UNIT 3 OSCILLATIONS

Structure

3.1 INTRODUCTION

Have you observed the motion of water on the ocean's surface? As a child you may have enjoyed playing on a swing. You might have seen the vibrations on guitar strings. There are many examples of such vibrations or oscillations around us. All in motions, which repeat themselves such as vibrations in your eardrum, guitar strings, in drums, in bells, quartz watches are examples of oscillatory motion. Some of these are obvious to us and some are not even detectable. However, the mathematical description is basically the same. In this unit, we will explain the simplest kind of oscillatory motion called the simple harmonic motion. You will learn about several systems that undergo this kind of motion. In addition, we discuss the compound pendulum and the torsional pendulum.

Suppose you set a swing in motion. If you stop pushing it, what do you observe after a while? It tends to stop. Why does this happen? Vibrations do not continue forever, i.e. real world oscillations are damped unless we continually pump the system (provide energy to the system). For example, in a pendulum clock, the spring acts as a pump, in a quartz watch, battery acts as a pump.

In most systems, we need to balance losses due to damping or friction. Dissipated energy is converted into heat and is lost.

You will, therefore, also study about damping in mechanical systems. In the next unit, we take up the study of sound.

Objectives

After studying this unit, you should be able to

- define a simple harmonic motion and the terms time period, frequency of periodic motion,
- calculate the time period, frequency of a simple harmonic motion,
- analyze the motion of a compound pendulum and torsional pendulum,
- distinguish between free and forced oscillations,
- analyze a free damped oscillation of a spring-mass system, and
- explain the phenomenon of resonance.

3.2 SIMPLE HARMONIC MOTION

A motion which repeats itself after equal intervals of time is called **periodic motion**. Many kinds of motion repeat themselves over and over; the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibration produced by a clarinet or an organ pipe, and the back and forth motion of the pistons in a car engine. We call this periodic motion or oscillation. Simple harmonic motion, abbreviated as SHM, is defined as the motion in which acceleration is directly proportional to the displacement from the mean position and the acceleration is directed opposite to the displacement.

A body is said to have Simple Harmonic Motion (SHM) if a body moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to its distance from the fixed point. This motion is in the form of to and fro reciprocating motion.

Examples of Simple Harmonic Motion are :

- reciprocating motion of piston in the gas cylinder,
- oscillation of simple pendulum, and
- vibration of simple mass and spring system.

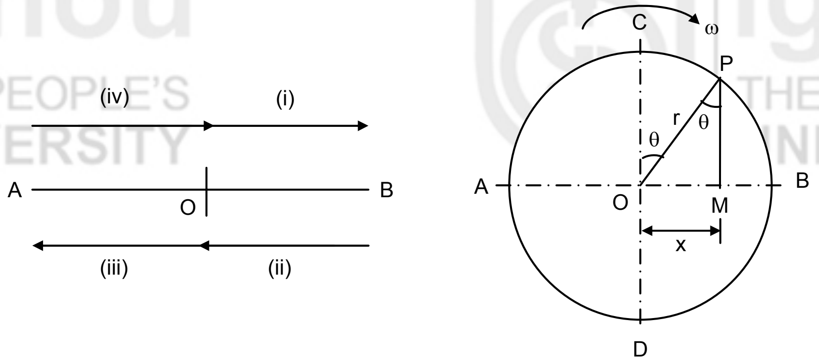
Simple harmonic motion is the most fundamental type of periodic motion.

The following important properties are to be noted if a particle is moving in Simple Harmonic Motion.

- The particular fixed point must be situated on the path of the motion.
- The acceleration of the particle is proportional to the displacement but in opposite direction.
- The acceleration of the particle, at any instant is always directed towards the fixed position.
- The frequency and period of motion are independent of amplitude.

3.2.1 Displacement, Velocity and Acceleration of a Particle Moving with Simple Harmonic Motion (SHM)

Let us consider a particle moving clockwise with a constant angular velocity ω along the circumference of a circle of radius *r*. The particle is initially at position *C* and at any time *t* shifts to position *P* making angle θ with the verticle. *M* is the foot of perpendicular drawn from *P* on the horizontal diameter *AB* of the circular orbit as shown in Figure 3.1.



(a) **(b)**

Figure 3.1 : Uniform Circular Motion

It is to be noted that :

(a) when the particle moves from C to B, the projection point M on diameter AB moves from O to B.

