UNIT 10 SUPERPOSITION OF WAVES

Structure

10.1 INTRODUCTION

In Unit 9 we considered the propagation of a single wave in space. We discussed many examples of waves and saw that all of them satisfy the same wave equation given by Eqn.(9.3). In some specific cases we obtained expressions for the phase velocity in terms of the physical constants of the medium.

It is possible that two or more waves pass through the same region of space simultaneously. This is what happens when we listen to an orchestra. Sound waves from various musical instruments in the orchestra travel towards us and reach our ears. Similar consideration holds good if two pulses are generated on a single stretched string. There will be time when two pulses will pass through the same region of the string. In such a situation, the displacement in the string at that time will be the algebraic sum of the displacements due to each one of the waves acting alone. This is the *principle of superposition.* The principle is useful in many branches of physics like acoustics, electromagnetism, optics etc.

Unlike the waves on a stretched string or water waves, the electromagnetic waves do not require a medium for their propagation. As a result, there is no question of displacement of a particle. In this case, the wave is described by specifying the electric or the magnetic field in space, which varies in space and time like the displacement. Like in the case of waves with associated displacement, the superposition principle holds good for the electric field or the magnetic field, when more than one sources generate waves that propagate in the same region of space.

Mathematically, the superposition principle arises because of the linearity of the second order differential wave equation (9.3). Since each of the waves must satisfy the wave equation separately in the same region of space, a linear combination of the waves must also do the same. In particular a sum of two such solutions also satisfy the wave equation. In this unit, we look at the principle of superposition closely and discuss some interesting consequences. In the following unit on interference and diffraction the same principle will be used to explain important optical effects.

Objectives

After reading this unit you should be able to :

- explain the superposition principle and formation of standing waves (SAQs 2–4),
- explain the difference between phase velocity and group velocity (SAQs 5–6),
- calculate the resultant displacement due to several sinusoidal waves by use of complex variables and by graphical method (SAQs 7–9), and
- explain the connection between the superposition principle and the linearity of the wave equation (SAQ 10).

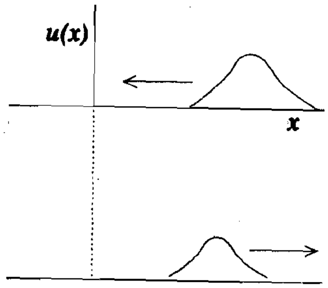

Figure 10.1 : Reflection of a wave pulse

10.2 STANDING WAVES

Consider a string fixed at one end at $x = 0$. Wave pulses are generated by keeping the string under tension and plucking it at some point. Figure (10.1) shows such a pulse travelling to the left. Till the pulse arrives at $x = 0$, its wave function is given by

$$y = f(x + vt)$$

Since the string is fixed at $x = 0$, we must have $y = 0$ for *all time* at $x = 0$. This can be achieved by adding to the above, a wave $g(x - vt)$ travelling to the right, so that the resultant displacement is given by

$$y = f(x + vt) + g(x - vt)$$

Since at $x = 0$, $y = 0$, we require

$$f(vt) = -g(-vt)$$

for all t. This is feasible if $g(\xi) = -f(-\xi)$ for all values of the argument ξ. Thus the total displacement is given by

$$y = f(x + vt) - f(vt - x)$$

The second term is a pulse reflected from the left. It is similar to the incident pulse but reversed in sign.

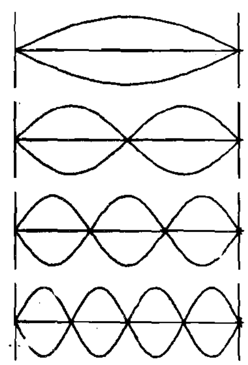

Figure 10.2 : Standing wave pattern on a string fixed at both ends

Suppose the string is fixed at two ends A and B as shown in Fig. 10.2. Instead of considering a general wave pulse, let us assume that on plucking the string, a sinusoidal wave is generated which propagates from A to B. We call this the incident wave and write it as

$$u_i = u_0 \sin \frac{2\pi}{\lambda}(x - vt) \tag{10.1}$$

On reaching B this wave is reflected back. Assuming that u_0, λ and v are unchanged by reflection, the reflected wave is

$$u_r = u_0 \sin \frac{2\pi}{\lambda}(x + vt) \tag{10.2}$$

Using the superposition principle, the total displacement is

$$\begin{aligned} u_t(x,t) &= u_i(x,t) + u_r(x,t) \\ &= u_0 \sin \frac{2\pi}{\lambda}(x - vt) + u_0 \sin \frac{2\pi}{\lambda}(x + vt) \\ &= 2u_0 \sin \frac{2\pi}{\lambda} x \cos \frac{2\pi}{\lambda} vt \end{aligned}$$

Therefore

$$u_t(x,t) = 2u_0 \sin kx \cos \omega t \tag{10.3}$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi v}{\lambda} = 2\pi v$. This is the wave function for a standing wave.

