UNIT 4 IDEAL FLUID FLOW

Structure

4.1 INTRODUCTION

In the previous unit, Fundamentals of fluid motion, you have studied different types of fluid flow, the concept of streamline acceleration and its components and derived the equation of continuity. In this unit we shall be dealing with the Ideal fluid flow and its related aspects

Ideal fluid flow is an important branch of fluid mechanics. Ideal fluid may be defind as that in which friction is absent, that is, its viscosity is zero. Thus, in ideal fluid tangential stresses or shear stresses are absent when fluid is in motion. Ideal fluid is an imaginary fluid conceived by mathematicians to simplify the problems of fluid motion. Ideal fluid flow has uniform flow velocity at a cross-section. It exhibits 'slip' condition at the fixed boundary whereas in real fluid flow, there is 'no slip' condition at the fixed boundary. That is, in real fluid flow if the boundary is at rest, the fluid particles are also at rest whereas in ideal fluid flow, fluid particles at the fixed solid boundary move. Hence shear stresses are absent in ideal fluid flow. Ideal fluid flow is also known as **potential flow.** Theory developed for ideal fluid flow can be applied to real fluid flow with certain degree of approximation.

Objectives

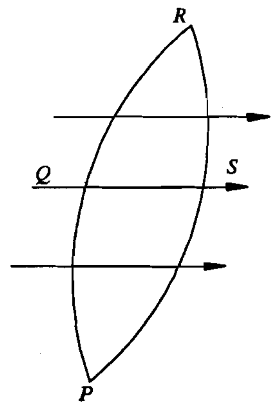
After studying this unit, you should be able to

* define stream function and velocity potential,
* Identify whether a flow is rotational or irrotational,
* define and compute circulation and vorticity,
* derive Laplace equation from fundamentals,
* describe vortex flow and also forced vortex & free vortex, and
* define flownet and enumerate its uses and applications.

4.2 STREAM FUNCTION (2-DIMENSIONAL)

The velocity components u and v are related to a scalar field function called stream function $\psi(x, y)$. This **stream function** is generally used for two-dimensional motions. It has the dimensions of volume per unit length per unit time.

Let $P(x_0, y_0)$ be a fixed point in the $x-y$ plane which has unit thickness in z-direction and $R(x, y)$ an arbitrary point in the same plane. Let there be two arbitrary paths *PQR* and *PSR* between points *P* and *R*.



Because of law of conservation of mass, no fluid is created or destroyed within the region bounded by these two paths.

Hence the flow rate entering across path *PQR* will be same as that of flow rate across *PSR*. The flux across *PQR* will be same as that across *PSR*. Since point *P* is fixed, flux is a function of position of *R* and time *t*. This flux is denoted by ψ known as **stream function** and written as $\psi = \psi(x, y, t)$

Above fact is represented here to relate velocity components with stream function.

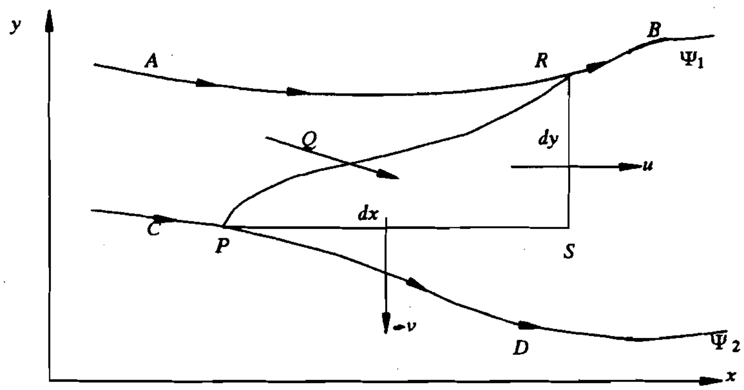


Figure 4.1 Relationship between velocity components and stream function

Let ψ_1 and ψ_2 be the neighbouring streamlines represented by *ARB* and *CPD* (figure 4.1). Points *R* and *P* are points on streamlines having stream functions of ψ_1 and ψ_2 and flux across *PQR* is $\psi_2 - \psi_1 = d\psi$. Along streamline *ARB* or *CPD*, or any streamline the stream function remains constant and hence $d\psi = 0$, because there cannot be any flow across it.

By law of conservation of mass,

flux across *PQR* = flux across *RS* + flux across *PS*

$$d\psi = u\,dy - v\,dx \tag{4.1}$$

By rules of partial differentiation, we get

$$d\psi = \frac{\partial\psi}{\partial x}\cdot dx + \frac{\partial\psi}{\partial y}\cdot dy \quad \text{and} \quad \psi = \int \frac{\partial\psi}{\partial x}\cdot dx + \int \frac{\partial\psi}{\partial y}\cdot dy + C \tag{4.2}$$

Comparing above equations (4.1) and (4.2), velocity components are defined as under:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \qquad (4.3)$$

This is valid for both rotational and irrotational flow. The rate of flow δQ between any two streamlines is defined as difference between the two stream functions

$$\delta Q = \psi_2 - \psi_1$$

The continuity equation for two-dimensional flow is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting values of u and v from equation (4.3), we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \qquad (4.4)$$

This proves that existence of ψ means a case of possible fluid flow. The flow may be rotational or irrotational. It shows that ψ is a continuous function of x and y and hence the order of differentiation can be interchanged.

Example 1 :

If for a two-dimensional flow, the stream function is given by $\psi = y\,(3x^2 - y^2)$, obtain the velocity components. What is the velocity at point (2, 1) ?

Solution :

By definitions of velocity components in ψ, $\psi = 3x^2y - y^3$, $u = \frac{\partial \psi}{\partial y} = 3x^2 - 3y^2$

and $$v = -\frac{\partial \psi}{\partial x} = -6xy$$

$$\vec{V} = \vec{i}\,(3x^2 - 3y^2) - \vec{j}\,(6xy)$$

$$u \text{ at point } (2, 1) = (3 \times 4) - (3 \times 1) = 12 - 3 = 9$$

$$v \text{ at point } (2, 1) = -6 \times 2 \times 1 = -12$$

Hence $$|V| = \sqrt{u^2 + v^2} = \sqrt{(9)^2 + (-12)^2} = \sqrt{81 + 144} = 15$$

We now give you some exercises based on the concepts discussed so far.