(b) when the particle moves from B to D, point M shifts from B to O.

(c) when the particle moves from D to A, point M moves from O to A, and

(d) when the particle moves from A to C, the point M shifts from A to O and that completes the cycle.

The motion of M along AB is repetitive and thus simple harmonic.

Thus, the displacement of the projection at any instant is given by

$$x = OM = r \sin \theta = r \sin \omega t \qquad \ldots (3.1)$$

when $\omega t = 0; \quad x = 0$

$$\omega t = \frac{\pi}{2} = 90^\circ; \quad x = r$$

$$\omega t = \pi = 180^\circ; \quad x = 0$$

The variation of displacement (x) with the angle of rotation ($\theta = \omega t$) has been shown in Figure 3.2(a).

$\therefore$ Maximum displacement, $x = r$, when $\omega t = 90^\circ$.

Velocity

If we differentiate the Eq. (3.1) w.r.t. time t, we get

$$v = \frac{dx}{dt} = \omega\, r \cos \omega t = \omega\, r \cos \theta$$

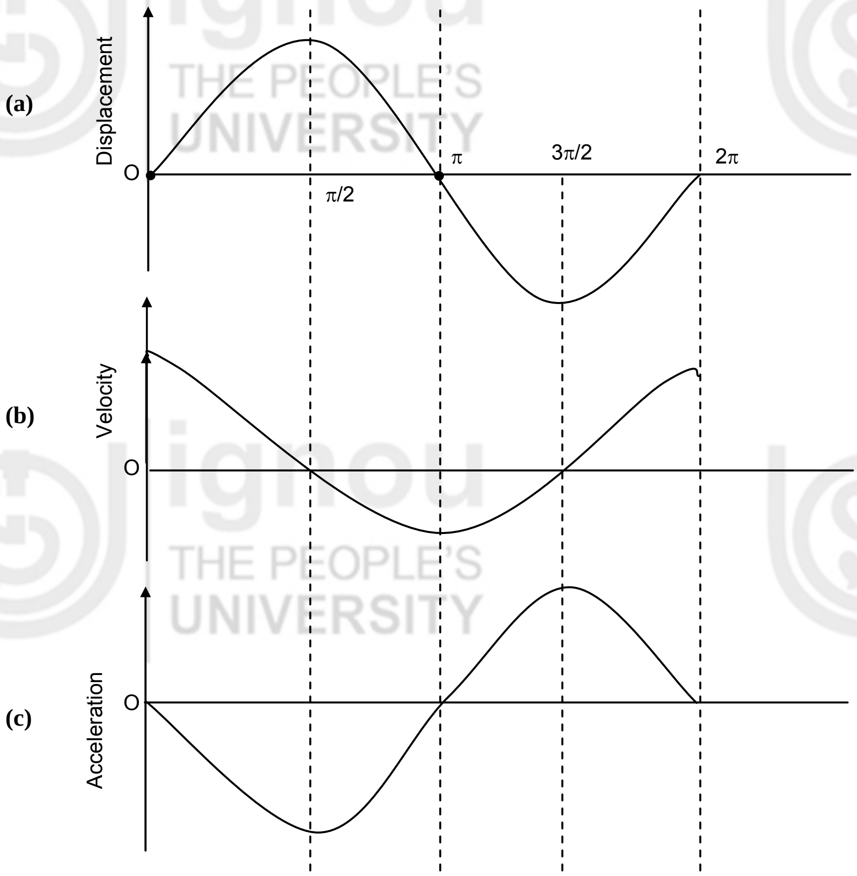


Figure 3.2

Velocity at any instant is expressed by the relation

$$v = \omega\, r \cos \omega t = \omega\, r \cos \theta \qquad \ldots (3.2)$$

when $\quad \omega t = 0; \quad v = \omega\, r$

$$\omega t = \frac{\pi}{2} = 90^o; \quad v = 0$$

$$\omega t = \pi = 180^o; \quad v = -\,\omega r$$

The variation of velocity v with angle of rotation $\theta = \omega t$ has been depicted in Figure 3.2(b).

The magnitude of maximum velocity $v_{max} = \omega\, r$, and this occurs when $\omega t = 0$ to 180^o.