SAQ 1 :

Does $u_t(x,t)$ satisfy the wave equation 9.3 of Unit 9 ?

Example 1

Two sound waves are described by pressure changes given by

$$\begin{aligned} P_1(x,t) &= P_m \cos(kx - \omega t + \pi/4) \\ P_2(x,t) &= (P_m/2) \sin(kx - \omega t + \pi/4) \end{aligned}$$

What is the total pressure at (a) $x = \lambda, t = 0$ (b) $x = \lambda, t = 2\pi/3\omega$.

Solution :

The resultant pressure wave is

$$P(x,t) = P_m \cos(kx - \omega t + \pi/4) + (P_m/2)\sin(kx - \omega t + \pi/4)$$

Putting $P_m = A\sin\delta$ and $(P_m/2) = A\cos\delta$, so that $A^2 = 5P_m^2/4$ and $\tan\delta = 2$, we get

$$P(x,t) = P_m \frac{\sqrt{5}}{2} \sin(kx - \omega t + \frac{\pi}{4} + \delta)$$

Remembering that $k\lambda = 2\pi$, we get on substituting the numerical values, $P(\lambda, 0) = (3/2\sqrt{2})P_m$ and $P(\lambda, 2\pi\omega/3) = -0.224P_m$. □

SAQ 2 :

A wave is given by

$$u(x,t) = u_0 \sin(kx - \omega t)$$

Upon reflection, its amplitude is halved and its sign is reversed. Show that the resultant can be written as a sum of a travelling wave and a standing wave.

Since A and B are fixed points the displacements at A and B should be zero. At A, $x = 0, u_t(x,t) = 0$. For the displacement to be zero at B, i.e. at $x = L$, we must have $\sin kL = 0$, i.e.,

$$\lambda = \frac{2L}{n} \tag{10.4}$$

where $n = 1, 2, 3 \ldots$. The fundamental mode of vibration is given by $n = 1$ while $n = 2, 3, 4 \ldots$ give the harmonics. The various modes of vibration for $n = 1, 2, 3, 4$ are shown in Fig.10.2.

Note that the wave pattern does not move and hence the name *standing waves.* $2u_0 \sin kx$ can be regarded as the amplitude of SHM of an element of the string located at the position x. Since the amplitude itself is a sine function of position, in addition to the end points, there are some points on the string where the displacement is zero at all times. Such points are called *nodes.* The locations of these points are given by

$$\sin \frac{2\pi}{\lambda} x = 0$$

The condition is satisfied at

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots \tag{10.5}$$

The nodes are thus equally spaced with a separation $\frac{\lambda}{2}$ between two adjacent ones.

Between two such nodes, there are points where the displacement is a maximum. Such points are called *anti nodes.* Unlike the nodes where the displacement has the same value (zero) at all times, at the antinodes, the displacement varies with time sinusoidally. At any given time however, the displacement at the antinode is the greatest (excepting at such times when the entire string has zero displacement). For antinodes, therefore

$$\sin \frac{2\pi}{\lambda} x = \pm 1$$

which is satisfied at values of x given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \tag{10.6}$$

The allowed frequencies of vibration are

$$\nu = \frac{v}{\lambda} = \frac{v}{2L} n \tag{10.7}$$

where $n = 1, 2, , 3, \ldots$. Thus in order to set up standing waves in the string, the external agency which is exciting the waves must vibrate with one of the allowed frequencies. The string is then said to *resonate* at such a frequency.

Example 2

A string which is fixed at both ends is found to resonate at 175 Hz and 210 Hz but at no intermediate frequency. What is the fundamental frequency and at what frequencies less that 175 Hz does the string resonate ?

