E1) In game theory, we try to identify the optimal strategy for each player. So, part (C) is the correct option.

E2) We know that a game is said to be fair if both lower value and upper value of the game are equal to zero. So, part (C) is the correct option.

E3) We know that saddle point is the point of intersection of the row of maximin value and column of minimax value. Hence the saddle point exists in a game if maximin value = minimax value. So, part (B) is the correct option.

E4) We know that among m strategies $A_1, A_2, ..., A_m$ available to a player A, a strategy say $A_i, (1 \le i \le m)$ is said to be a pure optimal strategy if player A has to use A_i if he/she wants to be in a comfortable position irrespective of the strategy adopted by his/her opponent B. That is, for A_i to be a pure optimal strategy, we should have

$$p_i = 1 \text{ and } p_j = 0 \text{ for all } j \ne i$$

where $p_1, p_2, ..., p_m$ are the probabilities associated with the strategies $A_1, A_2, ..., A_m$, respectively.

Hence, except for the part (A), all others are cases of pure strategies. So, the correct option is (A).

E5) As explained in the solution of E4, the correct option is (D).

E6) We solve the game by using the maximin-minimax principle. The following three steps are involved in applying this principle:

Step 1: We identify the minimum element of each row and then select the maximum of these minimum elements. It is the maximin value as indicated in Table 11.3.

Step 2: We identify the maximum element of each column and then select the minimum of these maximum elements. It is the minimax value as indicated in Table 11.3.

Table 11.3: Calculation of Maximin-Minimax Values

		Player B			**Step 1**	
		B_1	**B_2**	**B_3**	**Row Minima**	**Maximin Value**
Player A	**A_1**	2	4	3	2	$\max\{2, -3, 0\} = 2$
	A_2	1	–2	–3	–3	
	A_3	0	6	1	0	
Step 2	**Column Maxima**	2	6	3		
	Minimax Value	$\min\{2, 6, 3\} = 2$				

Step 3: We check whether the maximin and minimax values are equal or not. Here maximin value = minimax value = 2. So, the game is solved using the maximin-minimax principle, and the solution is:

Optimal strategy for player A is A_1 or $(1, 0, 0)$.

Optimal strategy for player B is B_1 or $(0, 0, 1)$.

The value of the game is 2.