Now, from the right angled triangle OPM (Figure 3.1(b)),

$$\cos \theta = \frac{PM}{OP}$$

$$= \frac{\sqrt{r^2 - x^2}}{r}$$

If we put the value of cos θ in the Eq. (3.2), we get

$$v = \omega \sqrt{r^2 - x^2}$$

Apparently the velocity will be maximum when $x = 0$, i.e. when the projection point lies at the mean position. We can also get, $v_{max} = \omega\, r$.

Acceleration

Again differentiating Eq. (3.2) w.r.t. time t, we get

$$\frac{dv}{dt} = a = -\,\omega^2\, r \sin \omega t$$

$$a = -\,\omega^2\, r \sin \theta$$

$$a = -\,\omega^2\, x \qquad (\text{since } x = r \sin \theta)$$

The acceleration is thus, directly proportional to distance x from the mean position. Further, the negative sign indicates that the direction of acceleration is opposite to the direction in which x increases, i.e. the direction is always directed towards the mean position.

when $\quad \omega t = 0; \quad a = 0$

$$\omega t = \frac{\pi}{2} = 90^o; \quad a = -\,\omega^2\, r$$

$$\omega t = \pi = 180^o; \quad a = 0$$

The variation of acceleration a with the angle of rotation $\theta = \omega t$ is shown in Figure 3.2(c). Maximum acceleration $a_{max} = -\,\omega^2\, r$ and it occurs at $\omega t = 90^o$, i.e. when the projection point is farthest from the mean position.

Example 3.1

A body oscillates with a simple harmonic motion along x-axis. Its displacement varies with time according to $x = 8 \cos \left(\pi t + \frac{\pi}{4}\right)$, where t is in second and angle is in radians.

(a) Determine amplitude, frequency and period of motion.

(b) Calculate velocity, and acceleration of the body at any time '*t*'.

(c) Using results of (b), determine the position, velocity and acceleration of the body at $t = 1$ second.

(d) Determine the maximum speed and acceleration.

(e) Find the displacement of the body between $t = 0$ and $t = 1$ second.

Solution

$$x = 8\cos\left(\pi t + \frac{\pi}{4}\right)$$

$$x = 8\sin\left(\pi t + \frac{3\pi}{4}\right)$$

$A = 8$; $\omega_n = \pi$ rad/sec.

so, $$f = \frac{\omega_n}{2\pi} = \frac{1}{2}\text{ Hz}$$

$$T = \frac{1}{f} = 2\text{ seconds}$$

$$v = \frac{dx}{dt} = -8\pi\sin\left(\pi t + \frac{\pi}{4}\right);\ a = \frac{dv}{dt} = -8\pi^2\cos\left(\pi t + \frac{\pi}{4}\right)$$

At $$t = 1;\ x = 8\cos\left(\pi + \frac{\pi}{4}\right) = 8\cos\left(\frac{5\pi}{4}\right) = -5.66\text{ m}$$

$$v = -8\pi\sin\left(\frac{5\pi}{4}\right) = 17.78\text{ m/s}$$

$$a = -8\pi^2\cos\left(\pi + \frac{\pi}{4}\right) = 55.8\text{ m/s}^2$$

$$v_{max} = 8\pi\text{ m/s},\ a_{max} = 8\pi^2\text{ m/s}^2$$

At $$t = 0;\ x_0 = 8\cos\left(0 + \frac{\pi}{4}\right) = 5.66\text{ m}$$

At $$t = 1\text{ sec},\ x = -2.83 \times 2 = -5.66\text{ m}$$

Hence, displacement from $t = 0$ to $t = 1$ second is

$$\Delta x = x - x_0 = -5.66 - 5.66 = -11.32\text{ m}$$

Since, the particles velocity changes sign during the first second, the magnitude of Δx is not the sum as the distance travelled in the first second.

3.2.2 Some Important Terms of SHM

Cycle

The movement of a particle executing SHM body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then to its extreme position in the other direction, and back to equilibrium position is called one **cycle of oscillation**. One revolution, i.e. angular displacement of 2π radian constitutes one cycle.

Amplitude

It is the magnitude of the maximum displacement of the particle in either direction of the mean position. The value of cosine function varies between $\pm$ 1 and so the displacement x (t) varies between $\pm A$.