SAQ 1

The stream function exists for

a) Two dimensional flow only.
b) Three dimensional flow.
c) Irrotational flow only.
d) Rotational flow only.
e) All types of flow.
f) Two-dimensional, rotational and irrotational flow.

Select the correct answer from the above.

SAQ 2

The x and y components of velocity in two-dimensional incompressible fluid flow are given by $u = 2xy$ and $v = a^2 + x^2 - y^2$.

Determine the expression for stream function.

4.3 VELOCITY POTENTIAL

Similar to stream function, there is another scalar field function $\phi(x,y)$ called **velocity potential.** It is used to define velocity components in irrotational flow. The definition of velocity potential is applicable to ideal fluid flow only.

The velocity vector is expressed as

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w \quad \text{...(For 3-dimensional flow)}$$

and

$$\vec{V} = \vec{i}u + \vec{j}v \quad \text{...(For 2-dimensional flow)}$$

It is also written as

$$\vec{V} = \text{grad}\,\phi = \vec{\nabla}\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

Therefore

$$u = \frac{\partial \phi}{\partial x},\ v = \frac{\partial \phi}{\partial y},\ \text{and } w = \frac{\partial \phi}{\partial z} \qquad (4.5)$$

Hence velocity component is defined as a gradient of velocity potential in that direction. It is to be noted that for a velocity potential to exist, the flow has to be irrotational. This is proved in the following discussion vide equation (4.17).

If we substitute this definitions of velocity components in equation of continuity, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = 0$$

Therefore,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

which is Laplace's equation in ϕ. This is to say that if the flow is irrotational, it must satisfy Laplace Equation.

Example 2 :

Find the velocity components and resultant velocity for the velocity potential given by $\phi = x^2 - y^2$. Determine stream function also.

Solution :

$$\phi = x^2 - y^2$$

and

$$u = \frac{\partial \phi}{\partial x} = 2x, \quad v = \frac{\partial \phi}{\partial y} = -2y$$

$$|V| = \sqrt{u^2 + v^2} = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$

By definition

$$u = \frac{\partial \psi}{\partial y} = 2x \ \text{ and } \ v = -\frac{\partial \psi}{\partial x} = -2y$$

Therefore

$$\psi = 2xy + \text{constant},\ \text{ and } \ \psi = 2yx + \text{constant}$$

Hence

$$\psi = 2xy \qquad \text{(constant is zero as at } x = 0 \text{ and } y = 0,\ u = 0,\ v = 0\text{).}$$

Example 3 :

Find the velocity potential, resultant velocity and its direction at point (2, 4) for the stream function represented by $\psi = x^2 + y^2$

Solution :

$$\text{Given } \psi = x^2 + y^2$$

Therefore $$u = \frac{\partial\psi}{\partial y} = 2y \text{ and } v = -\frac{\partial\psi}{\partial x} = -2x$$

$$|V| = \sqrt{u^2 + v^2} = \sqrt{4y^2 + 4x^2} = 2\sqrt{x^2 + y^2}$$

At point (2, 4), velocity components are $u = 2 \times 4 = 8$ units, and $v = -2 \times 2 = -4$ units.

Resultant velocity $= 2\sqrt{4 + 16} = 2\sqrt{20} = 2 \times 4.472$

$= 8.944$

and $$\tan\theta = \frac{v}{u} = \frac{-4}{8} = -\frac{1}{2}$$

$$\therefore \theta = \tan^{-1}(-0{\cdot}5) = 26°34'$$

that is $\tan(180 - \theta)$ or $\tan(360 - \theta) = -\tan\theta$

Therefore, resultant velocity makes an angle of $(180 - 26° 34') = 153° 26'$ with x-axis,

or $(360 - 26° 34') = 333° 26'$ with x-axis.

Streamlines are concentric circles as $\psi = x^2 + y^2$.

Example 4 :

The velocity potential is given by $\phi = x^2 - y^2$. Does this represent a possible flow field ? If it so, prove that the flow is irrotational.

Solution :

To have a possible flow, it must satisfy continuity equation

Given : $$\phi = x^2 - y^2$$

$$\therefore u = \frac{\partial\phi}{\partial x} = 2x, \quad v = \frac{\partial\phi}{\partial y} = -2y$$

Equation of continuity in 2-dimensional is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Hence $$\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) = 2 - 2 = 0.$$

This proves that this is a possible flow field.

Flow to be irrotational, it should satisfy Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Given $$\phi = x^2 - y^2$$

$$\therefore \frac{\partial\phi}{\partial x} = 2x, \text{ and } \frac{\partial^2\phi}{\partial x^2} = 2$$

Similarly, $$\frac{\partial\phi}{\partial y} = -2y \text{ and } \frac{\partial^2\phi}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2 - 2 = 0$$

This proves that flow is irrotational.

4.4 PROPERTIES OF STREAM FUNCTION AND VELOCITY POTENTIAL

From the above discussion, we may summarize the properties of stream function and potential function.

The properties of stream function (ψ) are:

(1) If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational. It exists for 2-Dimensional flows only.

(2) If stream function (ψ) satisfies the Laplace equation, it is a possible case of irrotational flow.

(3) If stream function (ψ) satisfies equation of continuity but does not satisfy Laplace equation, it is a case of rotational flow.

The properties of potential function (ϕ) are :

(1) If velocity potential (ϕ) exists, the flow should be irrotational. This means the velocity potential is applicable only for ideal fluid flow. It exists for 2D or 3D flows.

(2) If velocity potential (ϕ) satisfies the Laplace equation, it is a case of possible steady incompressible irrotational flow.

