UNIT 9 DIFFERENTIAL EQUATIONS

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9.1 INTRODUCTION

Analysis has been dominant branch of mathematics for 300 years, and differential equations is the heart of analysis. Differential equation is an important part of mathematics for understanding the physical sciences. It is the source of most of the ideas and theories which constitute higher analysis. Many interesting geometrical and physical problems are proposed as problems in differential equations, and solutions of these equations give complete picture of the state of these problems.

Differential equations work as a powerful tool for solving many practical problems of science as well as a wide range of purely mathematical problems. In Units 2 and 5, we defined first and higher order ordinary and partial derivatives of a given function. In this unit, we make use of these derivatives and we first introduce some basic definitions. Then we discuss the formation of differential equations . We also discuss various techniques of solving some important types of differential equations of the first order and first degree. We have discussed the method of separation of variables together with methods of solving homogeneous, exact and linear differential equations. Also, differential equations which are reducible to homogeneous form or equations reducible to linear form are considered.

Objectives

After studying this unit you should be able to

* distinguish between ordinary and partial differential equations and between the order and the degree of an equation,
* form a differential equation whose solution is given,
* use the method of separation of variables,
* identify homogeneous or linear equations and solve the same,
* verify whether a given differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact or not, and solve exact equations,and
* identify an integrating factor in some simple cases which makes the given equation exact.

9.2 PRELIMINARIES

In this section we shall define and explain the basic concepts in differential equations and illustrate them through examples. Recall that given an equation or relation of the type $f(x, y) = 0$, involving two variables x and y, where $y = y(x)$, we call x the **independent variable** and y the **dependent variable** (ref. Unit 1 of Block 1). Any equation which gives the relation between the independent variable and the derivative of the dependent variable with respect to the independent variable is a differential equation. In general we have the following definition.

Definition :

A differential equation is an equation that involves derivatives of dependent variable with respect to one or more independent variables.

For example,

$$x\frac{dy}{dx} + y = 0, \qquad \ldots (9.1)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz, \qquad \ldots (9.2)$$

$$\frac{\partial^2 z}{\partial x^2} + a^2\frac{\partial^2 z}{\partial y^2} = 0, \qquad \ldots (9.3)$$

are differential equations.

Differential equations are classified into various types. The most obvious classification of differential equations is based on the nature of the dependent variable and its derivative (or derivatives) in the equations. The following definitions give the various types of equations:

Definition :

A differential equation involving only derivatives (that is, derivatives with respect to a single independent variable) is called an **ordinary differential equation** (abbreviated as ODE). For example, equation (9.1) namely,

$$x\frac{dy}{dx} + y = 0$$

is an ordinary differential equation.

Definition :

A differential equation containing partial derivatives, that is, the derivative of a dependent variable with respect to two or more independent variables is called a **partial differential equation** (abbreviated as PDE).

For example, equations (9.2) and (9.3), namely

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz,$$

$$\frac{\partial^2 z}{\partial x^2} + a^2\frac{\partial^2 z}{\partial y^2} = 0,$$

are partial differential equations. Differential equations can be further broadly classified by their order and degree.

Definition :

The n^{th} **derivative** of a dependent variable with respect to one or more independent variables is called a derivative of order n, or simply an n^{th} order derivative. For example, $\frac{d^2y}{dx^2}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, are second order derivatives and $\frac{d^3y}{dx^3}$, $\frac{\partial^3 z}{\partial x^2 \partial y}$ are third order derivatives.

Definition :

The **order** of a differential equation is the order of the highest order derivative appearing in the equation. For example,

$$\left(\frac{dy}{dx}\right)^2 + y = 0 \text{ is of order } \textbf{one} \qquad \ldots (9.4)$$

(because the highest order derivative is $\frac{dy}{dx}$ which is of first order)

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = x^2 + 2, \text{ is of order } \textbf{two.} \qquad \ldots (9.5)$$

(highest order derivative is $\frac{d^2y}{dx^2}$).

$$\left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \text{ is of order } \textbf{three.} \qquad \ldots (9.6)$$

Definition :

The **degree** of a differential equation is the highest exponent of the derivative of the highest order appearing in it after the equation has been expressed in the form free from radicals and fractions as far as derivatives are concerned. For example, equations (9.1), (9.2), (9.3), and (9.5) are of first degree and equations (9.4) and (9.6) are of second degree.

Equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2} \qquad \ldots (9.7)$$

is of second degree. To determine its order, we make the equation free from radicals. We need to square both the sides so that

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

Since the highest exponent of the highest derivative, that is, $\frac{d^2y}{dx^2}$, is two and, by definition, the degree of equation (9.7) is two.

And now an exercise for you.

E 1

State the order and the degree of each of the following differential equations.

a) $\left(\frac{dy}{dx}\right)^2 = \frac{3x}{4y}$

b) $\left(\frac{dy}{dx}\right)^3 = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

c) $\sqrt{\frac{d^2y}{dx^2}} = 3\frac{dy}{dx}+x$

d) $\frac{d^3y}{dx^3} = \sqrt{\frac{dy}{dx}}$

e) $\left(\frac{d^2y}{dx^2}\right)^{1/3} = k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2}$

In this unit we will only be concerned with the study of certain types of ordinary differential equations which we shall simply refer to as differential equations, dropping the word "ordinary".

The principal task of the theory of differential equations is to find all the solutions of a given differential equation. But then it is natural to ask as to what exactly is the meaning of a solution of a differential equation. The answer to this question is given in the following definition.

Definition :

A **solution** or an **integral** of a differential equation is a relation between the variables, not involving the derivatives, such that relation and the derivatives obtained from it satisfy the given differential equation.

To illustrate this let us do the following examples.

Example 1 :

Prove that $y = cx^2$ ... (9.8)

is a solution of $xy' = 2y$. ... (9.9)

Solution :

Step 1 :

Differentiating both sides of equation (9.8) with respect to x, we get,

$$y' = 2cx \quad \text{... (9.10)}$$

where y' stands for $\frac{dy}{dx}$

Step 2 :

Substituting in equation (9.9) the values of y and y' obtained from equations (9.8) and (9.10)
respectively, we get an identity,

$$x \,.\, 2cx = 2cx^2$$

It should also be noted that a differential equation may have many solutions. For instance, each of the functions $y = \sin x$, $y = \sin x + 3$, $y = \sin x - 4/5$ is a solution of the differential equation $y'' = \cos x$. But we also know from our knowledge of calculus that any solution of the equation is of the form

$$y = \sin x + c, \qquad \ldots (9.11)$$

where c is a constant. If we regard c as arbitrary, then equation (9.11) represents the totality of all solutions of the equation.