Frequency

It is defined as the number of oscillations that are completed in one second. The symbol for frequency is f and its SI unit is hertz (abbreviated Hz), where

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}$$

Time Period

It is the time taken for one complete oscillation (one cycle). It is denoted by T :

$$T = \frac{1}{f}$$

Angular Frequency

It is defined as the angular velocity of the cyclic motion. It is denoted by ω and is measured in radians/second. It is also called circular frequency.

$$\omega = \frac{2\pi}{T}$$

Phase

The state or condition as defined by the vibrating particle in regard to its position, direction of motion is referred to as phase. It tells us the stage of the vibrating particle.

Phase Difference

The amount by which the phases of the particle differ is called phase difference.

Initial Phase

Eqs. (3.1) and (3.2) would be true if we start counting the time t when it is passing through the mean position. If, however, we start measuring time when the oscillating particle is already displaced from the mean position by some angle ϕ then ϕ is called the **initial phase**. In the following example, as shown in Figure 3.3, P_0 is the initial position of the particle at $t = 0$, and at any time t, the particle is occupying the position P. Here, the particle is moving in a circular orbit in a counter clockwise direction with a constant angular speed ω. The phase ϕ is an algebraic variable and can take both signs. However, if we impose the condition $y = A \sin \phi$ at $t = 0$, we have

$$y = A \sin (\omega\, t + \phi) \qquad \text{. . . (3.3)}$$

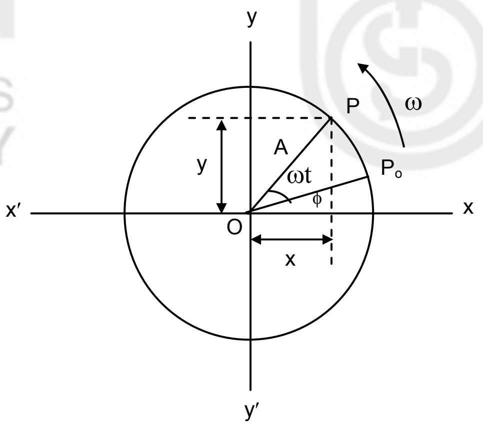
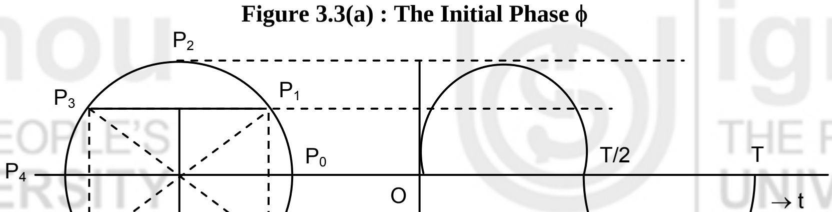


Figure 3.3(a) : The Initial Phase ϕ



(b) SHO with Initial Phase ϕ

Figure 3.3

However, since the system can always be made to oscillate again and *t* measured from the time when the oscillating particle is passing through the mean position, the introduction of the initial phase ϕ in the equation of only one oscillation can always be avoided.

In Figure 3.4, the curve represents a SHM having a phase of π/2.

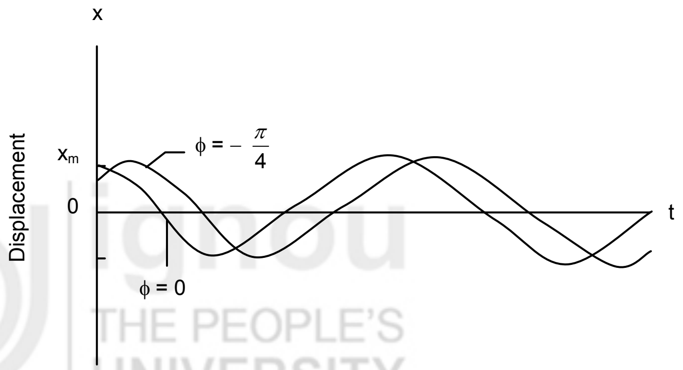


Figure 3.4

The curves in Figure 3.5 show two SHMs which have the same amplitude and period but have phase difference ϕ. Points with the same phase have the same value of *x*. A decrease in ϕ shifts one of the curves to the right as shown. Consider two harmonic motions as follows

$$x_1 = x_m \cos \omega t \qquad \text{. . . (3.4)}$$

$$x_2 = x_m \cos (\omega t + \phi) \qquad \text{. . . (3.5)}$$

The two harmonic motions are called synchronous because they have the same frequency or angular velocity ω. Two synchronous oscillations need not have the same amplitude, and they need not attain their maximum values at the same time. In Eq. (3.3), ϕ is called the phase angle. This means that the maximum of the second motion would occur ϕ radians earlier than that of the first motion. Note that instead of maxima, any other corresponding points can be taken for finding the phase angle.

