UNIT 3 FUNDAMENTALS OF FLUID MOTION

Structure

3.1 INTRODUCTION

In the previous units you have studied the physical properties of the fluid, pressure distribution on submerged plane and curved surfaces when the fluid is at rest. You also know the magnitude and direction of pressure as well as its measurement. The concept of buoyancy, metacentre was also introduced. Previous unit has also covered relative equillibrium of a liquid subjected to linear acceleration and rotation. In this unit, we shall be dealing with a few fundamentals of fluid motion and its related aspects. Fluid kinematics deals with space-time relationship without considering forces involved in the flow. When fluid is in motion, the relative position of each particle is not fixed from time to time. Depending upon the type of flow, type of fluid each particle will have its own velocity and acceleration at any instant of time. The terms velocity and acceleration apply at a point in a fluid. In rectanglar coordinate system, the velocity components along the x, y and z directions are denoted by u, v and w respectively. The velocity vector at a point is denoted by $\vec{V}$.

Objectives

After reading this unit, you should be able to

* identify different types of flow,
* differentiate between streamlines, pathlines and streaklines,
* determine the acceleration in a fluid flow, and
* derive the continuity equation for a fluid flow.

3.2 TYPES OF FLUID FLOW

There are 96 combinations in which fluid flow can be classified. However, the following combinations are usually encountered in our day to day life which are required to be known in detail.

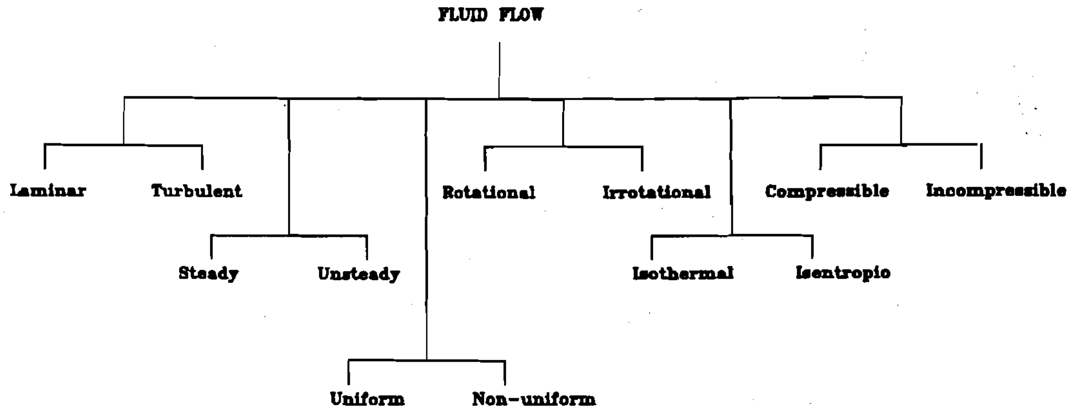


Figure 3.1 : Classification of Fluid Flow

3.2.1 Laminar and Turbulent Flows

If fluid particles move in smooth paths in layers or laminae with one layer sliding over an adjacent layer, the flow is said to be laminar. Paths of individual particles do not cross or intersect. Contrary to this, flow is said to be turbulent if the fluid particles move in irregular paths, there being momentum exchange from one portion of fluid to another.

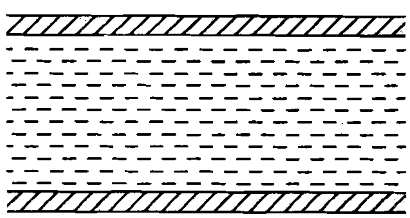


Figure 3.2 (a) : Laminar Flow

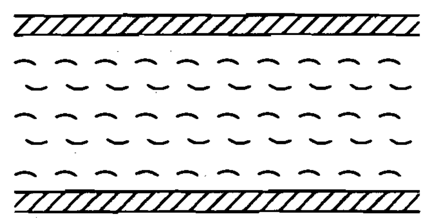


Figure 3.2 (b) : Turbulent Flow

3.2.2 Steady and Unsteady Flows

A flow is considered to be steady if the dependent fluid flow variables at any point in the flow do not change with time. Mathematically, this can be stated as

$$\frac{\partial}{\partial t}(\text{dependent fluid variables}) = 0 \qquad (3.1)$$

Thus, for steady flow

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 = \frac{\partial p}{\partial t} = \frac{\partial \rho}{\partial t} \qquad (3.2)$$

Generally velocity vector is represented as

$$\vec{V} = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w \qquad (3.3)$$

where u, v, w are the velocity components of the fluid, p is the pressure and ρ is the specific mass of the fluid.

For example, if the volume rate of fluid at a given cross section remains constant with time, the flow is said to be steady flow.

Whereas in unsteady flow or transient flow, fluid flow exhibits variations at a fixed point in a space with respect to time.

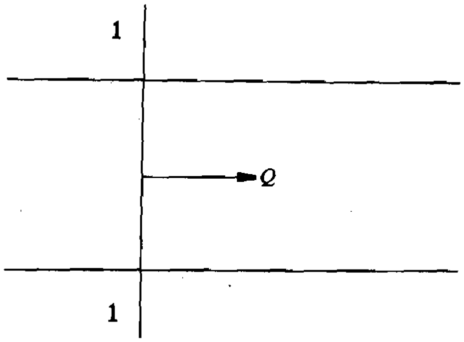


Figure 3.3 : Pipeline of constant cross section

Q is the volume rate of flow. The same can also be expressed as weight rate of flow

$W = \gamma Q$ where γ = specific weight of fluid in N/m^3

M = Mass rate of flow

$M = \rho \times Q$ where ρ = specific mass of fluid in kg/m^3

At cross section 1-1 in figure 3.3

and at time t_1, $Q = Q_1$.