If you have understood the above example, then you may now try the following exercise.

E 2

Verify that each function is a solution of the differential equation written next to it.

a) $y = x^2 + x + c;\ y' = 2x + 1$

b) $y = x^2 + cx;\ \ xy' = x^2 + y$

c) $y = A \sin 5x + B \cos 5x;\ y'' + 25y = 0$

d) $y = (x + c)e^{-x};\ \ y' + y = e^{-x}$

e) $\text{In} y = c_1 e^{x} + c_2 e^{-x};\ yy'' - y'^2 = y^2 \ln y.$

As illustrated above, a differential equation may have more than one solution. It may even have infinitely many solutions, which can be represented by a single formula involving arbitrary constants. Accordingly, we classify various tyoes of solutions of a differential equation as follows.

Definition :

The solution of the n^{th} order differential equation which contains n arbitrary constants is called its **general solution.**

Definition :

Anv solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a **particular solution.**

For example, $y = c_1 \sin 2x - c_2 \cos 2x$ involving two arbitrary constants c_1 and c_2 is the general solution of second order equation $\frac{d^2y}{dx^2} + 4y = 0$ whereas

$y = 2\sin 2x + \cos 2x$ is a particular solution (taking $c_1 = 2$ and $c_2 = 1$).

In some cases there may be further solutions of a given equation which cannot be obtained by assigning a definite value to the arbitrary constant in the general solution. Such a solution is called a **singular solution** of the equation. For example, the equation

$$y'^2 - xy' + y = 0 \qquad \ldots (9.12)$$

has the general solution, $y = cx - c^2$. A further solution of equation (9.12) is $y = \frac{x^2}{4}$.

Since the solution cannot be obtained by assigning a definite value to c in the general solution, it is a singular solution of equation (9.12).

After going through the definitions and illustrations given in Section 9.2 you would have definitely got the clue that a differential equation can also be derived from its solution by the process of differential, algebraic processes of elimination, etc. Now we shall discuss the method of finding the differential equation when its general solution is given.

9.3 FORMATION OF DIFFERENTIAL EQUATIONS

We begin by taking an example. The procedure involved will show clearly the relation between the number of constants in the general solution and the order of a differential equaqtion.

A differential equation whose general solution is given by

$$y = ax + bx^3 \qquad \ldots (9.13)$$

can be found as follows.

Differentiating equation (9.13) with respect to x, we get

$$y' = a + 3bx^2 \qquad \ldots (9.14)$$

Again differentiating equation (9.14) with respect to x, we obtain,

$$y'' = 6bx \qquad \ldots (9.15)$$

Solving equations (9.14) and (9.15) for a and b we get,

$$b = \frac{1}{6}\left(\frac{y''}{x}\right),\ a = y' - 3x^2 \cdot \left(\frac{1}{6}\frac{y''}{x}\right) = y' - \frac{1}{2}xy''$$

Substituting these values of a and b in equation(9.13), we get

$$y = xy' - \frac{1}{2}x^2 y'' + \frac{1}{6}x^2 y''$$

$$= xy' - \frac{1}{3}x^2 y''$$

Thus, $x^2 y'' - 3x y' - 3y = 0$ is the desired differential equation. Generalising the above procedure we obtain the following rule to find the differential equation when its general solution is given.

Rule

a) Differentiate the general solution. Let us refer to the resulting equation as the derived equation.

b) Differentiate the derived equation and obtain the second derived equation.

c) Differentiate the second derived equation and continue the process until the number of derived equations is equal to the number of independent arbitrary constants involved in the general solutions.

d) Finally eliminate all the constants of the general solution with the help of these derived equations, and get the desired equation.

Example 2:

By eliminating the constants h and k, find the differential equation of which

$$(x-h)^2 + (y-k)^2 = a^2 \qquad \ldots (9.16)$$

is a solution.

Solution :

Step 1 :

Differentiating equation (9.16) we get the first derived equation as

$$(x-h) + (y-k)\frac{dy}{dx} = 0 \qquad \ldots (9.17)$$

Step 2 :

Differentiating equation (9.17) we get the second derived equation as

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \qquad \ldots (9.18)$$

Step 3 :

Finally, we eliminate h and k from equation (9.16).
Equations (9.17) and (9.18) yield,

$$(y-k) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

and

$$(x-h) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

Substituting these values in the given relation (9.16) we obtain,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2,$$

as the required differential equation.

How about doing some exercises now?

E 3

Find the differential equations having the following as solutions.

a) $y = x^3 + c$

b) $y = c_1 e^x + c_2 e^{-x}$

c) $y = c_1 x^2 + c_2 x + c_3$

E 4

Find the differential equations whose solution is given by

$$y = e^x (A\cos x + B\sin x)$$

where A and B are arbitrary constants.

It will probably come as no surprise to you that differential equations which look similar are solved in similar ways. In fact, the equations are classified into various types which have similar mathematical characteristics. Any differential equation of the first order and first degree may be written in the form

$$\frac{dy}{dx} = f(x, y)$$

In the next section, we shall discuss various methods of solving first order and first degree differential equations.

9.4 METHODS OF SOLVING DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

In general, it may not be very easy to solve even the apparently simple equation $\frac{dy}{dx} = f(x, y)$. This is because no formulas exist for obtaining its solution in all cases. But, there are certain standard types of first order equations for which routine methods of solution are available. We now discuss four of them that have many applications. Let us discuss them one by one.

9.4.1 Separation of Variables

This is a technique for solving an important class of differential equations, namely, those of the form

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}, \qquad \ldots (9.19)$$

where $p(x)$ is a function of x only and $q(y)$ is a function of y only. Such an equation is called an equation with separable variables, or a **separable equation**. For example, the differential equation $\frac{dy}{dx} = \frac{(2x+1)}{e^y}$ is of the same type as equation (9.19) with $p(x) = 2x+1$ and $q(y) = e^y$.