Velocity

The velocity of a particle executing SHM is obtained by differentiating displacement (Eq. (3.3)) with respect to '*t*'. If we consider the displacement-time equation for SHM as

$$x(t) = x_m \cos (\omega t + \phi) \qquad \text{. . . (3.6)}$$

then velocity is,

$$v(t) = \left[\frac{d}{dt}\, x\,(t)\right] = \frac{d}{dt}\,[x_m \cos(\omega t + \phi)]$$

$$v\,(t) = -\,\omega\, x_m \sin(\omega\, t + \phi) \qquad \ldots (3.7)$$

The quantity $\omega\, x_m$ is called the velocity amplitude v_m. The curve of velocity with respect to time is shown in Figure 3.5(b). Comparing the curve in Figure 3.5(a) with that in Figure 3.5(b) it is evident that magnitude of the velocity is maximum when the magnitude of displacement is zero and vice-versa.

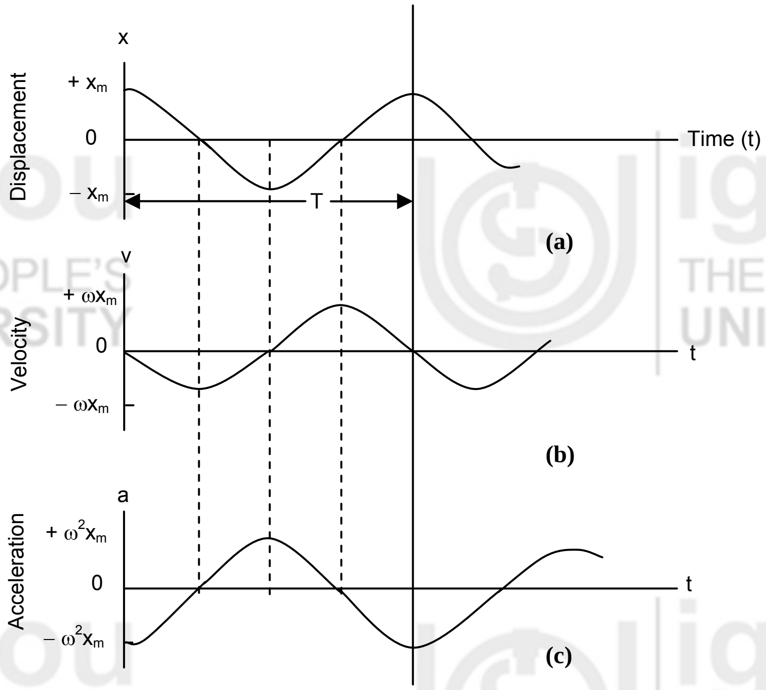


Figure 3.5

Acceleration

The acceleration of a particle executing SHM is obtained by differentiating velocity (Eq. (3.7)) with respect to 't'.

Since $\qquad v\,(t) = -\,\omega\, x_m \sin(\omega\, t + \phi)$,

the acceleration is $a\,(t) = \dfrac{d}{dt}\, v\,(t) = \dfrac{d}{dt}\,[-\,\omega\, x_m \sin(\omega\, t + \phi)]$

or $\qquad a\,(t) = -\,\omega^2\, x_m \cos(\omega\, t + \phi) \qquad \ldots (3.8)$

The positive quantity $\omega^2\, x_m$ is called the magnitude of acceleration a_m.

Comparing the curve in Figure 3.6(c) with that in Figure 3.6(b), it is evident that magnitude of the acceleration is maximum when the magnitude of velocity is zero and vice-versa.

We can combine Eqs. (3.8) and (3.6) to get

$$a\,(t) = -\,\omega^2\, x\,(t) \qquad \ldots (3.9)$$

Therefore, in SHM acceleration is proportional to the displacement but opposite in sign, and the two quantities are related by the square of the angular frequency.

Note : Figure 3.6(a) shows a typical displacement vs time cosine curve for an SHM oscillator. It could have been a sine curve. It just depends on where the oscillator is at time $t = 0$.

i.e. $\qquad x = x_m \cos 2\pi\, ft$

or $\qquad x = x_m \sin 2\pi\, ft$

x = displacement

x_m = amplitude

f = frequency

t = time

You can understand these ideas better with the help of the following examples.