At time t_2, $Q = Q_2$

Hence, $\frac{\partial Q}{\partial t} = 0.$

For turbulent flows, due to fluctuations in flow at any section (figure 3.4) if temporal mean velocity $\overline{u}$ does not vary with time, flow is called steady flow. If temporal mean velocity varies with time, the flow can be treated as unsteady flow

$$\overline{u} = \frac{1}{t}\int_0^t u\,dt \tag{3.4}$$

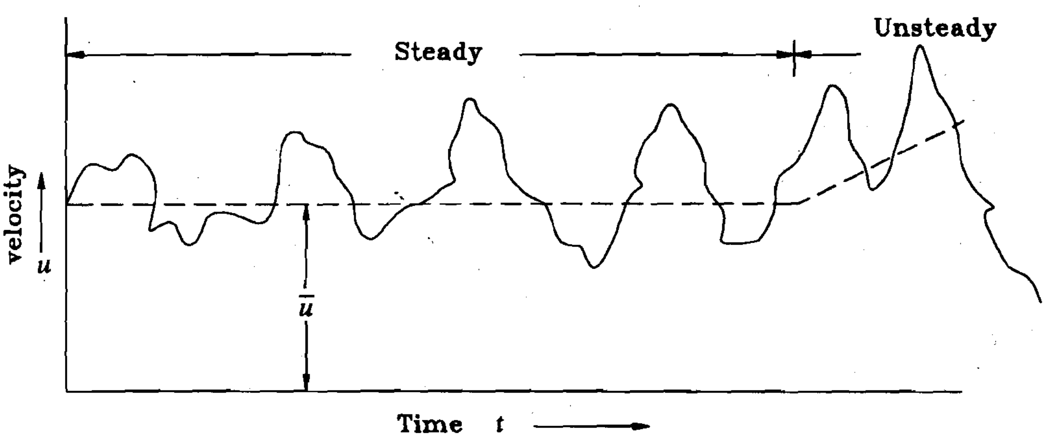


Figure 3.4 : Turbulent flow - Temporal Mean Velocity

3.2.3 Uniform Flow and Non-Uniform Flow

If at any given instant velocity vector of fluid remains same in magnitude and direction at all points, the flow is uniform. This stipulates the velocity components to be the same at different points in the flow. If the velocity varies from point to point at any instant, the flow is called non-uniform.

For uniform flow,

$$\frac{\partial}{\partial s}(\text{dependent fluid variables}) = 0 \quad i.e. \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0 \tag{3.5}$$

In case of ideal fluid flow, velocity at a fixed boundary is not zero while in the case of real fluids, the velocity at fixed boundary is zero. This is usually referred to as **No-slip** condition. In ideal fluids, at any section, all fluid particles move with the same velocity.

3.2.4 Isothermal and Isentropic Flow

If gas flows at constant temperature, flow is said to be isothermal. If during the gas flow,

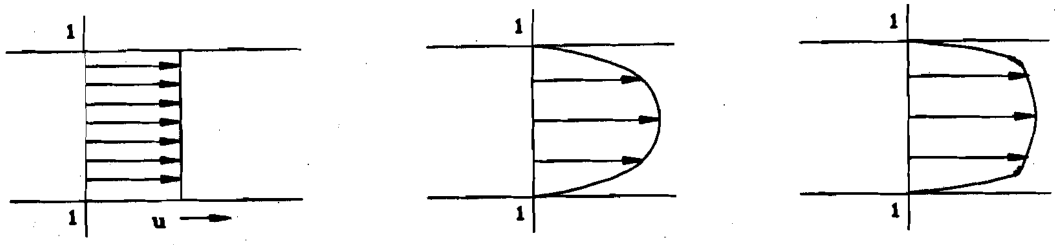


Figure 3.5 (a) Velocity in an ideal fluid (slip condition at the boundary)

Figure : 3.5 (b) Velocity in laminar flow (Real fluid)

Figure 3.5 (c) Velocity in turbulent flow (Real Fluid)

no heat enters or leaves the boundaries of the fluid, the fluid flow is adiabatic. Frictionless flow is called isentropic flow. Detailed description of these are beyond the scope of this course.

For compressible and incompressible fluid flows reader may refer any book on these topics. Rotational and irrotational flows are seperately treated in Unit 4.

3.3 METHODS OF FLUID FLOW ANALYSIS

All fluid flows must belong to any one of the three types of flow viz, one-dimensional flow, two-dimensional flow and three-dimensional flow.