Solution :

The highest common factor of 175 and 210 is 35. Thus the fundamental could be all factors of 35, viz., 1,5,7 and 35 Hz. However, as there is no resonance frequency between 175 Hz and 210 Hz, 35 Hz is the fundamental. The string will also vibrate at 70, 105 and 140 Hz. □

You may recall that the total energy density in a wave is given by $(1/2)mA^2\omega^2$, where A is the amplitude of the wave. For a standing wave we get

$$\begin{aligned} \langle E \rangle &= \frac{1}{2} m A^2 \omega^2 \\ &= \frac{1}{2} m (2u_0)^2 \omega^2 \sin^2 kx \end{aligned} \tag{10.8}$$

which gives

$$\begin{aligned} \langle E \rangle &= 2m\mu_0^2\omega^2 \qquad \text{for } x = \frac{\lambda}{4}, \frac{3\lambda}{4} \ldots \\ &= 0 \qquad \text{for } x = 0, \frac{\lambda}{2}, \lambda, \ldots \end{aligned} \tag{10.9}$$

Thus for the nodes $\langle E \rangle = 0$ and for the antinodes $\langle E \rangle$ is maximum. Hence no energy is transferred across the nodes. Unlike in the case of a travelling wave, the energy is not transferred from one point to another in a standing wave. It is merely redistributed within each lobe.

Like the waves on a stretched string, standing waves of sound can be set up in a pipe. In case of a pipe, the reflection is not complete, but if the frequency of the sound wave matches a resonant frequency, pronounced standing wave pattern may be established. If the pipe is open at both ends, a displacement antinode will occur at each of the ends. The modes are shown in Fig.(10.3) for an organ pipe open at both ends. In such a case, the allowed wavelengths are given by

$$\lambda = \frac{2L}{n}$$

where n is an integer $1, 2, \ldots$. The corresponding resonant frequencies are

$$\nu = \frac{vn}{2L} \tag{10.10}$$

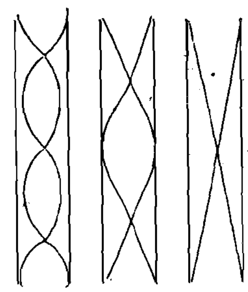

Figure 10.3 : Modes in an open organ pipe

For the case of a pipe closed at one end, the modes are shown in Fig.(10.4). The closed end is a node while the open end is an antinode. Thus

$$L = \frac{(2n-1)\lambda}{4}$$

The resonant frequencies are

$$\nu = \frac{v(2n-1)}{4L} \tag{10.11}$$

Example 3

A tube closed at one end and open at the other, resonates with a sound wave of frequency 135 Hz and 165 Hz but not with any intermediate frequency. Find the fundamental frequency and all the other frequencies below 135 Hz at which the pipe resonates.

Solution :

The highest common factor of 135 and 165 is 35. The fundamental frequency could therefore be 1, 3, 5 or 15. The first three are not acceptable for then there would have been resonant frequencies between 135 Hz and 165 Hz. 15 Hz is therefore the fundamental frequency. In case of a pipe with one end closed the separation between adjacent frequencies are twice the fundamental. Thus the other frequencies are 45, 75 and 105 Hz. □

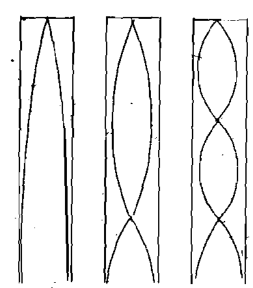

Figure 10.4 : Resonance in a pipe with one end closed

SAQ 3 :

A hollow tube which is open at both ends resonates at 640 Hz and 800 Hz but at no intermediate frequencies. Determine the fundamental frequency and all the other resonating frequencies below 640 Hz.

SAQ 4 :

A tube which is open at both ends has a fundamental frequency of 750 Hz. If one end of the tube is closed, what will be the three lowest frequencies at which the tube will resonate ?

The discussion of standing waves suggests a simple method for measuring the wavelength of a wave. All one has to do is to locate the nodes and antinodes. This idea was exploited by Wiener in 1890 to measure the wavelength of visible light. Since light is an electromagnetic wave (with wavelength between 400 nm and 700 nm), it will form standing waves if reflected by a plain mirror. Since the wavelength for visible light is very small, Wiener used an ingeneous idea to measure it. Figure 10.5 illustrates Wiener's experimental set up. Light incident normally on a plane mirror MM′ is reflected back giving rise to a standing wave pattern. In Fig.(10.5) the positions of the nodes are shown by a set of parallel lines. A thin transparent photographic film 10^{-6} cm thick was used to observe the standing waves. The film was inclined at an angle α with the mirror. The distance between two nodes on the plate will increase if α is decreased. On developing the film a series of dark lines indicating nodal lines was found. The distance between the successive nodes was measured by an ordinary optical microscope. Wiener found this distance to be $\approx 2.75 \times 10^{-2}$ cm corresponding to a value of $\alpha \approx 10^{-3}$ radians. Since α is small, we can replace it by its sine, so that