In order to solve the differential equation given by (9.19) we may proceed as follows:

Step 1 :

Rewrite equation (9.19) as

$$q(y)\,dy = p(x)\,dx \qquad \ldots (9.20)$$

Step 2 :

Integrate both sides of equation (9.20) and get

$$\int q(y)\,dy = \int p(x)\,dx$$

or

$$Q(y) = P(x) + c \qquad \ldots (9.21)$$

where $Q(y) = \int q(y)\,dy$ and $P(x) = \int p(x)\,dx$

Step 3 :

Solve equation (9.21) for y in terms of x.

Note that in Step 2, there is no need to write two constants since these can be combined into one as in equation (9.21).

We now take up an example to illustrate these steps.

Example 3 :

Solve $\frac{dy}{dx} = \frac{3x^2}{y^2}$

Solution :

Step 1 :

The given differential equation is

$$\frac{dy}{dx} = \frac{3x^2}{y^2} \qquad \dots (9.22)$$

Multiplying both sides of equation (9.22) by y^2, we get

$$y^2 \frac{dy}{dx} = 3x^2$$

or,
$$y^2\, dy = 3x^2\, dx \qquad \dots (9.23)$$

Step 2 :

Integrating both sides of equation (9.23), we get

$$\int y^2\, dy = \int 3x^2\, dx$$

$$\Rightarrow \frac{y^3}{3} = x^3 + c, \text{ where } c \text{ is a constant}$$

or,
$$y^3 = 3x^3 + c_1, \qquad \dots (9.24)$$

where
$$c_1 = 3c \text{ is a constant.}$$

Step 3 :

Solve equation (9.24) for y and get

$$y = [\, 3x^3 + c_1\,]^{1/3}$$

is the required solution of the differential equation (9.22).

Let us look at another example.

Example 4 :

Solve $\frac{dy}{dt} = t^3 y^2 + y^2 \qquad \dots (9.25)$

Solution :

Step 1 :

Note that right hand side of equation (9.25) is not of the form $\frac{p(t)}{q(y)}$.

However, if we write equation (9.25) as

$$\frac{dy}{dt} = (t^3 + 1)\, y^2 = \frac{(t^3 + 1)}{y^{-2}}, \qquad \dots (9.26)$$

we see that the right hand side of equation (9.26) is of the form

$$\frac{p(t)}{q(t)} \text{ with } p(t) = t^3 + 1 \text{ and } q(y) = y^{-2}$$

Separating the variables in equation (9.26), we get

$$\frac{1}{y^2}dy = (t^3+1)\,dt \qquad \ldots (9.27)$$

Step 2 :

Integrating both sides of equation (9.27), we get

$$\int \frac{1}{y^2}dy = \int (t^3+1)\,dt + c$$

or, $$\frac{-1}{y} = \frac{1}{4}t^4 + t + c, \qquad \ldots (9.28)$$

where c is the constant of integration.

Step 3 :

Solving equation (9.28) for y, we get

$y = -\dfrac{1}{t^4/4 + t + c}$ as the required solution of equation (9.25).

You may now try this exercise yourself.

E 5

Solve the following differential equations:

a) $\dfrac{dy}{dt} = \dfrac{5+t}{y^2}$

b) $\dfrac{dy}{dt} = y^2 - e^{2t}.y^2$

c) $\dfrac{dy}{dt} = 4(y-3)$

d) $\dfrac{dy}{dt} = \sqrt{\dfrac{y}{t}}$

Not all differential equations can be solved by the method of separation of variables. For example, we cannot solve,

$$\frac{dy}{dt} = t^2y^2 + 1$$

by the method of separation of variables because the expression t^2y^2+1 cannot be written in the form $\dfrac{p(t)}{q(y)}$. In such cases we have to look for other techniques of solving differential equations. In the next subsections we shall take up another technique of solving first order equations.

9.4.2 Homogeneous Differential Equations

You are already familiar with the definition of a homogeneous function. Recall that a function $f(x, y)$ is said to be homogeneous of degree n, if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y), \qquad \ldots (9.29)$$

for all values of x and y

We also say that any function of the form $\frac{y}{x}$ is homogeneous of degree zero since

$$\left(\frac{\lambda y}{\lambda x}\right) = \lambda^0 \left(\frac{y}{x}\right).$$

However, the function

$$g(x, y) = x^4 - x^3 + y^2 \text{ is not homogeneous for}$$

$$g(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3 + (\lambda y)^2 \neq \lambda^n f(x, y) \text{ for any } n.$$

From equation (9.29) a useful relation is obtained by letting $\lambda = \frac{1}{x}$.

This gives for a homogeneous expression of the n^{th} degree

$$\frac{1}{x^n} f(x, y) = f\left[1, \frac{y}{x}\right] = \phi\left(\frac{y}{x}\right) \text{ (say)}$$

or

$$f(x, y) = x^n f\left(1, \frac{y}{x}\right) = x^n \phi\left(\frac{y}{x}\right) \qquad \ldots (9.30)$$

A homogeneous equation of the first order is of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \text{ or } f(x, y)\,dx - g(x, y)\,dy = 0,$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree.

For example, $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ is a homogeneous differential equation. Here $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$ are both homogeneous functions of degree two. You must have noticed above that $\frac{y}{x}$ plays an important role in a homogeneous function. Thus we would expect that the substitution

$$\frac{y}{x} = v \quad \text{or} \quad y = vx \qquad \ldots (9.31)$$

might be effective in solving a homogeneous equation.

In fact, it will be seen that the substitution (9.31) in a homogeneous equation of the first order and first degree reduces the equation to an equation with separable variables. Let us, now, take up an example.

Example 5 :

Solve the differential equation

$$(x - 2y)\,dx + y\,dy = 0 \qquad \ldots (9.32)$$

Solution :

Step 1 :

Clearly the equation $\frac{dy}{dx} = \frac{2y - x}{y}$ is homogeneous. Putting $y = vx$ in equation (9.32), we get,

$$(x - 2vx)\,dx + vx\,(v\,dx + x\,dv) = 0 \quad (\text{since } dy = v\,dx + x\,dv)$$

$$\Rightarrow x(1 - 2v + v^2)\,dx + x^2 v\,dv = 0$$

$$\Rightarrow \frac{dx}{x} + \frac{v}{(v-1)^2}\,dv = 0 \qquad \ldots (9.33)$$

Step 2 :

In equation(9.33) the variables are now separated and therefore

$\int \frac{1}{x}dx + \int \frac{v}{(v-1)^2}\,dv = c_1$, c_1 being a constant.