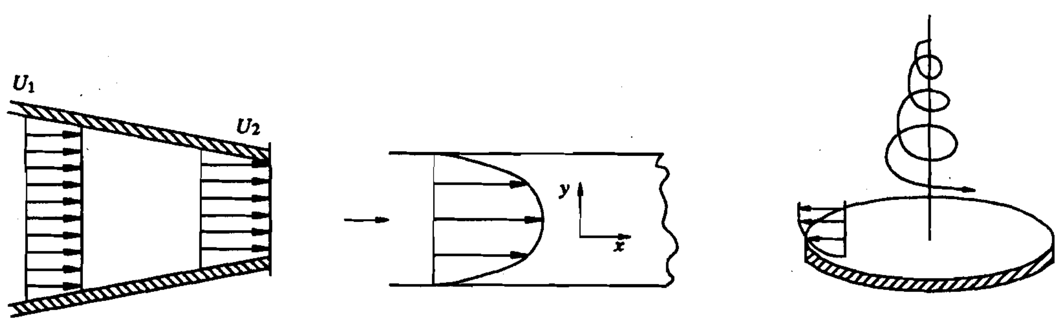
In one-dimensional flow, fluid property and parameters of flow are constant at any cross section normal to the flow. In otherwards, if the dependent variables are functions of only one space coordinate, say x it is 1-dimensional flow (figure 3.6(a)).

If the dependent variables are functions of two space coordinates x and y then, the flow is said to be 2-dimensional (figure 3.6(b)).

If the fluid flow parameters are functions of all the three space coordinates x, y and z, then the flow is said to be 3-dimensional (figure 3.6(c)).

Numerous examples for 3-dimensional flow are available in nature. For example, flow around trees, flow in a river, flow within fluid machines are examples of three-dimensional flows.

A flow is said to be axisymmetric if the velocity profile is symmetrical about the axis.



(a) Steady ideal fluid flow through a converging section
ONE-DIMENSIONAL FLOW

(b) Steady flow through a pipe
TWO-DIMENSIONAL FLOW

(c) Steady flow rotating in a fixed wall
THREE-DIMENSIONAL FLOW

Figure 3.6 : 1-2-3-Dimensional Flow

This means that the dependent variables cannot vary in the circumferential direction. An axisymmetric flow is essentially a two-dimensional flow.

SAQ 1

Identify the type of flow for problems listed in column A choosing the type from column B.

	A		B
(a)	flow in a pipe of constant cross section	(i)	non-uniform
(b)	$u = a + 2xt$	(ii)	unsteady and non-uniform
(c)	flow in a river during a rainstorm	(iii)	unsteady, non- uniform and turbulent
(d)	flow in a pipe of varying cross section	(iv)	uniform

SAQ 2

Select the correct answer from the multiple choices given

(a) One-dimensional flow is

- (i) steady uniform flow
- (ii) uniform flow
- (iii) flow which neglects changes in a transverse direction
- (iv) restricted to flow in a straight line
- (v) none of the above.

(b) The correct practical example of steady uniform flow is

- (i) motion of water around a boat in a lake.
- (ii) motion of river around bridge piers.
- (iii) pipeline carrying varying flow.
- (iv) constant flow through a reducing pipe section.
- (v) constant flow through a long, straight pipe.

3.4 DESCRIPTION OF FLOW FIELD

The value of velocity of a fluid particle can be treated as a continuous position of space and time. There are two approaches usually followed to describe the flow field. They are Lagrangian approach and Eulerian approach.

In the Lagrangian approach, the observer observes the movement of a single fluid particle instant to instant. In otherwords, the observer here traces the movement of individual fluid particles. As the fluid flow consists of movement of large number of individual fluid particles it is not only impractical to watch the movement of individual fluid particles but also cumbersome to keep track of their movement. The flow quantity say, velocity is defined as a function of time and function of position of the fluid particle at initial instant t_0. The initial position $\vec{r_0}$ of the particle at instant t_0 is given by

$$\vec{r_0} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k} \quad (3.6)$$

At a later time t, the particle is at position $\vec{r}$ which is $\vec{r}\,(\vec{r_0},\ t)$. The velocity vector

$$\vec{V} = \vec{V}\,(\vec{r_0},t) = \left(\frac{\partial \vec{r}}{\partial t}\right)_{\vec{r} = \vec{r_0}} \quad (3.7)$$

$$\text{or} \quad u = \left(\frac{\partial x}{\partial t}\right)_{x = x_0},\ v = \left(\frac{\partial y}{\partial t}\right)_{y = y_0} \ \text{and} \ w = \left(\frac{\partial z}{\partial t}\right)_{z = z_0} \quad (3.8)$$

Thus, one can very well realise how cumbersome this approach can be. This approach finds its place in standard works on Classical Mechanics of Fluid which is very much in the realm of Mathematicians.

Whereas in Eulerian approach, the observer sits at one place and watches the movement of fluid particles as they flow past by his side. Here, the observer notes the movement of many fluid particles as they flow past the observer. Thus, the velocity therefore is a function of space and time. This approach is much more practical and is adopted by Engineers in their study of Fluid Mechanics.

$$\vec{V} = \text{velocity vector} = \vec{V}(x, y, z, t) = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w \quad (3.9)$$

where $u = u(x,y,z,t)$; $v = v(x,y,z,t)$ and $w = w(x, y, z, t)$

This approach is much easier to follow and is widely used in Fluid Mechanics.

3.5 STREAMLINES

Streamline : Streamline is a line tangent to which at every point indicates the direction of flow. We can say that streamline is an instantaneous picture of entire flow field.