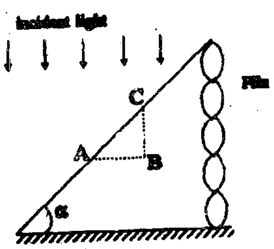

Figure 10.5 : Wiener's experiment

$$\frac{\lambda}{2} = \mathrm{BC} = \mathrm{AC}\sin\alpha = \mathrm{AC}\alpha$$

Taking the measured values of AC and α, Wiener measured the wavelength to be 550 nm, which is the wavelength corresponding to the green colour. Wiener not only demonstrated the existence of standing waves for light but also obtained a value of λ consistent with the mean λ for visible light.

10.3 GROUP VELOCITY

In Unit 8, we briefly mentioned the phenomenon of beats, which occurs when two SHMs of slightly different frequencies superpose on each other. It was found that the resultant amplitude becomes a sinusoidal function with an angular frequency equal to the difference between the frequencies of the two SHMs. Suppose the two oscillators concerned are used to generate wave disturbances. Because of the linearity of the wave equation, superposition principle applies.

Let us consider two harmonic waves of frequencies ω_1 and ω_2. The corresponding wave numbers will also differ. Let us denote them by k_1 and k_2 respectively. If we assume that the two waves have the same amplitude u_0 and that both are travelling in the x–direction, the superposition principle gives the resultant wave disturbance to be

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) \\ &= u_0\cos(k_1x - \omega_1 t + \phi_1) + u_0\cos(k_2x - \omega_2 t + \phi_2) \\ &= 2u_0\cos(\frac{\Delta kx - \Delta\omega t + \Delta\phi}{2})\cos(kx - \omega t + \theta) \end{aligned} \quad (10.12)$$

where we have used

$$k = \frac{k_1 + k_2}{2} \qquad \omega = \frac{\omega_1 + \omega_2}{2} \qquad \text{and } \theta = \frac{\phi_1 - \phi_2}{2}$$

and $\Delta k = k_2 - k_1$ $\Delta\omega = \omega_2 - \omega_1$ and $\Delta\phi = \phi_2 - \phi_1$. The assumption of equal amplitudes is not crucial to our discussion. If we do not assume this, only the algebra gets more complicated. We can write $u(x,t)$ as

$$u(x,t) = u_m\cos(kx - \omega t + \theta) \quad (10.13)$$

where

$$u_m = 2u_0\cos\left(\frac{1}{2}(\Delta kx - \Delta\omega t + \Delta\phi)\right) \quad (10.14)$$

It is trivial to see that this equation yields the standing waves and beats as special cases. Equation (10.13) describes a wave motion with an average frequency ω (which reduces to ω_1 if $\omega_1 \approx \omega_2$) and a modulated amplitude u_m given by Eqn.(10.14). The modulated amplitude is a function of space and time.

Beats are obtained as a solution to Eqns.(10.13) and (10.14) if we assume that the beat frequency, $\Delta\omega = \omega_2 - \omega_1$, is small compared to the average frequency ω. In this case, the resultant wave disturbance has a fast moving frequency ω and a short wavelength $2\pi/k$. This is called the *carrier wave.* The carrier wave propagates at the average phase velocity given by $v = \omega/k$. The amplitude factor u_m is also a wave. This is called the *amplitude modulation wave.* At $x = 0$, it has the time dependence given by

$$u_m(0,t) = 2u_0\cos\left(\frac{1}{2}(\Delta\omega t + \Delta\phi)\right)$$

This is the time variation obtained for beats in Unit 8. At any other x, the time dependence remains the same, but the phase is shifted by Δkx. Obviously, this phase difference is due to the time required by the beat to travel from the source to the position x. To find the speed with which the beat propagates, we note that the phase of the beat at $x = 0$, $t = 0$ will appear at x at time t. Thus $\Delta\omega t - kx$ = constant. Differentiating this, we find that the beat propagates with a speed

$$v_b = \frac{\Delta\omega}{\Delta k}$$

In general, Eqns.(10.13) and (10.14) describe a wave propagation where the amplitude itself is modulated in a wave like fashion. This wave is shown in Fig. 10.6 where we have plotted the time dependence and supressed the x dependence. We could also have suppressed the time dependence and plotted the x dependence. We would obtain a similar variation with the faster variation coming from k (instead of ω and a slower variation coming from Δk (instead of $\Delta\omega$). In a general case both variations are present. Thus we see that the amplitude is now a function of space and time and gives rise to a wave envelope of a modulated frequency $\Delta\omega$ which advances in space and time and travels at a certain velocity which is called the ***group velocity*** or the modulation velocity. The word group indicates the presence of more than one wave. Frequently we refer to this as a ***wave packet***