$\Rightarrow \ln|x| + \int \frac{v}{(v-1)^2}\,dv = c_1,$

For finding the integral $\int \frac{v}{(v-1)^2}dv$, we substitute $v - 1 = t$, so that $dv = dt$ and equation (9.33) becomes

$$\ln|x| + \int \frac{1+t}{t^2}\,dt = c_1$$

$$\Rightarrow \ln|x| + \int \frac{1}{t^2}dt + \int \frac{1}{t}\,dt = c_1$$

$$\Rightarrow \ln|x| - \frac{1}{t} + \ln|t| = c_1$$

$$\Rightarrow \ln|x| - \frac{1}{v-1} + \ln|(v-1)| = c_1 \qquad \ldots (9.34)$$

(since $(v-1) = t$).

It is convenient to replace the constant c_1 by $\ln c$, where $c > 0$. Then equation (9.34) becomes,

$$\ln\left|\frac{x(v-1)}{c}\right| = \frac{1}{v-1} \qquad \ldots (9.35)$$

Step 3 :

Replacing v by $\frac{y}{x}$ in equation (9.35), we get,

$$\ln\left|\frac{y-x}{c}\right| = \frac{x}{y-x}$$

which is the required solution of equation (9.32).

Let us consider another example.

Example 6 :

Solve $$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \qquad \ldots (9.36)$$

Solution :

Step 1 :

The given equation (9.36) is homogeneous, since both numerator $f(x, y) = 2xy$ and the denominator $g(x, y) = x^2 - y^2$ are homogeneous functions of degree 2.

We may rewrite equation (9.36) as

$$\frac{dy}{dx} = \frac{(2y/x)}{1 - (y/x)^2} \qquad \ldots (9.37)$$

$$\text{or,} \quad v + x\frac{dv}{dx} = \frac{2v}{1-v^2}, \quad \text{where } v = y/x,$$

or,

$$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = v\frac{(1+v^2)}{1-v^2}$$

Therefore,

$$\int \frac{1-v^2}{v(1+v^2)}dv = \int \frac{dx}{x} + c, \qquad \ldots (9.3$$

where c is a constant of integration. To integrate equation (9.38), we write it as

$$\int \left[\frac{1}{v} - \frac{2v}{1+v^2}\right]dv = \int \frac{dx}{x} + c$$

$$\Rightarrow \ln|v| - \ln|(1+v^2)| = \ln|x| + \ln|A|,$$

where A is another constant.

Thus, we get $$\ln\left|\frac{v}{1+v^2}\right| = \ln|Ax|$$

$$\Rightarrow \frac{v}{1+v^2} = Ax$$

Step 3 :

Replacing v by $\frac{y}{x}$ in the last equation, we obtain

$$\frac{xy}{x^2+y^2} = Ax \text{ or } x^2 + y^2 = \frac{1}{A}y$$

as the required solution of equation (9.36).

You may now try the following exercises.

E 6

Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

b) $x\frac{dy}{dx} = x\tan\frac{y}{x} + y$

c) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

d) $\left[x + y\sin\frac{y}{x}\right]dx - x\sin\frac{y}{x}dy = 0$

E 7

Solve the following equations:

a) $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

b) $x^2 y\,dy + (x^3 + x^2 y - 2xy^2 - y^3)\,dx = 0$

c) $(2\sqrt{xy} - x)\,dy + y\,dx = 0$

d) $(6x^2 + 2y^2)\,dx - (x^2 + 4xy)\,dy = 0$

Equations Reducible to Homogeneous Form

Sometimes it may happen that a given equation is not homogeneous but can be reduced to homogeneous form by considering a certain change of variables. For example, consider the equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + D} \qquad \ldots (9.39)$$

This can be reduced to homogeneous form by changing the variables x, y to X, Y respectively, where

$$x = X + h,$$

$$y = Y + k,$$

with h and k as constants to be so chosen as to make the given equation homogeneous. With these new variables we have

$$\frac{dy}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX} \quad \left(\text{because } \frac{dX}{dx} = 1\right)$$

Therefore, equation (9.39) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + D)} \qquad \ldots (9.40)$$

which will be homogeneous provided h and k are so chosen so that

$$ah + bk + c = 0$$

$$Ah + Bk + D = 0$$

Solving the last two equations for h and k, we get,

$$h = \frac{bD - Bc}{aB - Ad} \quad \text{and } k = \frac{Ac - aD}{aB - Ab}$$

which always have meanings except when

$$aB - Ab = 0 \text{ that is when } \frac{a}{A} = \frac{b}{B}.$$

So, with these values of h and k, the given equation can be reduced to homogeneous equation and the resulting equation

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

can now easily be solved by means of the substitution $Y = vX$. But what happens to the equation if $\frac{a}{A} = \frac{b}{B}$?

In such cases we let $\frac{a}{A} = \frac{b}{B} = r$, say.

Then $a = Ar$ and $b = Br$ and the given equation becomes

$$\frac{dy}{dx} = \frac{r(Ax + By) + C}{Ax + By + D} \qquad \ldots (9.41)$$

We, then, put $Ax + By = Z$ and its differential with respect to x, that is,

$A + B\frac{dy}{dx} = \frac{dZ}{dx}$ in equation (9.41) to get

$$\frac{1}{B}\left[\frac{dZ}{dx} - A\right] = \frac{rZ + c}{Z + D} \text{ or } \frac{dZ}{dx} = B\frac{rZ + C}{Z + D} + A,$$

so that the variables are separated and hence the equation can be solved.
Let us illustrate the above discussion with the help of the following examples.