Since, the component of velocity normal to a streamline is zero, there can be no flow across a streamline. Further, the instantaneous velocity at a point in a fluid flow field

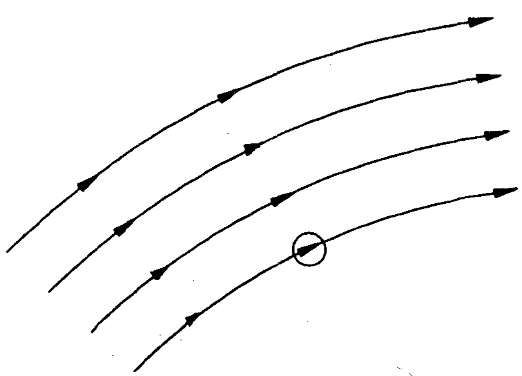


Figure 3.7 (a) : Streamlines

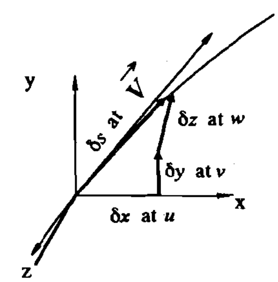


Figure 3.7 (b) : Expanded view of portion circled at Figure 3.7 (a)

must be unique both in magnitude and direction. This clearly means that the same point cannot belong to more than one streamline.

If we consider steady flow, the velocity components do not change with time. Hence, the velocity vector also does not change with time. This essentially means that the same streamline pattern holds good at all times. Whereas in unsteady flow the streamline pattern changes from instant to instant.

3.6 DIFFERENTIAL EQUATION OF A STREAMLINE

Let us consider an elementary displacement element $\vec{\delta s}$ along a streamline where the velocity $\vec{V}$ is given by

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \quad \text{and} \quad \vec{\delta s} = \delta x\,\vec{i} + \delta y\,\vec{j} + \delta z\,\vec{k} \tag{3.10}$$

From the definition of streamline it is clear that $\vec{V}$ must be directed along $\vec{\delta s}$. Hence,

i.e.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & v & w \\ \delta x & \delta y & \delta z \end{vmatrix} = 0 \tag{3.11}$$

or $$(v\delta z - w\delta y)\,\vec{i} - (u\delta z - w\delta x)\,\vec{j} + (u\delta y - v\delta x)\,\vec{k} = 0$$

which gives

$$\frac{\delta x}{u} = \frac{\delta y}{v} = \frac{\delta z}{w} \tag{3.12}$$

This is called the differential equation of a streamline in three-dimensional flow.

For a 2-D flow in the $x - y$ plane

$$\frac{\delta x}{u} = \frac{\delta y}{v} \quad \text{or} \quad \frac{\delta y}{\delta x} = \frac{v}{u} \tag{3.13}$$

which in the limit $\delta x \rightarrow 0$ becomes

$$\frac{dy}{dx} = \frac{v}{u} \tag{3.14}$$

We can clearly say that the slope of a plane streamline is the ratio of velocity components.

3.6.1 Stream Tube

A bunch of closely spaced streamlines bound by a imaginary tubular stream surface is called a streamtube (figure 3.8).

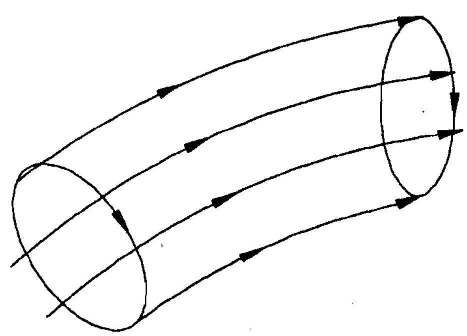


Figure 3.8 : Stream tube

As there cannot be any flow across a streamline it is quite obvious that the flow enters the stream tube at one end and leaves at the other end.

Example 1 :

The velocity vector $\vec{V}$ is given by $\vec{V} = \vec{i}\,x - \vec{j}\,y$. Determine the equation of streamlines. Plot them.

Solution :

We have $\vec{V} \times \vec{\delta s} = 0$

where $\vec{V} = \vec{i}\,x - \vec{j}\,y$

and $\vec{\delta s} = \vec{i}\,\delta x + \vec{j}\,\delta y$

i.e. $(\vec{i}\,x - \vec{j}\,y) \times (\vec{i}\,dx + \vec{j}\,dy) = 0$

i.e. $(xdy + ydx)\,\vec{k} = 0$

or $$\frac{dx}{x} = -\frac{dy}{y}$$

Integrating we get $xy = C$ where C is an arbitrary constant.

∴ The equation of stream lines is $xy = C$.

On plotting it, we see that it represents stream lines of steady 2-dimensional flow at a 90°

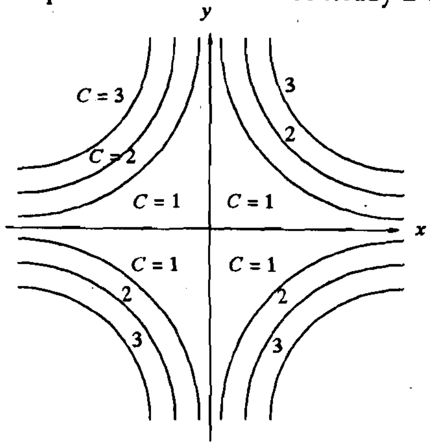


Figure 3.9 : Streamline pattern

corner (figure 3.9). We can also see that velocity vector $\vec{V} = \vec{i}\,x - \vec{j}\,y$ is everywhere tangent to the stream line $xy = C$.

3.7 PATH LINE AND STREAK LINE

3.7.1 Path Line

Path line is the path taken by a single fluid particle at a given time. Path line thus shows the direction of the velocity of same particle at successive instants of time (figure 3.18).