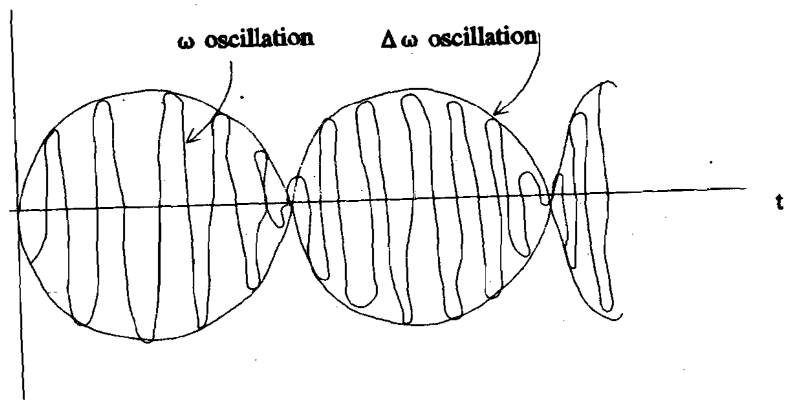

Figure 10.6 : Combination of two harmonic waves

To calculate the group velocity, recall that the displacement describing a wave is expressible in a form $(x \pm vt)$. Equation (10.13) is an equation which has the same form. From this equation we get, for the phase velocity

$$v = \omega/k \tag{10.15}$$

From Eqn.(10.14), which represents the modulation, the group velocity is

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \tag{10.16}$$

Clearly, for a monochromatic wave there is no modulation as $u_m = u_0$; we have only the phase velocity. Moreover, for a monochromatic wave $\omega = vk$, $v_g = v$, i.e., the group velocity is the same as the phase velocity. However, for light propagating in a medium we have to distinguish between group and phase velocity because light of different wavelengths have different speeds. This phenomenon is called ***dispersion*** and the medium is termed dispersive. If we regard vacuum as a medium, it is a nondispersive medium (so $v_g = v$).

We will now obtain a relationship between the phase velocity and the group velocity in a dispersive medium. Since $\omega = vk$

$$v_g = \frac{d}{dk}(vk) = v + k\frac{dv}{dk} \tag{10.17}$$

As the wave number k is related to the wavelength λ through $k = 2\pi/\lambda$, this equation can be written as

$$v_g = v - \lambda\frac{dv}{d\lambda} \tag{10.18}$$

You should realise that the wavelength in this equation is the wavelength of light in the medium, and not the wavelength in vacuum. The two are of course related by the refractive index of the medium. However, we should be careful in using the refractive index because it also depends on the frequency of light.

For most dispersive media, the index of refraction increases with frequency, i.e., decreases with increase of wavelength. As a result $d\mu/d\lambda_0$ is negative. For such media, the group velocity is smaller than the phase velocity.

Clearly, the group velocity itself is a function of the frequency. However, if a given modulated wave occupies a narrow range of frequencies, the group velocity may be considered well defined for the packet.

SAQ 5 :
In terms of the wavelength λ_0 of light in vacuum and the refractive index μ of the medium, show that the group velocity and the phase velocities are related by

$$\frac{1}{v_g} = \frac{1}{v} - \frac{\lambda_0}{c}\frac{d\mu}{d\lambda_0} \tag{10.19}$$

Example 4
The refractive index of a block of glass varies with wavelength according to the empirical relation

$$\mu = 1.5 + \frac{3 \times 10^4}{\lambda_0^2}$$

where λ_0 is the vacuum wavelength expressed in nm. Find the group velocity at $\lambda_0 = 500$ nm.

Solution :
Differentiating the above expression

$$\frac{d\mu}{d\lambda_0} = -\frac{6 \times 10^4}{\lambda_0^3}$$

Substituting the values of λ_0 in Eqn.(10.19), the group velocity is found to be 1.61×10^8 m/s. □

Example 5
Surface waves in liquids at long wavelengths move with a phase velocity given by $v = \sqrt{g\lambda/2\pi}$. Find the group velocity.