Example 7:

Solve the differential equation

$$(2x + 3y - 6)\,dy = (6x - 2y - 7)\,dx \qquad \ldots (9.42)$$

Solution:

Step 1:

Putting $x = X + h,\ y = Y + k$, in the given equation,

$\frac{dy}{dx} = \frac{6x - 2y - 7}{2x + 3y - 6}$, we get

$$\frac{dY}{dX} = \frac{6(X + h) - 2(Y + k) - 7}{2(X + h) + 3(Y + k) - 6} = \frac{6X - 2Y}{2X + 3Y} \qquad \ldots (9.43)$$

provided that

$$6h - 2k - 7 = 0 \qquad \ldots (9.44)$$

and

$$2h + 3k - 6 = 0 \qquad \ldots (9.45)$$

Step 2:

Solving equations (9.44) and (9.45) for h and k we get,

$$h = \frac{3}{2} \text{ and } k = 1.$$

Step 3:

Equation (9.43) is in homogeneous form, thus putting $Y = vX$ in equation (9.43), we get

$$v + X\frac{dv}{dX} = \frac{6X - 2vX}{2X + 3vX} \text{ or, } -\frac{dX}{X} = \frac{6v + 4}{2(3v^2 + 4v - 6)}dv$$

Since the variables have been separated on integration, we get

$$\ln|c| - 2\ln|X| = \ln\left|(3v^2+4v-6)\right|,$$

where c is a constant of integration.

Note that the integral on the right hand side is of the form

$$\int \frac{f'(x)}{f(x)}\,dx \qquad \text{(see Unit 6)}$$

which has led us to $2\ln|X| + \ln\left|(3v^2+4v-6)\right| = \ln|c|$

$$\Rightarrow X^2(3v^2+4v-6) = c \qquad \ldots(9.46)$$

Substituting back the value of $v = \left(\frac{Y}{X}\right)$ in equation (9.46), we get

$$X^2\left[\frac{3Y^2}{X^2}+\frac{4Y}{X}-6\right] = c$$

or $$3Y^2+4XY-6X^2 = c \qquad \ldots(9.47)$$

Step 4 :

But $Y = y-k = y-1$ and $X = \frac{2x-3}{2}$.

Thus, replacing these values of X and Y in equation (9.47), we get

$$3(y-1)^2+4(y-1)\left(\frac{2x-3}{2}\right)-6\left(\frac{2x-3}{2}\right)^2 = c$$

or $$3y^2+4xy-6x^2-12y+18x = c+\frac{9}{2}$$

is the required solution.

Example 8 :

Solve $\quad (2x+y+1)\,dx+(4x+2y-1)\,dy = 0.$

Solution: Step 1:

Rewrite the given equation as

$$\frac{dy}{dx} = -\frac{2x+y+1}{2(2x+y)-1} \qquad \ldots(9.48)$$

Step 2:

Putting $2x+y = t$ and $2+\frac{dy}{dx} = \frac{dt}{dx}$, we get

$$\frac{dy}{dx} = \frac{dt}{dx}-2 = -\frac{t+1}{2t-1}$$

or, $$\frac{dt}{dx} = \frac{3(t-1)}{2t-1} \qquad \ldots(9.49)$$

Step 3:

Since equation (9.49) is in variable separable form, we integrate both sides and get,

$$\int dx = \int \frac{2t-1}{3(t-1)}\,dt = \frac{2}{3}\int\frac{t}{t-1}\,dt-\frac{1}{3}\int\frac{1}{t-1}\,dt$$

$$= \frac{2}{3}\int\left(1+\frac{1}{t-1}\right)dt-\frac{1}{3}\int\frac{1}{t-1}\,dt$$

$$\Rightarrow x = \frac{2}{3}\int dt + \frac{1}{3}\int \frac{dt}{t-1} + c_1$$

$$\Rightarrow 3x = 2t + \ln|(t-1)| + c_1, \qquad \ldots(9.50)$$

where $c = 3c_1$ is a constant.

Step 4:
Replacing back the value of t in terms of x in equation (9.50), we get

$$3x = 2(2x+y) + \ln|(2x+y-1)| + c$$

or $x + 2y + \ln|(2x+y-1)| + c = 0$

as the required solution.

You may now try this exercise.

E8

Solve the following differential equations:

a) $(x-2y+4)\,dx + (2x-y+2)\,dy = 0$

b) $(2x+3y-1)\,dx - 4(x+1)\,dy = 0$

c) $(2x+3y)\,dx + (y+2)\,dy = 0$

d) $(2x-3y+2)\,dx + 3(4x-6y-1)\,dy = 0$

In Unit 4 you have studied the total differential of a given function. Now, in the next sub-section, we shall make use of this to define an exact differential equation. We shall also discuss the method of solving it.

9.4.3 Exact Differential Equations

Suppose that we are given a function $g(x, y) = c$. Then its total differential is given by

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0.$$

For instance, the equation $xy = c$ has the total differential

$$ydx + xdy = 0,$$

Now, we consider the reverse situation, that is, given the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0, \qquad \ldots(9.51)$$

can we find a function $g(x, y) = c$, such that

$$dg = Mdx + Ndy,$$

where $$\frac{\partial g}{\partial x} = M \text{ and } \frac{\partial g}{\partial y} = N$$

If so, we say that equation (9.51) is an **exact differential equation.**

It can be shown that the necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

We will not be proving this result here. However, we shall be making use of this to verify whether a given differential equation of type (9.51) is exact or not.

Various steps involved in solving an exact differential equation $Mdx + Ndy = 0$, are as follows.

Step 1:

Integrate M with respect to x, regarding y as a constant.

Step 2:

Integrate with respect to y those terms in N which do not involve x.

Step 3:

The sum of the two expressions obtained in Steps 1 and 2 equated to a constant is the required solution.

Let us illustrate this method with the help of an example.

Example 9:

Solve the equation

$$(1 - \sin x \tan y)\, dx + (\cos x \sec^2 y)\, dy = 0 \qquad \ldots (9.52)$$

Solution:

The given equation is of the form

$$Mdx + Ndy = 0 \text{ with } M = 1 - \sin x \tan y, N = \cos x \sec^2 y.$$

Now, $\dfrac{\partial M}{\partial y} = -\sin x \sec^2 y$,

and $\dfrac{\partial N}{\partial x} = -\sin x \sec^2 y$,

so that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$,

which shows that equation (9.52) is an exact equation.

To solve equation (9.52), we proceed as follows:

Step 1:

Integrating M w.r.t. x regarding y as a constant, we have

$$\int (1 - \sin x \tan y)\, dx = x + \tan y . \cos x.$$

Step 2:

We integrate those terms in N w.r.t. y which do not involve x. But there is no such term because $N = \cos x \sec^2 y$.