(3) If Laplace equation is not satisfied by velocity potential, the flow is either rotational or non-existing.

All real fluid flows are rotational while ideal fluid flows are irrotational.

Example 5 :

For a two-dimensional flow, stream function is given by $\psi = 2xy$, prove that the flow is irrotational.

Solution :

$$u = \frac{\partial \psi}{\partial y} = 2x \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -2y .$$

For $x - y$ plane, vorticity is given by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-2y) - \frac{\partial}{\partial y}(2x) = 0 - 0 = 0$$

Hence, flow is irrotational.

4.5 ROTATIONAL AND IRROTATIONAL FLOWS

The ideal fluid flow consists of two types differing from each other both physically and mathematically. They are rotational and irrotational flow. If the fluid particles within a flow have a rotation about any axis, the flow is said to be rotational. If fluid particles do not suffer rotation, the flow is an irrotational one. The non-uniform velocity distribution of real fluids close to the boundary causes particles to deform with a small degree of rotation whereas, the flow is considered irrotational if the velocity distribution is uniform across a section in a flow field.

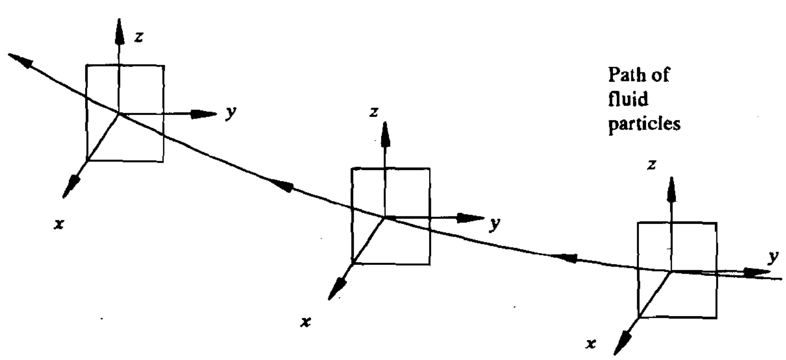


Figure 4.2 Irrotational Motion

The Prandtl hypothesis (which provides an important link between ideal fluid flow and real fluid flow) states that for fluids of low viscosity (e.g. air, water), the effects of viscosity are appreciable only in a narrow region surrounding the fluid boundary. For incompressible flow situations, results of ideal fluid flow can be applied to real fluid flow to a satisfactory degree of approximation.

To understand rotational and irrotational flow, one has to understand the concepts of circulation and vorticity.

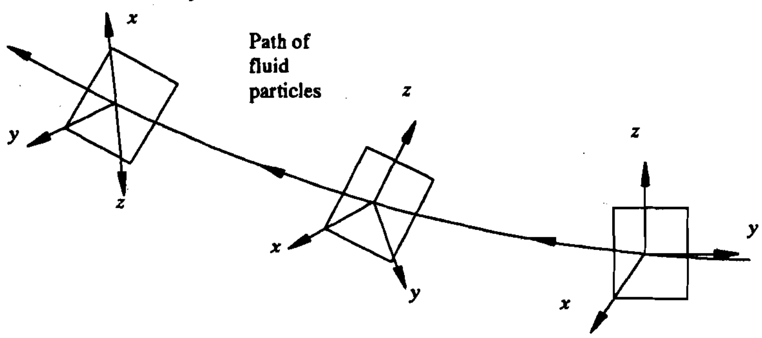


Figure 4.3 Rotational motion

Now would you like to try your hand at these exercises?

SAQ 3

Tick (√) the correct answer or most appropriate response from among the alternatives given:

Irrotational flow is such that

a) Circulation is zero.

b) Stream function exists.

c) Vorticity is not zero.

d) Velocity potential satisfies Laplace equation.

SAQ 4

Following are the velocity components

a) $u = 3x + 5y + 6z,\ v = 5x - 3y,\ w = 5x + 5y + 5$

b) $u = 4xy,\ v = 5yz,\ w = 6yz + z^2$

Do these two cases represents irrotational flow? Determine velocity potentials.

4.6 CIRCULATION OF FLOW: CIRCULATION AND VORTICITY

Circulation Γ (gamma) is defined as the line integral of a velocity around a closed curve in a fluid flow.

For a three-dimensional flow

$$\Gamma = \oint \vec{V} \cdot \vec{dL} = \oint (udx + vdy + wdz) \quad (4.6)$$

For a two-dimensional flow

$$\Gamma = \oint (udx + vdy) \tag{4.7}$$

$$\Gamma = \oint \vec{v_L} \cdot \vec{dL} = \oint v \cos \alpha dL \tag{4.8}$$

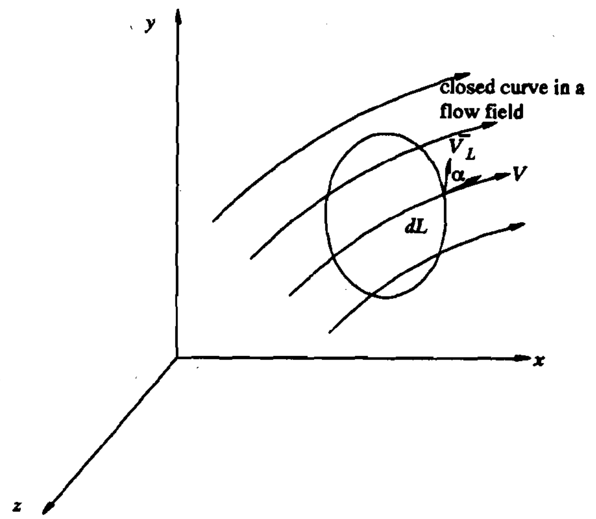


Figure 4.4 : Circulation around a curve

Consider a rectangular element having sides δx and δy parallel to x and y axis. We may obtain circulation around any closed curve with the help of above equation (4.8). Convention is that anticlockwise direction of circulation is taken as positive.