Figure 3.10 : Path line

$$u = \frac{dx}{dt},\ v = \frac{dy}{dt} \text{ and } w = \frac{dz}{dt}$$

or $x = \int u dt,\ y = \int v dt$ and $z = \int w dt$ (3.15)

This constitutes the equation of the path line.

3.7.2 Streak Line

A streak line is the locus of locations at an instant of time of all the fluid particles that has passed through a fixed point in the flow field.

SAQ 3

A streamline is a line

(i) connecting the mid points of flow cross sections

(ii) defined for uniform flow only

(iii) drawn normal to the velocity vector at every point

(iv) tangent to which at every point indicates the direction of flow

(v) showing the path of the particle.

SAQ 4

Velocity components for a three-dimensional flow is given by $u = -x$, $v = 2y$ and $w = 3 - z$. Find the equation of streamline passing through (1,1,2).

3.8 TRANSLATION, DEFORMATION AND ROTATION OF A FLUID ELEMENT IN MOTION

As a fluid element moves, it can be translated, deformed or rotated as shown in figure 3.11. Of the four possible modes of changes the fluid element undergoes the translation and rotation does not impose any stress on the system and hence not of much use in fluid dynamics. The angular deformation is shear strain which gives rise to tangential stresses and thus very important to be considered.

The linear deformation results in change in volume. Let us derive expressions for all these four possible changes the fluid element undergoes during its motion.

3.8.1 Translation

Let the velocity at the left hand lowermost corner of the fluid element have components u along x-axis and v along y-axis. In time δt, the element will occupy new position such that along x-axis it has moved a distance $u\delta t$ and along y-axis a distance $v\delta t$. This amply describes the translation of the fluid element [figure 3.11(a)].

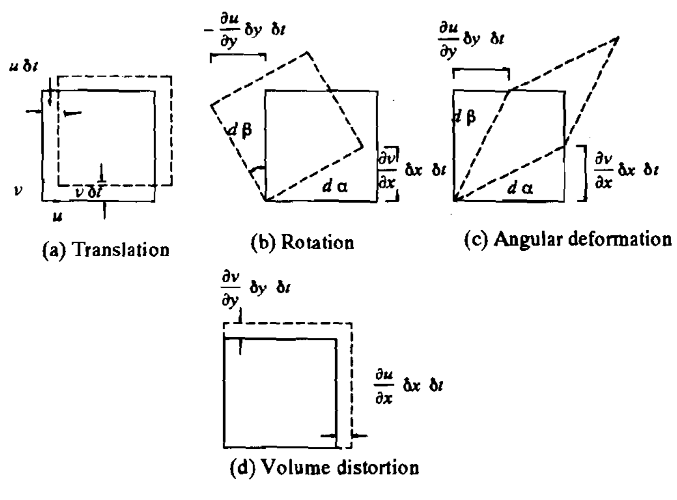


Figure 3.11 : Changes the fluid element undergoes while in motion

3.8.2 Rotation

Let us consider anti-clockwise rotation as positive [figure 3.11 (b)]. The rotation depends on the rate $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$. For rotation as shown in figure 3.11 (b) $\frac{\partial v}{\partial x}$ is positive since v increases with increase in x. Whereas $\frac{\partial u}{\partial y}$ is negative because, u is in negative direction with increase in y.

$$d\alpha = \frac{\frac{\partial v}{\partial x}\delta x \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t \tag{3.16}$$

Similarly

$$d\beta = -\frac{\partial u}{\partial y}\delta t$$

Therefore, the average rate of rotation in the positive direction will be

$$\omega_z = \left(\frac{d\alpha + d\beta}{2}\right)\frac{1}{\delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3.17}$$

Similarly the rotation about axes parallel to x and y can be obtained

as $$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \text{ and } \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{3.18}$$

This rotation is related to vorticity which will be further explained in Unit 4.

The resultant rotation will be

$$\vec{\omega} = \vec{i}\,\omega_x + \vec{j}\,\omega_y + \vec{k}\,\omega_z \tag{3.19}$$

or $$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V}) = \frac{1}{2}\text{curl } \vec{V} \tag{3.19}$$

It will be shown in Unit 4 that, for irrotational flow these components of rotation are zero.

3.8.3 Angular Deformation

In figure 3.11 (c) $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ are positive. $d\alpha$ is positive while $d\beta$ is negative. This is because the face is rotated clockwise. Thus, the rate of angular deformation is

$$e_z = (d\alpha - d\beta)\frac{1}{\delta t} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \text{ parallel to } z\text{-axis}$$

Similarly $$e_x = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \text{ parallel to } x\text{-axis}$$

and $$e_y = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \text{ parallel to } y\text{-axis} \tag{3.20}$$

As stated earlier, these rates of angular deformations cause tangential stresses.

3.8.4 Volume Deformation

The rates of linear deformation of each of the three sides of a cube [figure 3.11 (d)] of sides δx, δy and δz are $\frac{\partial u}{\partial x}\delta x$, $\frac{\partial v}{\partial y}\delta y$ and $\frac{\partial w}{\partial z}\delta z$ respectively. The increment in volume by each of these deformation are $\frac{\partial u}{\partial x}\delta x\delta y\delta z$, $\frac{\partial v}{\partial y}\delta x\delta y\delta z$ and $\frac{\partial w}{\partial z}\delta x\delta y\delta z$. Therefore, the rate of increase of entire volume per unit volume will be

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{3.21}$$

This equation is called dilatation or div $\vec{V}$ or $\nabla \cdot \vec{V}$. It will be proved later in this unit that for incompressible fluid flow, div $\vec{V} = 0$. This relation is also the elementary continuity equation.