Step 3:

The required solution is the sum of expressions obtained from steps 1 and 2. That is,

$$x + \tan y \cos x = c.$$

In order to check your calculations you can find the total differential of the equation obtained as a result of Step 3, and get back the original differential equation.

In practice differential equations are rarely exact but can often easily be transformed into exact equations on multiplications by some suitable function known as **integrating factor**. In other words, the integrating factor is a function which when multiplied with a non-exact differential equation makes it exact. In general such a function exists but is difficult to obtain, except for certain cases.

Consider for example, the equation

$$xdy - ydx = 0$$

which is not exact, since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Here $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-y) = -1 \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

But looking at the form of the equation, we can guess that the equation becomes exact on multiplication by $\frac{1}{y^2}$ throughout. For, then, we get

$$\frac{x}{y^2}dy - \frac{1}{y}dx = 0,$$

which is exact, as can be readily verified by using the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Let us consider another illustration.

Example 10:

Show that the differential equation $\frac{1}{y}dy + \left[\frac{1}{x} - \frac{x}{y}\right]dx = 0$ becomes exact on multiplication by xy, and solve it.

Solution:

The differential equation $\frac{1}{y}dy + \left[\frac{1}{x} - \frac{x}{y}\right]dx = 0$ . . . (9.53)

is not exact for $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Multiplying equation (9.53) by xy, we get

$$xdy + (y - x^2)\,dx = 0 \qquad \text{. . . (9.54)}$$

This is, now exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y - x^2) = 1 = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x).$$

It can, then be verified easily that the solution of equation (9.54) is given by

$$\left[xy - \frac{x^3}{3}\right] = \text{constant.}$$

Remark :

In some cases the integrating factor can be found by inspection as illustrated above by two simple examples. Obviously guessing comes from practice only. However, rules for finding the integrating factor do exist for other cases. But we do not intend to consider these rules for determining the integrating factor in this course. Here we will restrict our discussion only to some simple type of equations for which we can find the integrating factor by inspection.

You may now try the following exercises:

E 9

Show that $(y - 2x^3)\,dx - x(1 - xy)\,dy = 0$ becomes exact on multiplication by $\frac{1}{x^2}$ and solve it.

E 10

Solve the following differential equations:

a) $(1 + yx)\,x\,dy + (1 - yx)\,y\,dx = 0$

b) $2xy\,dx + (x^2 + 1)\,dy = 0$

c) $\left[\frac{\ln(\ln y)}{x} + \frac{2}{3}xy^3\right]dx + \left[\frac{\ln x}{y \ln y} + x^2 y^2\right]dy = 0.$

In the next sub-section we introduce you to find method of solving the most important type of differential equations. These equations are important because of their wide range of applications. In these equations the derivatives of the highest order is a linear function of the lower order derivatives. Let us now study them.

9.4.4 Linear Differential Equations

We begin with the definition of linear differential equation.

Definition :

Linear differential equation is a name given to those differential equations which contain the dependent variables and its derivatives in its first degree.

For example, the equation

$$\frac{dy}{dx} + \frac{2y}{x} = x^2$$

is a linear differential equation, however $y\frac{dy}{dx} + x^2 = 5$ is not linear because of the presence of the term $y\frac{dy}{dx}$, the product of depend variables and its derivative.

The general form of the first order linear differential equation is

$$\frac{dy}{dx} + p(x)\,y = q(x)\,, \qquad \ldots (9.55)$$

where $p(x)$ and $q(x)$ are functions of the independent variable x alone, or are constants. From our discussion in Sub-section 9.4.3 you will at once see that the equation (9.55) can be solved by the use of an integrating factor. If, on multiplication with a suitable integrating factor, the given equation can be expressed as an exact derivative, then the resulting equation can be integrated directly. In general, to solve such differential equation we proceed as follows.

Multiply both sides of this equation by $e^{\int p dx}$ (an integrating factor), we get

$$e^{\int p dx}\left[\frac{dy}{dx}+py\right] = q\,e^{\int p dx} \qquad \ldots (9.56)$$

The left hand side, now, is the differential coefficient of $y\,e^{\int p dx}$, that is, equation (9.56) is nothing but

$$\frac{d}{dx}\left[y\,e^{\int p dx}\right] = qe^{\int p dx} \qquad \ldots (9.57)$$

Integrating both sides of equation (9.57) with respect to x, we get

$$y\,e^{\int p dx} = \int q\,e^{\int p dx}\,dx + c \qquad \ldots (9.58)$$

where c is a constant of integration.

They may, again, be written after multiplying both sides by $e^{-\int p dx}$ as

$$y = e^{-\int p dx}\left[\int q\,e^{p dx}\,dx + c\right],$$

which is the required solution of equation (9.55).

The factor $e^{\int p dx}$, on multiplying by which equation (9.55) becomes an exact differential, serves the purpose of integrating factor of the differential equation (9.55).

Let us now solve a few example.

Example 11:

Solve the differential equation

$$\cos x\,\frac{dy}{dx} + y\sin x = 1 \qquad \ldots (9.59)$$

Solution:

Step 1:

Dividing both sides of (9.59) by $\cos x$ we get,

$$\frac{dy}{dx} + y\,\frac{\sin x}{\cos \mathrm{x}} = \frac{1}{\cos x}$$

or $$\frac{dy}{dx} + y\tan x = \sec x \qquad \ldots (9.60)$$

which is of the form

$$\frac{dy}{dx} + p(x)\,y = q(x),$$

with $$p(x) = \tan x \text{ and } q(x) = \sec x.$$

Step 2:

Now, $e^{\int p dx} = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x$.

Step 3:

Multiply both sides of equation (9.60) by $\sec x$, we obtain

$$\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x$$

or $$\frac{d}{dx}(y \sec x) = \sec^2 x. \qquad \ldots (9.61)$$

Step 4:

Integrating both sides of equation (9.61), we get

$$y \sec x = \int \sec^2 x \, dx + c$$

where c is an arbitrary constant.

Thus, $y \sec x = \tan x + c$

or, $$y = \frac{\tan x}{\sec x} + \frac{1}{\sec x} c = \sin x + c \cos x$$

is the required solution of equation (9.59).