Solution :
Since $k = 2\pi/\lambda$, we have $v = \sqrt{g/k}$. Using

$$v_g = v + k\frac{dv}{dk}$$

we get group velocity to be $v/2$. □

SAQ 6 :
Index of refraction of a substance is found to vary inversely as the vacuum wavelength. Show that the group velocity is half the phase velocity.

10.4 SUPERPOSITION OF MANY WAVES

It is clear from our previous discussions that it would be a formidable task if we had to calculate the total displacement of a large number of sinusoidal waves. In this section we will illustrate two methods for calculating the total displacement when many sinusoidal waves superpose.

Consider N sinusoidal waves. The total displacement is

$$\begin{aligned} U = & u_1 \sin(k_1 x - \omega_1 t + \phi_1) + u_2 \sin(k_2 x - \omega_2 t + \phi_2) \\ + & \ldots + u_N \sin(k_N x - \omega_N t + \phi_N) \end{aligned} \quad (10.20)$$

It would be impossible to obtain a simple analytical expression for U. So we make some simplifying assumptions, which are not really crucial to the basic physics involved. We assume that all k_1, ω_i and u_i are the same. Then

$$U = u \sum_{j=1}^{N} \sin(kx - \omega t + \phi_j) \quad (10.21)$$

We are trying to find the total displacement of N waves all having the same amplitudes, frequencies and propagation vectors but with different phases ϕ_j. Even in this case it is not possible to get a compact expression for U. If however $\phi_j = j\phi$, i.e., if the successive waves have constant phase diifference between them, considerable simplification can be achieved.

SAQ 7 :

Using elementary trigonometry calculate the sum

$$\begin{aligned} U = & \ u \sin(kx - \omega t + \phi) + u \sin(kx - \omega t + 2\phi) \\ + & \ \ldots\ldots + u \sin(kx - \omega t + N\phi) \end{aligned}$$

Now that you have gone through the lengthy and cumbersome algebra, we will illustrate an alternative method for calculating the total displacement. The method makes use of the complex representation of the exponential function

$$\exp(i\theta) = \cos\theta + i \sin\theta$$

Thus $\sin\theta$ is the imaginary part of $\exp(i\theta)$. We may therefore write Eqn.(10.21) as

$$\begin{aligned} U &= u\Im\left[\exp(ikx - i\omega t + \phi) + \exp(ikx - i\omega t + 2\phi) + \ldots + \exp(ikx - i\omega t + N\phi)\right] \\ &= u\Im\left[\exp(ikx - i\omega t + \phi)(1 + \exp(i\phi) + \exp(2i\phi) + \ldots + \exp(i(N-1)\phi)\right] \end{aligned}$$

where $\Im$ stands for the imaginary part of the argument. Summing the geometric series, we get

$$U = u\Im\left[\exp(ikx - i\omega t + \phi)\frac{1 - \exp(iN\phi)}{1 - \exp(i\phi)}\right]$$

which can be written as

$$U = u\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}} \sin(kx - \omega t + \frac{N+1}{2}\phi) \quad (10.22)$$

which is a very compact form for U.

We see that the argument of the sine is the same $(kx - \omega t)$ with a changed phase $(N+1)\phi/2$, and the amplitude is multiplied by $\sin(N\phi/2)/\sin(\phi/2)$. In the next unit we will see that this factor determines the intensity in a multiple slit diffraction grating.

There is another equally instructive way of obtaining the same result using the graphical method. In this method, we use the fact that a sine or a cosine function may be regarded as the projection of a vector on one of the axes. Suppose we wish to add two wave disturbances.

$$E_1 = E_0 \sin \omega t$$
$$E_2 = E_0 \sin(\omega t + \phi)$$

We represent the waves graphically by rotating vectors or *phasors*. E_1 can be regarded as the y - component of a vector of length E_0 which makes an angle ωt with the x - axis (Fig. 10.7). Likewise, E_2 is the y - component of a vector of same length which makes an angle $(\omega t + \phi)$ with the x - axis. Thus $E_1 + E_2$ is the projection of the sum of two vectors as shown in Fig.(10.7).