ABCD represents rectangular fluid element. At centre *O*, velocity components u and v are represented. At mid-points of the sides of the rectangle, corresponding velocities are written at points 1,2,3 and 4.

Hence circulation around an elementary rectangle is given by

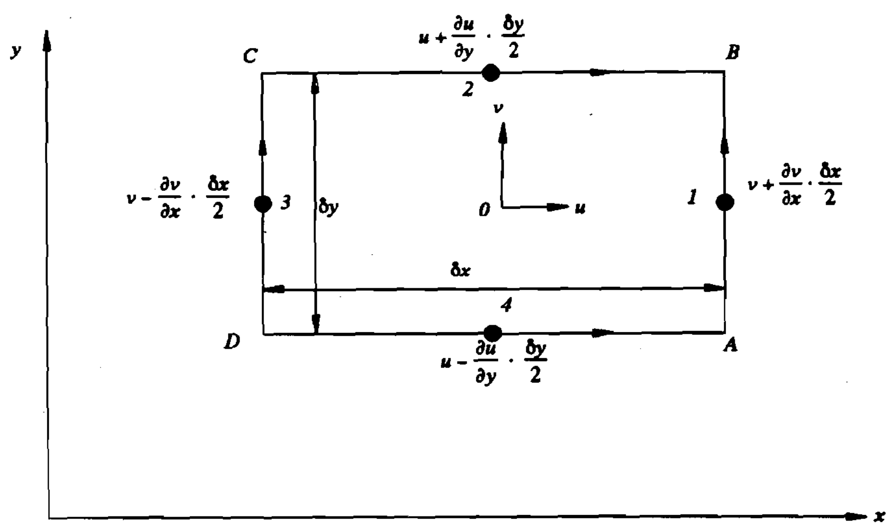


Figure 4.5 : Circulation in x - y plane

$$\delta \Gamma = \left(u - \frac{\partial u}{\partial y} \cdot \frac{\delta y}{2}\right) \delta x + \left(v + \frac{\partial v}{\partial x} \cdot \frac{\delta x}{2}\right) \delta y - \left(u + \frac{\partial u}{\partial y} \cdot \frac{\delta y}{2}\right) \delta x - \left(v - \frac{\partial v}{\partial x} \cdot \frac{\delta x}{2}\right) \delta y \tag{4.8a}$$

On simplifying, the above equation reduced to

$$\delta \Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \delta x \, \delta y$$

or
$$\lim_{\delta x\, \delta y \to 0} \frac{\delta \Gamma}{\delta x \delta y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta \text{ (zeta)} \tag{4.9}$$

which is the circulation in $x-y$ plane and along an axis parallel to z-axis. The limiting value of circulation per unit area is known as **Vorticity**.

Similarly, vorticity along axes parallel to x and y are

$$\xi \text{ (xi)} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad \text{and} \quad \eta \text{ (eta)} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad (4.10)$$

The above equations are 3-vorticity components of vorticity parallel to x, y and z axis.

For a rotational flow, vorticity ξ (xi), η (eta) and ζ (zeta) are not equal to zero, whereas in irrotational flow these components are equal to zero.

Thus for irrotational flow these components

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 = 2\,\omega_x$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 = 2\,\omega_y$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 = 2\,\omega_z \qquad (4.11)$$

Mathematically, vorticity can be defined as curl of velocity vector $\vec{V}$;

$$\vec{\zeta} = \nabla \times \vec{V} = \text{curl } \vec{V}$$

where $\vec{\zeta} = \vec{i}\xi + \vec{j}\eta + \vec{k}\zeta = \vec{\Omega}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Also sometime vorticity is expressed by Greek letter Ω and absolute value of vorticity is $|\,\Omega\,| = |\,\zeta\,| = \sqrt{\xi^2 + \eta^2 + \zeta^2}$

Example 6 :

Find the vorticity for the fluid motion having velocity $\vec{V}$ as

$$\vec{V} = \vec{i}\,(2A\,xz) + \vec{k}\,.[A\,(c^2 + x^2 - z^2\,)]$$

where c is numerical constant.

Solution :

For a given velocity, velocity components are $u = 2A\,xz$, $v = 0$ and $w = A\,(c^2 + x^2 - z^2)$

Now the components of vorticity are

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}[A\,(c^2 + x^2 - z^2)] - \frac{\partial}{\partial z}(0) = 0 - 0 = 0$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial}{\partial z}(2Axz) - \frac{\partial}{\partial x}[A\,(c^2 + x^2 - z^2)]$$

$$= 2Ax - 2Ax = 0$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2Axz) = 0 - 0 = 0$$

Hence the flow is irrotational

Example 7 :

The velocity components for a flow are given as $u = 6y$, $v = 0$, $w = 0$. Calculate vorticity.

Solution :

This flow is one-dimensional

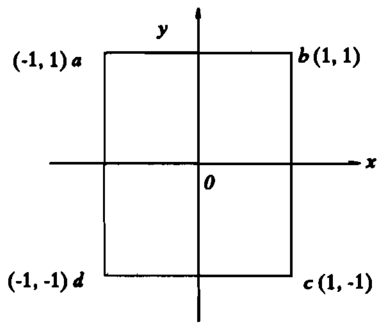
The vorticity will be $\xi = 0, \eta = 0$ and $\zeta = -6$

$$\therefore \quad |\zeta| = \sqrt{\xi^2 + n^2 + \zeta^2} = \sqrt{(-6)^2} = 6$$

If you have followed the ideas introduced in this section, then you should be able to solve this exercise.

SAQ 5

Find the circulation around the square enclosed by the lines $x = \pm 1$, $y = \pm 1$ for a two-dimensional flow given by $u = x + y$, $v = x^2 - y$ at centre O.