Example 3.2

The equation of an oscillating particle is given by $x = 2\sin\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$ where x is in cm and t is in seconds. Find

(i) the period of the oscillation (T)

(ii) the maximum velocity

(iii) the maximum acceleration

(iv) the initial displacement.

Solution

Here x_m = 2 cm

$$\omega = \frac{\pi}{2} \text{ and } \phi = \frac{\pi}{4}$$

We know $\omega = 2\pi f = \frac{2\pi}{T}$

Therefore, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$

$\therefore$ $T = 4$ s

$$|V_m| = \omega\, x_m$$

$$= 2 \times \frac{\pi}{2} = 3.1 \text{ cm/s}$$

$$|a_{\max}| = \omega^2 x_m$$

$$= 2 \times \frac{\pi^2}{4}$$

$$= 4.9 \text{ cm/s}^2$$

Now given is, $x = 2\sin\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$

Therefore at $t = 0$, we have $x = 2\sin\left(0 + \frac{\pi}{4}\right)$

or $x = \frac{2}{\sqrt{2}} = 1.4$ cm

Example 3.3

Write the equation of a simple harmonic motion with an amplitude of 5 cm if 150 oscillations are performed in one minute and the initial phase is 45°.

Solution

Now given data,

Amplitude, $x_m = 5\text{ cm} = \frac{5}{100} = 0.05\text{ m}$

Time period, $T = 60/150$ s/oscillation = 0.4 s

Angular frequency, $\omega = \frac{2\pi}{0.4} = 5\pi$ rad/s

Phase angle, $\phi = \frac{\pi}{4}$ radian

Thus $$x = 0.05 \sin\left(5\pi t + \frac{\pi}{4}\right)$$

where x is in meter and t is in seconds.

Example 3.4

A horizontal platform vibrates up and down with a simple harmonic motion of amplitude 20 cm. At what frequency will an object kept on the platform just lose contact with the platform?

Solution

It is given that, $x_m = 20\text{ cm} = 0.2\text{ m}$

$f_{\max} = ?$

The net force on the object is given by $\pm (mg - R)$

The object will lose contact with the platform if the normal reaction R becomes zero.

When this happens, the net force on the object is mg and, therefore, its acceleration is g in a downward direction.

The body, therefore, loses contact with the platform if the downward acceleration of the platform exceeds g. This would happen at the maximum upward displacement position.

$$\omega^2 r = g$$

or $$\omega^2 = \frac{9.8}{0.2}$$

or $$\omega^2 = 49$$

or $$\omega = 7$$

or $$2\pi f = 7$$

$$f = \frac{7}{2\pi}$$

$$f_{max} = 1.1\text{ Hz}$$

Example 3.5

A cylindrical block of area A and height h and density ρ floats with its axis vertical in a liquid of density σ in which it is immersed to a depth d. If the block is given a small downward displacement y and released, prove that it will execute simple harmonic oscillations and find an expression for the time period.

Solution

In equilibrium, the net force along the vertical axis must be = 0

Therefore, $$+ Ah\rho g - Ad\sigma g = 0$$

$$\Rightarrow \qquad h\,\rho = d\,\sigma$$

When under additional downward displacement *y*, net force acting on the cylinder is

$$F = -A\,(d + y)\,\sigma\,g + Ah\,\rho\,g$$

$$F = -Ay\,\sigma\,g$$

$$a = \frac{F}{m} = \frac{(-Ay\,\sigma\,g)}{Ah\,\rho}$$

Therefore, $$a = \frac{(-\sigma\,gy)}{h\,\rho}$$

Since the densities ρ and σ of the solid and the liquid are constants as are the height *h* of the solid and the acceleration due to gravity *g*, the acceleration is directly proportional to the displacement and is oppositely directed.

Therefore the motion is simple harmonic.

For this simple harmonic motion,

$$\frac{(\sigma g)}{(h\,\rho)} = \omega^2$$

$$\frac{\sigma\,g}{h\,\rho} = \left(\frac{2\pi}{T}\right)^2$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{h\,\rho}{\sigma\,g}}$$

You should now solve a few problems on your own to check your progress.

SAQ 1

(a) A body moving with SHM has a velocity of 3 m/s when 375 mm from the mid-position and an acceleration of 1 m/s^2 when 250 mm from the mid-position. Calculate the period and the amplitude.