3.9 ACCELERATION : TOTAL ACCELERATION— LOCAL AND CONVECTIVE COMPONENTS OF ACCELERATION

To obtain equations of motion for a fluid flow we have to make use of Newton's second law of motion. This needs proper definition of acceleration in terms of time t and coordinates x, y, z of space.

Our knowledge of elementary mechanics tells us that velocity is the rate of change of displacement with time. Thus, one writes the velocity component say u as a function of x, y, z and t. This is in close agreement with Eulerian approach which is used in Fluid Mechanics. It is also a well known fact that any property of the fluid element which is a function of space and time will change as it moves from point to point due to (i) passage of time and (ii) variation in its position.

Let us suppose that the fluid element is at a position (x, y, z) at any time t. $F(x, y, z, t)$ describes completely its position at that instant. After an increment of time δt, the fluid element has reached a new position $(x + u\delta t,\ y + v\delta t,\ z + w\delta t)$. Thus, the new value of the function is the old value of the function plus the change due to time and space variations.

$$F(x + u\delta t\ ,\ y + v\delta t,\ z + w\delta t) = F(x,y,z,t) + \frac{\partial F}{\partial t}(\delta t) + \frac{\partial F}{\partial x}(u\delta t) + \frac{\partial F}{\partial y}(v\delta t) + \frac{\partial F}{\partial z}(w\,\delta t) \qquad (3.22)$$

This can be rewritten as

$$F + \left(\frac{DF}{Dt}\right)\delta t = F + \frac{\partial F}{\partial t}\delta t + \frac{\partial F}{\partial x}(u\delta t) + \frac{\partial F}{\partial y}(v\delta t) + \frac{\partial F}{\partial z}(w\delta t) \qquad (3.22)$$

This simplifies to

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \qquad (3.23)$$

In the above expression $\frac{D}{Dt}$ stands for total derivative or substantial derivative.

or

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (3.24)$$

Vectorically this equation can be expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla \qquad (3.25)$$

The first term on the right hand side of the equation gives rate of change of a quantity at a fixed point whereas the second term gives the rate of change of quantity at a fixed time t. We will come to this point a little later.

If the above function F represents velocity vector $\vec{V}(x, y, z, t)$ then $\frac{D\vec{V}}{Dt}$ will indicate acceleration that the fluid element undergoes as it passes a given point. Thus, the accceleration vector $\vec{a}$ is

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\ \vec{V} \qquad (3.26)$$

where

$$\vec{a} = \vec{i}\,a_x + \vec{j}\,a_y + \vec{k}\,a_z \qquad (3.27)$$

The three components of acceleration vector are

a_x, acceleration along x–axis is $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

a_y, acceleration alon y–axis is $\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$

a_z, acceleration along z-axis is $\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$ (3.28)

In the above, the terms $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}$ represent "local accelerations" at a point where as the remaining terms $\vec{V}\cdot\nabla\ \vec{V}$ or $\vec{V}$ grad $\vec{V}$ represent convective acceleration.

$\frac{\partial u}{\partial t}$ is the local acceleration i.e. the time derivative of u at a certain place $A(x,y,z)$ in the field of flow. In steady flow, it is zero. The difference between substantial acceleration and local acceleration is "Convective Acceleration" so called because of the change in the velocity component due to convection movement of the fluid particle. This is generally not zero even in steady flow.

Further if we consider two-dimensional steady flow, the acceleration components a_x and a_y becomes

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

and

$$\frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad (3.29)$$

3.10 EQUATION OF CONTINUITY

Let us consider a rectangular element of fluid having lengths δx, δy and δz parallel to the coordinate axes x,y,z. Let the velocity components at the centroid $P(x, y, z)$ be u, v, w. Let ρ be the specific mass of fluid.

The rate at which the fluid is entering the surface $ABCD$ is $\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\ \delta z$. This is based on the assumption that the rate of change of mass is linear.

Similarly, the rate at which the fluid is leaving the surface $EFGH$ is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\ \delta z.$$