4.7 LAPLACE EQUATION

Thus for an incompressible, irrotational flow :-

By equation of continuity div $\vec{V}$ or $\nabla \cdot \vec{V} = 0$

i.e $$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\vec{i}\,u + \vec{j}\,v + \vec{k}\,w) = 0$$

which gives the equation of continuity for three-dimensional fluid flow as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4.12)$$

As defined earlier substituting the values of u, v and w in equation (4.12), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = 0$$

i.e. $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (4.13)$$

Above equation (4.13) is known as **Laplace's equaiton**. Hence, to obtain solution for any incompressible, irrotational flow, one has to solve Laplace's equation with the prescribed boundary conditions.

In case of two-dimensional flow the equation (4.13) will be

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (4.14)$$

Components of rotation as derived in earlier chapter are given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \text{about } x\text{-axis}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad \text{about } y\text{-axis}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad \text{about } z\text{-axis} \qquad (4.15)$$

For the flow to be irrotational, each component of rotational or vorticity should be zero.

For a two-dimensional flow in $x - y$ plane,

$\omega_z = 0$ and substituting the values of u and v in ψ

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

Hence we get

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad (4.16)$$

Equation (4.16) is known as Laplace's equation for a two-dimensional flow.

Similarly by substituting the values of u and v in ϕ, in ω_z, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x\, \partial y} - \frac{\partial^2 \phi}{\partial x\, \partial y} = 0 \qquad (4.17)$$

Hence if a flow is irrotational, a velocity potential must exist and it should satisfy Laplace's equation also. Further it proves that ϕ is a continuous function of x and y. Therefore order of differentiation can be interchanged.

Here are some exercises for you.

SAQ 6

State whether the following statements are true or false. If false then correct it.

a) Potential function satisfies Laplace equation whereas stream function for rotational flow only satisfies the Laplace equation.

b) Laplace equation holds good only for irrotational flow.

c) $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ can be used for irrotational and rotational flow.

d) In case of real fluid, the flow immediately near the boundary is rotational one.

4.8 VORTEX FLOW – FORCED VORTEX AND FREE VORTEX

4.8.1 Vortex Motion

Flow having circular streamlines or flow along curved path of a rotating mass of a fluid is known as **vortex motion.** The vortex flow is of two types namely

(a) Forced vortex or Rotational flow

(b) Free vortex or Irrotational flow

The flow in free vortex is known as **Rankine combined vortex** as it is a combination of above two types.

(a) Forced Vortex

In this type of vortex flow, application of external torque is required. When a container filled with liquid is rotated about its vertical axis, the liquid surface no longer remains horizontal and pressure variations occur due to centrifugal effects.

The vorticity of flow everywhere within the liquid is non-zero and finite. Hence the flow is called **rotational** .

From elementary mechanics, it is known that the tangential velocity at radius r is angular velocity (ω) times radius r. If V is the tangential velocity and ω (omega) is the angular velocity, then

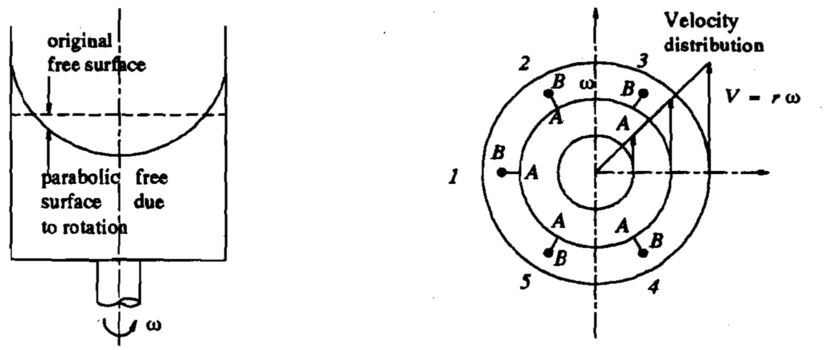
$$V = r\omega \tag{4.18}$$

that is $V \propto r$

Therefore velocity increases as we move away from the centre and streamlines are concentric circles.

Figure 4.6 illustrates the distinguishing characteristics of a forced-vortex or rotational vortex. As the streamlines are circular, the velocity vector being every where tangential has no radial component.

An expression for its stream function may be obtained as below for radial co-ordinates (r, θ)



(a) Free Surface in a rotating Tank

(b) Streamline pattern and velocity distribution in a horizontal plane

Figure 4.6 : Characteristics of a Forced-vortex

$$d\psi = \frac{\partial \psi}{\partial r} \cdot dr + \frac{\partial \psi}{\partial \theta} \cdot d\theta$$

$$\psi = \int \frac{\partial \psi}{\partial r} \cdot dr + \int \frac{\partial \psi}{\partial \theta} \cdot d\theta \tag{4.19}$$

$V = r\omega$ but

$$= -\frac{\partial \psi}{\partial r} \tag{4.20}$$

$$\text{and radial component } V_r = \frac{\partial \psi}{r \partial \theta} = 0 \tag{4.21}$$

Substituting these values and integrating,

we get

$$\psi = -\frac{r^2 \omega}{2} + \text{constant} \tag{4.22}$$

Constant is zero as at $r = 0$, $\psi = 0$.

Rotational aspects can be visualized by dropping a small match stick denoted by AB. Numbers 1, 2, 3, 4 and 5 represents the various positions of match sticks as it moves along with the rotating liquid. The direction of AB changes its orientation from place to place which clearly indicates the rotation of the object about its mass centre. This illustrates that vorticity exists at every point indicating aspects of rotational flow.

(b) **Free Vortex**

The flow in a free vortex is a combination of rotational and irrotational flow and known as **Rankine combined vortex**. The flow within the core is rotational while outside the core, the flow is irrotational.

Figure 4.7 shows the characteristics of a free-vortex or irrotational vortex flow. In free-vortex, no external torque is necessary to produce this motion. Examples of this are a cyclone, vortex in a river during flood at a deep pit, emptying of a sink etc.