Let us consider another example.

Example 12:

Solve the equation $(x + 2y^3)\frac{dy}{dx} = y \qquad \ldots (9.62)$

Solution:

Step 1:

The equation involves y^3 and hence it is not linear, but if we view y as the independent variable, and rewrite equation (9.62) as

$$y \frac{dx}{dy} = x + 2y^3$$

or

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2, \qquad \ldots (9.63)$$

then equation (9.63) is linear with x as the dependent variable and is of the form

$$\frac{dx}{dy} + p(y)x = q(y) \text{ with } p(y) = -\frac{1}{y} \text{ and } q(y) = 2y^2.$$

Step 2 :

The procedure for solving equation (9.63) is exactly same as before. Integrating factor of equation (9.63) is $e^{\int p(y)dy}$

$$= e^{-\int \frac{1}{y} dy} = e^{-\ln y} = y^{-1} = \frac{1}{y}$$

Step 3 :

Multiplying both sides of equation (9.63) by $\frac{1}{y}$, we get

$$\frac{1}{y}\left[\frac{dx}{dy} - \frac{x}{y}\right] = 2y$$

or $$\frac{d}{dy}\left[x \cdot \frac{1}{y}\right] = 2y \qquad \ldots (9.64)$$

Step 4 :

Now integrating both sides of equation (9.64) we get

$$x\frac{1}{y} = y^2 + c, \text{ where } c \text{ is a constant.}$$

Thus, $x = y(c + y^2)$ is the required solution of equation (9.62).

You may now try this exercise.

E 11

Solve the following differential equations:

a) $\frac{dy}{dx} + \frac{2}{x}y = 3$

b) $\frac{dy}{dx} + \frac{2x}{x^2 - 1}y = e^x$

c) $\frac{dy}{dx} + y \tan x = x^2 e^x \cos x$

d) $x \ln x \frac{dy}{dx} + y = 2 \ln x$

e) $(2x - 10y^3)\frac{dy}{dx} + y = 0$

Equations Reducible to Linear Form

Sometimes, equations which are not linear can be reduced to the linear form by suitable transformations of the variables. One such equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \qquad \ldots (9.65)$$

named **Bernoulli's equation** after James Bernoulli, who studied it in 1695. Clearly, equation (9.65) is not linear as it contains a power of y (dependent variables) which is not unity. However, this equation can be reduced to the linear form as follows:

Dividing equation (9.65) by y^n we get,

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \qquad \ldots (9.66)$$

Put $y^{-n+1} = z$,

so that $(-n + 1)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

or $\qquad y^{-n}\frac{dy}{dx} = \frac{1}{-n+1}\frac{dz}{dx}$

Substitute in equation (9.66) the values of $y^{-n}\frac{dy}{dx}$ and y^{-n+1} in terms of z to obtain

$$\frac{dz}{dx} + P(1-n)z = Q(1-n),$$

which is linear with z as the new dependent variable.

We now illustrate this method with the help of an example.

Example 11:

Solve the equation

$$dy + 2xy\,dx = xe^{-x^2} y^3\,dx \qquad \ldots (9.67)$$

Solution:

Step 1:

Equation (9.67) can be written as

$$\frac{dy}{dx} + 2xy = xe^{-x^2} y^3, \qquad \ldots (9.68)$$

which is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

with $\quad P(x) = 2x, Q(x) = xe^{-x^2}$ and $n = 3$.

Step 2:

Divide equation (9.68) throughout by y^3 so that

$$y^{-3}\frac{dy}{dx} + 2xy^{-2} = xe^{-x^2} \qquad \ldots (9.69)$$

Let $\quad z = y^{-2}$

Then, $\quad \dfrac{dz}{dx} = (-2)y^{-2-1}\dfrac{dy}{dx} \qquad \ldots (9.70)$

$$= -2y^{-3}\frac{dy}{dx}$$

Therefore, $\quad y^{-3}\dfrac{dy}{dx} = -\dfrac{1}{2}\dfrac{dz}{dx} \qquad \ldots (9.71)$

Substituting in equation (9.69) values of y^{-2} and $y^{-3}\dfrac{dy}{dx}$ from equations (9.70) and (9.71) respectively, we have

$$-\frac{1}{2}\frac{dz}{dx} + 2xz = xe^{-x^2}$$

or, $\quad \dfrac{dz}{dx} - 4xz = -2xe^{-x^2} \qquad \ldots (9.72)$

which is linear with z as the dependent variable.

Step 3:

To solve equation (9.72) we multiply both sides of equation (9.72) by $e^{-\int 4x\,dx} = e^{-2x^2}$ (integrating Factor) to get

$$e^{-2x^2}\frac{dz}{dx} - 4xze^{-2x^2} = -2x\,e^{-x^2}\,e^{-2x^2}$$

$$= -2xe^{-3x^2}$$

or $\quad \dfrac{d}{dx}(ze^{-2x^2}) = -2xe^{-3x^2}$

Integrating both sides, we get

$$ze^{-2x^2} = -\int 2xe^{-3x^2}\,dx + c \qquad (9.73)$$

For finding $-\int 2xe^{-3x^2}\,dx$, let $t = -3x^2$

so that $\quad -2x\,dx = \dfrac{dt}{3}$

Therefore, in terms of t, $-2\int xe^{-3x^2}\,dx$ becomes

$$\int e^{t}\frac{dt}{3} = \frac{1}{3}e^{t} = \frac{1}{3}e^{-3x^2}$$

Thus, from equation (9.73) we get,

$$z\,e^{-2x^2} = \frac{1}{3}e^{-3x^2} + c$$

or

$$z = \frac{1}{3}e^{-x^2}\,e^{-3x^2} + ce^{2x^2}$$

$$= \frac{1}{3}e^{2x^2} + ce^{2x^2}$$

Step 4:

But $z = y^{-2}$

Therefore, $\quad y^{-2} = \dfrac{1}{3}e^{-x^2} + ce^{2x^2}$

or, $\quad 3y^{-2} = e^{-x^2} + c_1 e^{2x^2}$, where $c_1 = 3c$

is the required solution of equation (9.67).

And now an exercise for you.