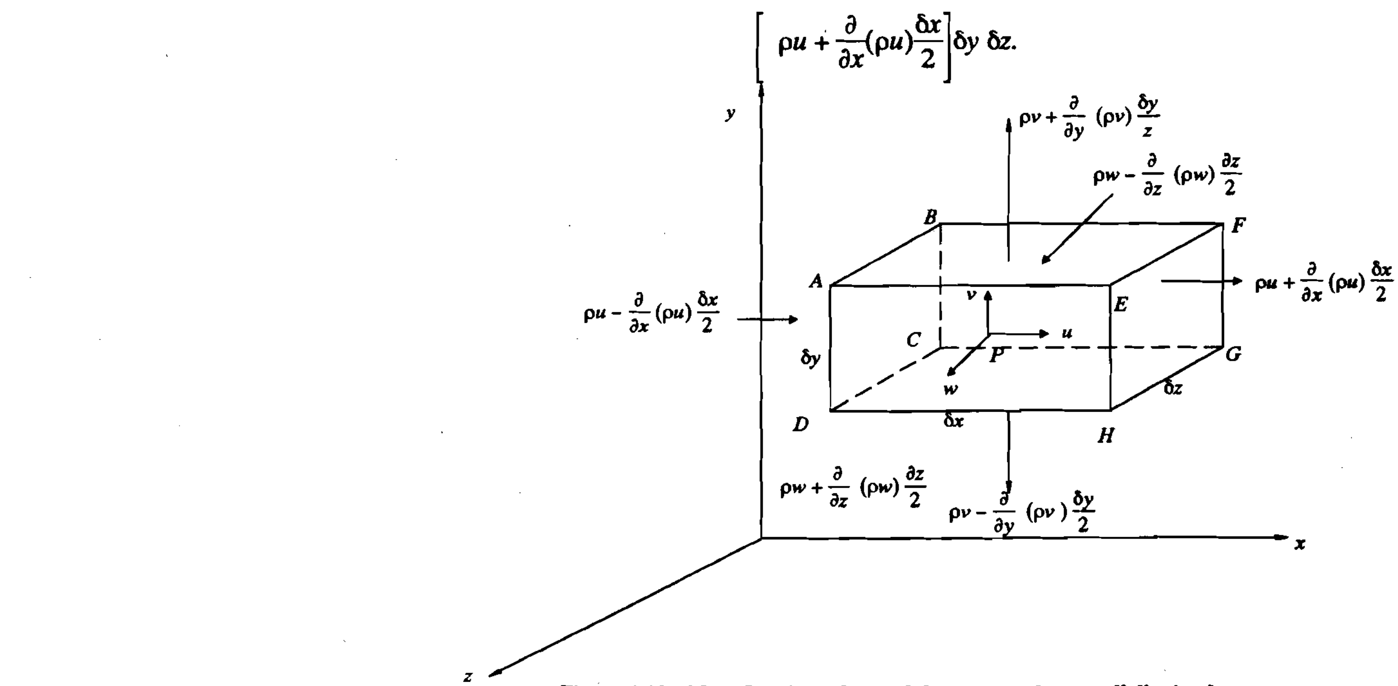


Figure 3.12 : Mass flow in and out of the rectangular parallellopiped

Therefore, the net rate at which the fluid is gained between the two faces in the x direction is the difference between the rate at which the fluid is entering the surface $ABCD$ and the rate at which the fluid is leaving the surface $EFGH$.

Thus, the rate of gain of mass in x direction is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\ \delta z$$

$$= -\frac{\partial}{\partial x}[\rho u]\delta x\ \delta y\ \delta z \qquad (3.30)$$

Similarly, the rates at which the fluid is gained between the faces along y and z axes will be

$$-\frac{\partial}{\partial y}[\rho v]\delta x\ \delta y\ \delta z \text{ and } -\frac{\partial}{\partial z}[\rho w]\delta x\ \delta y\ \delta z \text{ respectively.}$$

Thus, the rate at which the matter is gained in the element is the sum of the three gains, that is

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\ \delta y\ \delta z \qquad (3.31)$$

The rate at which the mass is increasing is

$$\frac{\partial}{\partial t}(\rho\ \delta x\ \delta y\ \delta z) \qquad (3.32)$$

$\rho\delta x\delta y\delta z$ represents the mass contained in the rectangular parallellopiped.

So, the rate at which the mass is increasing in the element must equal to the rate at which the matterr is gained within the fluid element.

$$\therefore \quad \frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\ \delta y\ \delta z$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (3.33)$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \qquad (3.34)$$

$$\text{or} \quad \nabla \cdot (\rho\vec{V}) + \frac{\partial \rho}{\partial t} = 0 \qquad (3.35)$$

These two equations (3.34 & 3.35) are known as **Equation of Continuity.**

Equation (3.34) is the general form of continuity equation while the equation (3.35) is the vectorial form of the continuity equation.

For incompressible fluid flow specific mass remains constant and the equation of continuity becomes

$$\nabla \cdot \vec{V} = 0 \text{ or div } \vec{V} = 0 \qquad (3.36)$$

$$\text{or} \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (3.37)$$

Thus, for incompressible fluid flow the continuity equation is given by equation (3.36) or (3.37).

For any possible fluid motion, steady or unsteady, uniform or non-uniform, the equation of continuity must be satisfied.

Example 2 :

A fluid flow field is given by $\vec{V} = x^2y\,\vec{i} + y^2z\,\vec{j} - (2xyz + yz^2)\vec{k}$. Prove that it is a case of possible steady incompressible flow. Calculate the velocity and acceleration at the point (2, 1, 3).

Solution :

In the given flow field

$u = x^2y, \; v = y^2z$ and $w = -(2xyz + yz^2)$

(i) For a case of possible steady incompressible flow, the elementary continuity equation must be satisfied.

$$\text{i.e.} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{i.e..} \qquad 2xy + 2yz - 2xy - 2yz = 0$$

Hence, the given flow field is a possible case of steady incompressible flow since it satisfies the continuity equation.