Torque T can be expressed as,

$$\vec{T} = \vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times \vec{M}) \qquad (4.23)$$

where $\vec{r}$ is position vector and $\vec{M}$ is momentum which can be written as

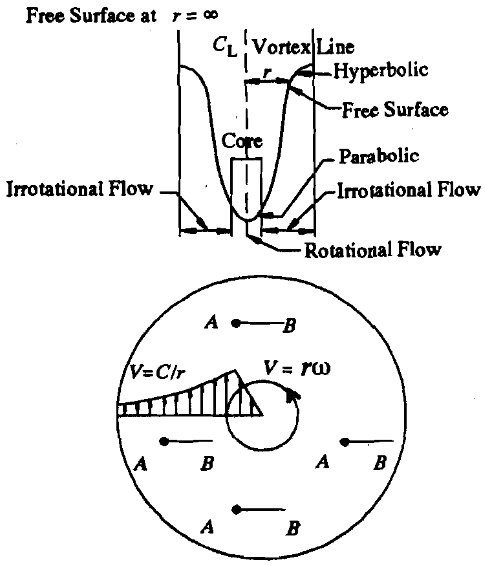
$$\vec{M} = m\,\vec{V}$$

In above, $\vec{V}$ is velocity vector and m is the mass of fluid.
Thus equation (4.23) becomes

$$\vec{T} = \frac{d}{dt}(\vec{r} \times m\,\vec{V}) \qquad 4.24)$$

Writing equation (4.24) in scaler form, we get

$$T = \frac{d}{dt}\,(rmV) \qquad (4.25)$$



(b)Velocity distribution
Figure 4.7 : Characteristics of a Free Vortex

If external torque is zero, equation (4.25) gives,

$$T = 0 = \frac{d}{dt}\,(rmV)$$

or

$$\frac{d}{dt}(rV) = 0 \qquad (4.26)$$

$$\text{that is } rV = C = \text{constant} \qquad (4.27)$$

This means that V increases as r decreases while in forced vortex V increases as r increases ($V = r\omega$)

The circulation around a circular streamline is given by

$$\Gamma = 2\pi r \cdot V = 2\pi C \quad \text{as} \left(V = \frac{C}{r} \right)$$

$$= \text{constant} \qquad (4.28)$$

The circulation is constant for all the streamlines as it is independent of radial distance. The stream function for free vortex may be obtained as under.

$$\psi = \int \frac{\partial \psi}{\partial r} \cdot dr + \int \frac{\partial \psi}{\partial \theta} \cdot d\theta$$

and Γ (Gamma) = $V \times 2\pi r$ (circulation)

therefore
$$V = \frac{\Gamma}{2\pi r} = -\frac{\partial \psi}{\partial r}$$

and $$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

substituting these values,

$$\psi = \int \left(-\frac{\Gamma}{2\pi r}\right) dr + 0$$

$$= -\frac{\Gamma}{2\pi} \log r + \text{constant} B$$

Let at $r = r_0, \psi = 0$,

hence constant $$B = \frac{\Gamma}{2\pi} \log r_0$$

Now, $$\psi = \frac{\Gamma}{2\pi} \log \frac{r_0}{r} \quad (4.29)$$

The circulation Γ which is constant for a vortex is knwon as **vortex strength.**

Irrotational aspects can be visualized by dropping small match stick *AB* (figure 4.6). Its positions is shown at various points as the liquid rotates. Its orientation remains the same with respect to its mass centre. There is no rotation. This indicates irrotational aspects of the flow.

4.9 FLOW NET

As stated earlier, along streamline the value of stream function is constant. Similarly if we draw lines having equal values of velocity potential, we get equipotential lines along which velocity potential remains constant that is $d\phi = 0$.

A grid obtained by drawings series of equipotential lines and streamlines is known as **Flow net.** Flow net is one of the methods for solving and analyzing two-dimensional irrotational flow problems.

The following derivation proves that equipotential lines are normal to the stream lines.

Along a stream line, $d\psi = 0$ and along an equipotential line, $d\phi = 0$.

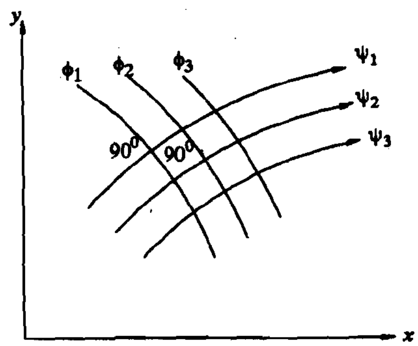


Figure 4.8 : Flow net for a 2-dimensional flow

As ψ and ϕ are functions of two independent variable x and y, i.e $\psi = \psi(x, y)$ and $\phi = \phi(x, y)$

By rules of partial differentiation,

$$d\psi = \frac{\partial \psi}{\partial x} \cdot dx + \frac{\partial \psi}{\partial y} \cdot dy = 0$$

which gives $$\frac{dy}{dx} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} \quad \text{for constant } \psi. \quad (4.30)$$

Similarly $$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy = 0$$

Hence $$\frac{dy}{dx} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} \quad \text{for constant } \phi \quad (4.31)$$

By definition, $$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

and $$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

Substituting these in equations (4.30) and (4.31)

We obtain $$\frac{dy}{dx} = \frac{v}{u} \quad \text{for constant } \psi$$

and $$\frac{dy}{dx} = -\frac{u}{v} \quad \text{for constant } \phi$$

Now, $\frac{dy}{dx}$ represents slopes of streamlines and equipotential lines.

Therefore $$\left(\frac{dy}{dx} \text{ for } \psi = \text{constant}\right) \times \left(\frac{dy}{dx} \text{ for } \phi = \text{constant}\right) = -1 \qquad ...(4.32)$$

which proves the fact that streamlines and equipotential lines are normal to each other. In other words, they intersect at right angles.

Flow net is a graphical solution for two-dimensional, irrotational flow. Combination of equipotential lines and streamlines give rise to small, tiny squares. There can be only one flow net for a given flow. With the help of flow net, pressure and velocity can be found. Spacing between streamlines and equipotential lines can be varied. To get accurate results, one must draw finer net work.