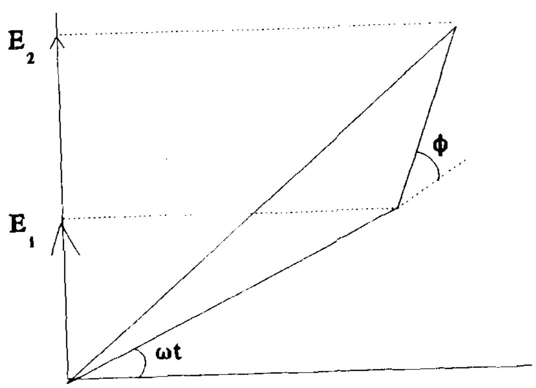

Figure 10.7 : Addition of wave disturbance by phasor method

This can be readily extended to N vectors. For simplicity, we consider the case where the disturbances $E_1, E_2, \ldots, E_N$ have a phase difference ϕ between any adjacent pair. When the vector are drawn end to end as shown in Fig.(10.8), the figure generated is N sides of a polygon. You can easily see that the angle between the resultant E and the x - axis is $(N-1)\phi/2 + \omega t$. Thus the projection of the resultant on the y - axis is

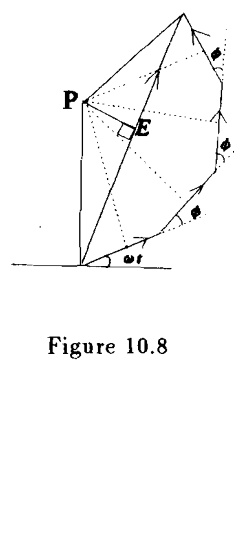

Figure 10.8

$$R \sin\left[(N-1)\frac{\phi}{2} + \omega t\right]$$

To determine the length of the resultant, we proceed as follows. Draw perpendicular bisectors to all the sides and let them meet at P (Fig.10.8). The angle between any two adjacent bisectors is ϕ. Thus the angle that the two end points O and A_N makes with P is $N\phi$. The resultant OA_N is given by $R = 2OP\sin(N\phi/2)$. Using the fact that $OP = E_0/2\sin(\phi/2)$. Thus the net disturbance is

$$E = E_0 \frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}} \sin(kx - \omega t + \frac{N+1}{2}\phi) \qquad (10.23)$$

which is the same as Eqn.(10.22).

Example 6
Find the resultant of

$$\begin{aligned} E_1 &= 5\sin(\omega t) \\ E_2 &= 5\sin(\omega t + 30^o) \\ E_3 &= 5\sin(\omega t + 60^o) \\ E_4 &= 5\sin(\omega t + 90^o) \end{aligned}$$

Solution :
In this case $N = 4$. The angle between two adjacent phasors is $\phi = 30^o$. Thus the amplitude is $5(\sin 60^o)/(\sin 15^o) = 16.73$. The resultant phase is $(\omega t + 45^o)$. Thus the resultant wave is $16.73\sin(\omega t + 45^o)$. □

SAQ 8 :
Add the following wave disturbances by method of phasors.

$$\begin{aligned} E_1 &= 5\sin(\omega t) \\ E_2 &= 5\sin(\omega t + 15^o) \\ E_3 &= 5\sin(\omega t + 30^o) \\ E_4 &= 5\sin(\omega t + 45^o) \end{aligned}$$

SAQ 9 :
Add the following wave disturbances

$$\begin{aligned} E_1 &= 10\sin(\omega t) \\ E_2 &= 10\sin(\omega t + 120^o) \\ E_3 &= 10\sin(\omega t + 240^o) \end{aligned}$$

10.5 WAVE EQUATION AND SUPERPOSITION

We had briefly mentioned that the superposition principle is a consequence of the linearity of the differential equation. In this section, we will try to clarify this statement.

A differential equation in u with respect to the variable t is said to be linear if the equation does not contain any power other than the first power of $t, du/dt, d^2u/dt^2$ etc. If the constant term in the differential equation is zero, the equation is called homogeneous. In our case, the equation is a linear, homogeneous, second order partial differential equation in two variables x and t :

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \quad (10.24)$$

Suppose $u_1(x,t)$ and $u_2(x,t)$ are solutions to the above equation. Then, we have

$$\frac{\partial^2 u_1}{\partial t^2} = v^2 \frac{\partial^2 u_1}{\partial x^2}$$

$$\frac{\partial^2 u_2}{\partial t^2} = v^2 \frac{\partial^2 u_2}{\partial x^2}$$

If we add these equations, we get

$$\frac{\partial^2 (u_1 + u_2)}{\partial t^2} = v^2 \frac{\partial^2 (u_1 + u_2)}{\partial x^2}$$

which is of the same form as Eqn.(10.23), with the identification $u = u_1 + u_2$. As a matter of fact, any linear combination of u_1 and u_2 is also a solution. The special case of the sum of two solutions is called superposition.