(ii) Velocity at (2, 1, 3)

$$\vec{V} = x^2y\,\vec{i} + y^2z\,\vec{j} + (-2xyz - yz^2)\,\vec{k}$$

$$= 2^2 \cdot 1\,\vec{i} + 1^2 \cdot 3 \cdot \vec{j} + (-2 \cdot 2 \cdot 1 \cdot 3 - 1 \cdot 3^2)\,\vec{k}$$

$$= 4\vec{i} + 3\vec{j} - 21\vec{k}$$

Resultant velocity $= \sqrt{4^2 + 3^2 + (-21)^2}$ units

$= \sqrt{16 + 9 + 441}$ units $= \sqrt{466}$ units $= 21.59$ units

(iii) Acceleration at (2, 1, 3)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = x^2y \cdot 2xy + y^2z \cdot x^2 + (-2xyz - yz^2) \cdot 0$$

$$= 2x^3y^2 + x^2y^2z = 2 \cdot (2)^3 \cdot 1^2 + 2^2 \cdot 1^2 \cdot 3 = 16 + 12 = 28 \text{ units}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = x^2y \cdot 0 + y^2z \cdot 2yz - (2xyz + yz^2)y^2$$

$$= 2y^3z^2 - 2xy^3z - y^3z^2 = 2 \cdot 1^3 \cdot 3^2 - 2 \cdot 2 \cdot 1^3 \cdot 3 - 1^3 \cdot 3^2$$

$$= 18 - 12 - 9 = -3 \text{ units}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = x^2y \cdot (-2yz) + y^2z(-2xz - z^2) - (2xyz + yz^2)\,(-2xy - 2yz)$$

$$= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + 4x^2y^2z + 4xy^2z^2 + 2xy^2z^2 + 2y^2z^3$$

$$= 123 \text{ units}$$

$$\therefore \qquad \vec{a} = 28\vec{i} - 3\vec{j} + 123\vec{k}$$

Resultant acceleration $= \sqrt{(28)^2 + (-3)^2 + (123)^2}$ units

$= 126.18$ units

Example 3 :

Given below are two components of velocity of a 3-Dimensional steady incompressible fluid flow. Determine the third velocity component.

$$u = x^2 + y^2 + z^2, \quad v = xy^2 - yz^2 + xy$$

Solution :

Since, they are the velocity components of a 3-dimensional steady, incompressible fluid flow they must satisfy the continuity equation.

$$\text{viz} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{i.e.} \qquad 2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

$$\text{i.e.} \qquad \frac{\partial w}{\partial z} = -3x - 2xy + z^2$$

Integrating we get,

$$w = (-3xz - 2xyz + \frac{z^3}{3}) + \text{constant of integration}$$

Here, the constant of integration can not be a function of z. But, it can be a function of x and y.

$$\therefore \quad w = (-3xz - 2xyz + \frac{z^3}{3}) + f(x, y)$$

is the missing velocity component.

SAQ 5

The following velocity components for steady, incompressible flow are

(i) $u = 2x - 3y$, $v = x - 2y$ and $w = 0$

(ii) $u = 2x^2 - xy + z^2$, $v = x^2 - 4xy + y^2$ and $w = 2xy - yz + y^2$

(iii) $u = 2x^2 + y^2$ and $v = -4xy$

Is the equation of continuity satisfied in these cases?

3.11 SUMMARY

In this unit, we have learnt that

* There are so many ways the fluid flow can be classified.
* In one-dimensional flow the dependent fluid variables are functions of only one space coordinate.
* In 2-dimensional and 3-dimensional flows they are functions of 2 and 3 space coordinates respectively.
* Eulerian approach to describe the flow field is more practical and hence used extensively in fluid mechanics.
* Streamline is a line tangent to which at every point gives the direction of flow.
* Translation and Rotation of fluid element do not impose any stress on the system.
* Angular deformation gives rise to tangential stresses.
* Acceleration consists of local and convective components.
* For any possible fluid motion, steady or unsteady, uniform or non-uniform, the equation of continuity must be satisfied.

3.12 ANSWERS TO SAQs

SAQ 1

(a) → (iv)

(b) → (ii)

(c) → (iii)

(d) → (i)

SAQ 2

(a) → (iii) because in one-dimensional flow only one velocity component in the direction of flow exists.

(b) → (v)

SAQ 3

(iv) by the definition of streamline.

SAQ 4

The differential equation of a streamline is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

In this case it is $\frac{dx}{-x} = \frac{dy}{2y} = \frac{dz}{3-z}$

i.e. $\frac{dx}{-x} = \frac{dy}{2y}$

Integrating $-\log x = \frac{1}{2}\log y + A$

Since, streamline is passing through $x = 1, y = 1$

i.e. $0 = 0 + A$ i.e. $A = 0$

$$\text{i.e. } x^{-1} = \sqrt{y} \text{ or } x = \frac{1}{\sqrt{y}} \qquad (1)$$

Considering the second differential equation

$$\frac{dx}{-x} = \frac{dz}{3-z}$$

Integrating $-\log x = -\log(3-z) + B$

Since streamline is passing through $x = 1$, $z = 2$, $B = 0$

$$x = 3 - z \qquad (2)$$

$$\therefore \quad x = \frac{1}{\sqrt{y}} = 3 - z$$

is the equation of streamline passing through (1,1,2).

SAQ 5

(i) Equation of continuity for a 3-D steady, incompressible fluid flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

i.e. $2 - 2 + 0 = 0$. Hence, this satisfies the continuity equation.

(ii) $4x - y - 4x + 2y - y = 0$. Hence this satisfies the continuity equation.

(iii) This is a 2-D flow

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4x - 4x = 0$. This too satisfies the continuity equation.