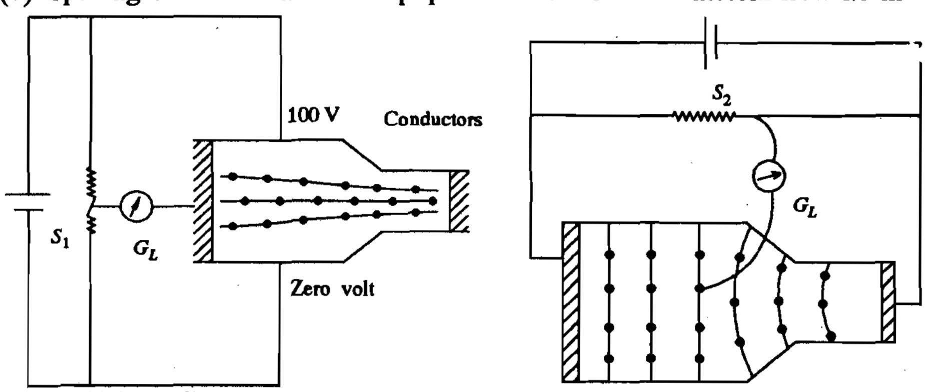
4.10 PLOTTING OF FLOW NET - METHODS

The plane ideal fluid flow pattern can be plotted with the help of followings techniques :

(1) Graphical Method
(2) Analytical Method
(3) Analogue Method
(4) Numerical Method
(5) Direct Method
(6) Hydraulic Models

(1) **Graphical Method** :- It is the simplest but it requires experience and skilled drawing. In graphical plotting, the properties of flow net should be kept in mind.

(i) Equipotential lines should meet streamlines at right angles
(ii) Rigid boundary is considered as one of the streamlines
(iii) Equipotential line should meet rigid boundary orthogonally.
(iv) Spacing of streamlines and equipotential lines in uniform flow should be equal and should result in perfect squares.
(v) Spacing of streamlines and equipotential line in non-uniform flow i.e in



(a) Streamlines (b) Equipotential lines

Figure 4.9 : Electrical analogy method

converging or diverging boundaries should be done skillfully so that the resulting pattern will have near-squares

(vi) Diagonal lines drawn through the squares in uniform flow will result in perfect squares and near-squares in non-uniform flow.

(2) **Analytical Method :-** It is also known as mathematical method. It requires the equations of stream function and velocity potential in terms of x and y. By assigning different constant values of ψ and ϕ and plotting them will give rise to flownet.

(3) **Analogue Method :-** If the characteristics of two or more systems which are apparently different, can be expressed in identical mathematical forms, they are said to be analogous. The technique of electrical analogy is based on the similarity of flow of electric current through conductors in an electrical system on one hand and flow of water in a pipe in hydraulic system on other hand.

Heat flow is also analogous to water flow in idealized circumstances. Electrical potential V is identical to ϕ and field strength vector E stands for velocity field u. Discharge Q corresponds to electric current i. Both the systems use Laplace equation for their solution.

(4) **Numerical Method :-** It requires to express Laplace's equation in finite-difference form and solving for the distribution of ϕ numerically by trial and error.

(5) **Direct Method :-** It involves velocity measurements and plotting velocity profile by actual measurements at number of axial stations. From velocity profiles, points of equal discharges along the length are obtained and they are joined to get a streamline.

(6) **Hydraulic Models :-** Hele-Shaw apparatus is used in which by injecting a dye streamlines can be traced. The flow net is completed by drawing equipotential lines on the streamlines obtained as above.

4.11 USES OF A FLOW NET

Some of the important uses of flow net analysis are as under :

(1) For given boundaries, the velocity and pressure can be obtained if velocity and pressure at any reference section are known.

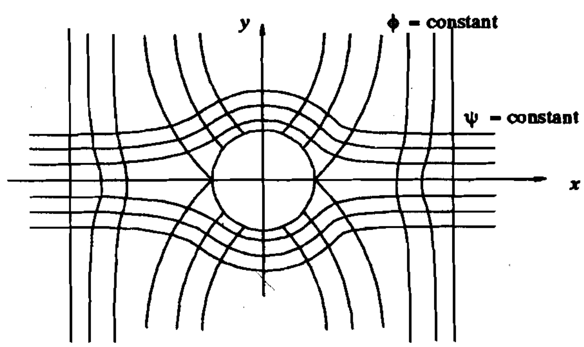


Figure 4.10 : Flow net for a flow past a cylinder (Irrotational flow)

(2) Seepage loss in earth dam and canals can be estimated.

(3) Uplift pressure below the weir floor can be estimated.

(4) Streamlining of the outlet's shape can be done.

4.12 APPLICATIONS

(i) **Flow Past a Circular Cylinder :-**

On the basis of ideal fluid flow theory, when a cylinder is placed in a free stream or uniform flow, it should experience no drag force. According to D'Alembert, he discovered that if a cylinder or any other symmetrical body is placed in a real fluid, it experiences a drag force. This paradox is called **D'Alembert Paradox.**

In actual flow, the cylinder experience viscous effects of the fluid particularly in the region close to the surface of the body. The flow pattern in the vicinity of the cylinder boundary deviates appreciably from the pattern of ideal fluid flow analysis.

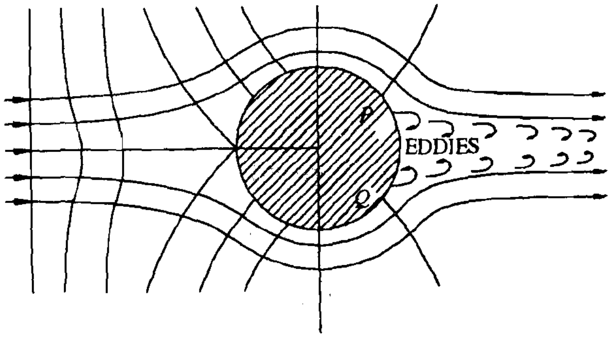


Figure 4.11 : Actual flow pattern for flow over a cylinder (Real fluid flow)

The pressure distribution on the surface is also not the same that is developed by ideal fluid flow theory. Figures (4.10) and (4.11) show this difference. The flow separates at points P and Q and vortex rings are formed on the downstream. The pressure distribution mainly deviates in the downstream region.