SAQ 10 :

Is the superposition principle valid for the following equation ?

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 \sin u}{\partial x^2}$$

10.6 SUMMARY

In this unit we discussed the principle of superposition, according to which, when two or more waves traverse the same region of space independently of each other, the resultant displacement at given point is given by the vector sum of the displacements caused by each wave independently. For an electromagnetic wave it is the electric or the magnetic field that has to be added vectorially.

We discussed the formation of standing waves in a stretched string and in organ pipes. Standing waves are formed by the superposition of an incident wave with a reflected wave of the same amplitude. We calculated the positions of nodes and antinodes in a standing wave pattern and found that these positions could be experimentally confirmed. The formation of standing wave was used by Wiener to determine the wavelength of light.

If the superposing waves have their frequencies within a small range, we can think of a wave packet which moves like a bundle of waves. This wave packet moves with a well defined velocity called the group velocity. For a dispersive medium the group velocity is quite different from the phase velocity.

We discussed the mathematical problem of calculating the total displacement when many waves superpose. Under some very simplifying assumptions we described two simple methods for obtaining the total displacement. These were the graphical method and a method based on the complex representation of the trigonometric functions. In the graphical method it was found that the individual waves could be regarded as projections of rotating vectors on one of the axes. Using elementary geometry, it was shown that the resultant wave disturbance is given as the projection of the vector sum of the rotating vectors on the axis.

10.7 ANSWERS TO SAQs

1. The reflected wave is given by

$$u_r(x,t) = -(u_0/2)\sin(kx+\omega t)$$

The resultant is therefore

$$u_0\sin(kx-\omega t) - (u_0/2)\sin(kx+\omega t)$$

which can be expressed as

$$[u_0\sin(kx-\omega t) - u_0\sin(kx+\omega t)] + (u_0/2)\sin(kx+\omega t)$$

The first two terms of the above result in a standing wave given by $2u_0\cos kx\sin\omega t$, while the third term is clearly a travelling wave.

2. Differentiating $u_t(x,t) = 2u_0\sin kx\cos\omega t$ we get

$$\begin{aligned}\frac{\partial u}{\partial u_t} &= -2u_0\omega\sin kx\sin\omega t\\ \frac{\partial^2 u_t}{\partial t^2} &= -2u_0\omega^2\sin kx\cos\omega t\\ \frac{\partial^2 u_t}{\partial x^2} &= -2u_0k^2\sin kx\cos\omega t\end{aligned}$$

Using these, the wave equation is seen to be satisfied. The phase velocity is $v=\omega/k$.

3. For a tube with both ends open, all multiples of the fundamental are resonating frequencies. The fundamental frequency is 160 Hz. The tube also resonates at 320 and 480 Hz.

4. $v/2L = 750$ gives the fundamental frequency when one end is closed to be $v/4L = 375$ Hz. As the other resonating frequencies are spaced at twice the fundamental, the tube resonates at 1125 and 1875 Hz.

5. The inverse of the group velocity may be written as

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{dk}(\frac{\omega}{v}) = \frac{1}{v} - \frac{\omega}{v^2}\frac{dv}{d\omega}$$

Use $v = c/\mu$ and $\omega = 2\pi c/\lambda_0$ to get the required answer.

6. Since $\mu = A/\lambda_0^2$, $d\mu/d\lambda_0 = -\mu/\lambda_0$. Use this in equation (10.19) to show that the group velocity is half of v.

7. To sum this series, it is convenient to group the terms. Sum the first and the last term, the second and last but one term, etc. The answer has been given in the text.

8. Following the method of example 6, the resultant is $19.15 \sin(\omega t + 22.5°)$.

9. The phasors form a closed figure (in this case a triangle). Hence the resultant is zero.

10. Consider two solutions u_1 and u_2. Write the corresponding equations and add them. You get

$$\frac{\partial^2}{\partial t^2}(u_1 + u_2) = v^2 \frac{\partial^2}{\partial x^2}(\sin u_1 + \sin u_2)$$

This equation does not have the form of the original equation because $\sin u_1 + \sin u_2 \neq \sin(u_1 + u_2)$. Thus the given equation does not lead to superposition.