E 12

Solve the following differential equations:

a) $dy + y\,dx = 2xy^2 e^x\,dx$

b) $dx + \dfrac{2}{y}x\,dy = 2x^2y^2\,dy$

c) $2\dfrac{dy}{dx} - \dfrac{y}{x} = 5x^3y^3$

d) $x^3\dfrac{dy}{dx} - x^2y + y^4\cos x = 0$

e) $y(x^2y + e^x)\,dx - e^x\,dy = 0$

We now conclude this unit by giving a summary of what we have covered in it.

9.5 SUMMARY

In this unit, we have covered the following points

1 (a) An equation involving derivatives of dependent variable w.r. to one or more independent variables is called a **differential equation.**

The order of a differential equation is the order of highest derivative appearing in it.

(c) The degree of a differential equation is the highest exponent of the highest order derivative in it after removing radicals/fractions.

2. A differential equation of the form $\frac{dy}{dx} = \frac{p(x)}{q(y)}$ is called a **separable equation** if its solution is obtained by direct integration.

3. The equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree is called **homogeneous equation** and put $y = vx$ to solve it.

4. The differential equation $M\,dx + N\,dy = 0$ is said to be **exact** if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and its solution is $\int M\,dx + \int N\,dy = \text{constant}$.
(treating y as constant) (terms independent of x)

5. (a) Equations of the form $\frac{dy}{dx} + p(x)y = q(x)$ is called a **linear differential equation** and $e^{\int p dx}$ is its integrating factor.

(b) Equations of the type $\frac{dy}{dx} + p(x)y = q(x)y^n$ are reduced to linear form by the substitution $y^{1-n} = z$

9.6 SOLUTIONS/ANSWERS

E 1

a) 1, 2

b) 1, 6 *Hint*: First square both the sides, the highest order derivative is $\frac{dy}{dx}$. and its power is 6.

c) 2, 1

d) 3, 2

e) 2, 2

E 3

a) $y' = 3x^2, y' = \frac{dy}{dx}$

b) $y'' - y = 0, y'' = \frac{d^2y}{dx^2}$

c) $y''' = 0, y''' = \frac{d^3y}{dx^3}$

E 4

Here $y' = e^x[(A+B)\cos x - (A-B)\sin x]$

$$= e^x[B\cos x - A\sin x] + y$$

$$y'' = 2e^x[B\cos x - A\sin x]$$

$$= 2[y' - y]$$

Hence, $y'' - 2y' + 2y = 0$

E 5

a) $y = \sqrt[3]{15t + 3t^2/2 + c}$

b) $y = \dfrac{2}{e^{2t} - 2t + c}$ or $y = 0$

c) $y = 3 + ce^{4t}$

d) $y = (\sqrt{t} + c)^2$ or $y = 0$

Hint : $y^{-1/2}dy = t^{-1/2}dt$, thus, $y^{1/2} = t^{1/2} + c$.

E 6

a) $y = x\sqrt{2\ln|x| + c}$

b) $\sin\left(\dfrac{y}{x}\right) = xc$

c) $y + \sqrt{x^2 + y^2} = cx^2$

d) $\ln|x| + \cos\left(\dfrac{y}{x}\right) = c.$

E 7

a) $-(x - y)^2 = c\,x\,e^{-y/x}$

b) $-\ln\dfrac{c(y - x)}{x^4(y + x)} = \dfrac{2x}{y + x}$

c) $\ln y + \sqrt{\dfrac{x}{y}} = c.$ *Hint* : Put $y = vx$

therefore, $x(x\sqrt{v} - 1)(vdx + xdv) + vx\,dx = 0$

or, $2v^{3/2}dx + x(2\sqrt{v} - 1)dv = 0$

or, $\dfrac{2}{x}dx + \dfrac{2\sqrt{v} - 1}{v^{3/2}}dv = 0$

$$\frac{2}{x}dx + \left[\frac{2}{v} - v^{-3/2}\right]dv = 0$$

$$2\ln x + 2\ln v + 2v^{-1/2} = c_1$$

or, $\ln xv + v^{-1/2} = c$

Putting $v = \dfrac{y}{x}$

$$\ln y + \sqrt{\frac{x}{y}} = c$$

d) $(2x + y)(2y - 3x) = cx$

E 8

a) $(x+y-2)^3 = c(x-y+2)$

b) $(y-2x-3)^4 = c(x+1)^3$

c) $(y+2x-4)^2 = c(x+y-1)$

d) $x+6y+\ln(2x-3y) = c$

E 9

$xy^2 = cx+2y+2x^3$

E 10

a) $\ln y - \frac{1}{xy} = c$

Hint: Use the substitution $xy = t$ then variables are separated.

b) $y = \frac{c}{(1+x^2)}$

c) $(\ln x)\ln(\ln y) + \frac{1}{3}x^2y^3 = c$

since $$\left[\frac{\ln(\ln y)}{x} + \frac{2}{3}xy^3\right]dx + \left[\frac{\ln x}{y\ln y} + x^2y^2\right]dy = 0$$

$$\Rightarrow \left[\frac{\ln(\ln y)}{x}dx + \frac{\ln x}{y\ln y}dy\right] + \left[\frac{2}{3}xy^3dx + x^2y^2dy\right] = 0$$

$$\Rightarrow d[\ln x\ln(\ln y)] + \frac{1}{3}d(x^2y^3) = 0$$

$$\text{or } (\ln x)\ln(\ln y) + \frac{1}{3}x^2y^3 = c$$

E 11

a) $y = x + \frac{c}{x^2}$

b) $y = \frac{x-1}{x+1}e^x + \frac{c}{x^2-1}$

c) $y = e^x(x^2-2x+2)\cos x + c\cos x$

d) $y\ln x = c + (\ln x)^2$

e) $xy^2 = 2y^5 + c.$

Hint : Given equation can be written as $\frac{dx}{dy} + \frac{2}{y}x = 10y^2$, which is linear with x as dependent variable.

E 12

a) $1 = ye^x(c-x^2)$

Hint. Use the substitution $\frac{1}{y} = z$

b) $x^{-1}y = c + 2y^3$

Hint : Put $\frac{1}{x} = z$

c) $xy^{-2} + x^5 = c$

d) $x^3 = (c+3\sin x)y^3$

e) $x^3y + 3e^x = cy.$.