(ii) **Flow through a sluice outlet**

Figure 4.12 shows a flow underneath a sluice gate. Here channel bottom, sluice gate and free surface represent boundary conditions in drawing flow net

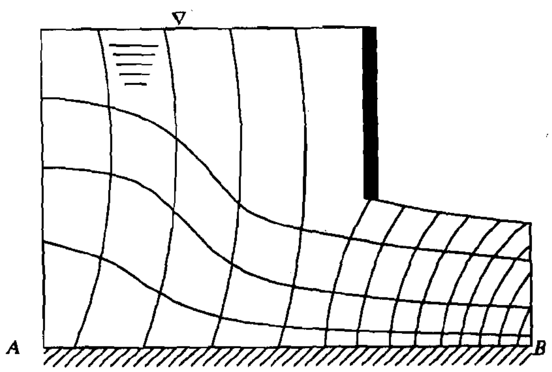


Figure 4.12 : Flow through sluice outlet

4.13 SUMMARY

To sum up what has been done in this unit

* Stream function and velocity potential are important scalar field functions to describe fluid flow
* Velocity potential exists for irrotational flow only
* Solution of Laplace equation will give ϕ-lines and ψ-lines (equipotential lines and streamlines)
* Flow net is a graphical solution of Laplace equation and can be drawn for irrotational flow
* Rotational and Irrotational aspects are the important phenomena of ideal fluid flow.

4.14 ANSWERS TO SAQs

SAQ 1

(f)

SAQ 2

$$u = \frac{\partial \psi}{\partial y} = 2xy \text{ (given)} \qquad \text{(i)}$$

Integrating equation (i) with respect to y.

$$\psi = \frac{2xy^2}{2} + C_1 + f(x) \qquad \text{(ii)}$$

And $$v = -\frac{\partial \psi}{\partial x} = a^2 + x^2 - y^2$$

i.e. $$\frac{\partial \psi}{\partial x} = -a^2 - x^2 + y^2 \qquad \text{(iii)}$$

Integrating equation (iii) with respect to x,

$$\psi = -a^2 x - \frac{x^3}{3} + xy^2 + C_2 + f(y) \qquad \text{(iv)}$$

Comparing equations (ii) and (iv),

$$xy^2 + C_1 + f(x) = -a^2 x - \frac{x^3}{3} + xy^2 + C_2 + f(y)$$

Hence $$f(x) = -a^2 x - \frac{x^3}{3}$$

$$f(y) = 0$$

$C_1 = C_2 = C$ as $u = 0$ when $x = 0$ and $y = 0$ but $v = a^2$ when $x = 0$ and $y = 0$

Hence $$\psi = -a^2 x - \frac{x^3}{3} + xy^2 + C$$

SAQ 3

(a) and (d)

SAQ 4

(a) For condition of irrotationality, curl $\vec{V}$ must be zero

i.e.

$$\text{curl } \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x + 5y + 6z & 5x - 3y & 5x + 5y + 5 \end{vmatrix}$$

$$\begin{aligned} \text{curl } \vec{V} = \; & \vec{i}\left[\frac{\partial}{\partial y}(5x + 5y + 5) - \frac{\partial}{\partial z}(5x - 3y)\right] \\ & - \vec{j}\left[\frac{\partial}{\partial x}(5x + 5y + 5) - \frac{\partial}{\partial z}(3x + 5y + 6z)\right] \\ & + \vec{k}\left[\frac{\partial}{\partial x}(5x - 3y) - \frac{\partial}{\partial y}(3x + 5y + 6z)\right] \\ = \; & \vec{i}\,[5 - 0] - \vec{j}\,[5 - 6] + \vec{k}\,[5 - 5] \\ = \; & 5\vec{i} + \vec{j} \end{aligned}$$

Hence the flow is rotational and velocity potentials do not exist for rotational

(b) For irrotational flow, curl $\vec{V}$ should be zero

$$\text{curl } \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy & 5yz & 6yz + z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(6yz + z^2) - \frac{\partial}{\partial z}(5yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(6yz + z^2) - \frac{\partial}{\partial z}(4xy)\right] + \vec{k}\left[\frac{\partial}{\partial x}(5yz) - \frac{\partial}{\partial y}(4xy)\right]$$

$$= \vec{i}\,[6z - 5y] - \vec{j}\,[0 - 0] + \vec{k}\,[0 - 4x] = \vec{i}\,(6z - 5y) - \vec{k}\,4x$$

which is not zero. Therefore the flow is rotational. Hence velocity potentials not exist.

SAQ 5

$$u = x + y, \; v = x^2 - y$$

$$\Gamma = \text{circulation} = \oint \vec{V_L} \cdot \vec{d_L}$$

$$= \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u\,dx + \int_d^a v\,dy$$

$$= \int_{-1}^{1} (x + y)dx + \int_{1}^{-1} (x^2 - y)dy + \int_{1}^{-1} (x + y)dx + \int_{-1}^{1} (x^2 - y)\,dy$$

Putting the values of x and y at proper places and using equation (4.8a) for integration.

$$\text{We get } \Gamma = \int_{-1}^{1} (x + 1)dx + \int_{1}^{-1} (1 - y)dy + \int_{1}^{-1} (x - 1)dx + \int_{-1}^{1} (1 - y)dy$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{1} + \left[y - \frac{y^2}{2}\right]_{1}^{-1} + \left[\frac{x^2}{2} - x\right]_{1}^{-1} + \left[y - \frac{y^2}{2}\right]_{-1}^{1} = 4$$

SAQ 6

(a) False. Potential function satisfies Laplace equation whereas stream function for irrotational flow only satisfies in the Laplace equation.

(b) True

(c) $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ can be used only for irrotational flow.

(d) True

Notes