(b) A harmonic motion is of amplitude 100 cm and period of 2 sec. Find the maximum values of velocity and acceleration.

(c) Two motions of the same frequency are such that when one reaches its maximum displacement, the displacement of the other is zero. What is the phase difference between the two motions?

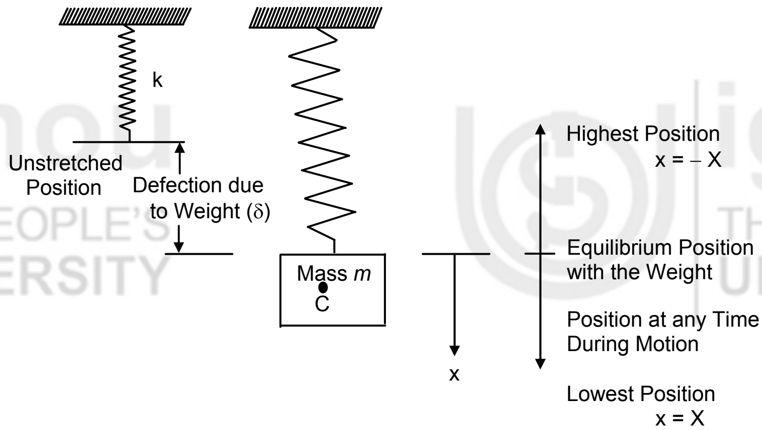
(d) The rectilinear motion of a point is given by $a = -9\,x$, where *a* is the acceleration and *x* is the displacement. The amplitude is 2 in. Find (i) the period and frequency, and (ii) displacement, velocity and acceleration after 21.5 s.

(e) Amplitude is the maximum repeating absolute value of the oscillation in terms of displacement only and not in terms of velocity and acceleration. State whether TRUE or FALSE.

We now describe the simple harmonic motion of a commonly used physical system : the spring-mass system.

3.2.3 Linear Vertical Spring-Mass System

Let us consider the oscillations of a mass attached to a vertical spring (Figure 3.6). In this system we assume that the spring is massless, air friction is neglected and the entire mass considered is attached at the bottom of the spring. Before the mass is attached to the spring the unstretched position of the spring is shown in Figure 3.6(a). When the mass *m* is attached to the spring it extends and the mass moves down until the spring force balances the gravity force on the mass. This equilibrium position is also shown in Figure 3.6(b). This position is called **Static deflection**. Now if we pull the mass up or down from this equilibrium position and release it, then it will start oscillating about this equilibrium position. The oscillation of the mass will be simple harmonic in nature. The word linear means that force in the spring is proportional to *x* rather than to some power of *x*.



(a) **(b)**

Figure 3.6 : Spring-mass System

In the static deflection condition, $mg = k\,\delta$

$$k/m = g/\delta \qquad \ldots (3.10)$$

where m = mass in kg,

g = acceleration due to gravity in m/s^2,

k = spring constant in N/m, and

δ = static deflection in m.

For a displacement x from the equilibrium position, the restoring force acting on the mass equals

$$F = -k\,(\delta + x) + mg$$

$$= -k\,(\delta + x) + k\,\delta = -kx \qquad (\because mg = k\delta)$$

Now we can write the equation of motion for the above spring-mass system by applying Newton's second law of motion,

$$F = ma$$

$$-kx = m\,\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 \qquad \ldots (3.11)$$

Comparing Eq. (3.11) with SHM Eq. (3.2) we see that both the equations are similar, i.e. **free undamped oscillations of a spring-mass system are simple harmonic in nature**. Undamped means, no damping forces (resisting forces) are acting on the system.

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \ldots (3.12)$$

ω_n is called the **angular frequency** and is expressed in rad/s. Subscript *n* stands for natural, i.e. natural angular frequency. It is called natural because the oscillation is maintained only by the restoring force developed in the spring when the mass is displaced from its equilibrium position. These types of oscillations are called **free oscillations**, which are maintained only by the restoring forces acting within the system.

Frequency $$f = \frac{\omega_n}{2\pi} = \left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}} \qquad \ldots (3.13)$$

Time period $$T = \frac{1}{f}$$

3.2.4 Linear Horizontal Spring-Mass System

If we consider a horizontal spring, the analysis will be the same as given above. The only difference will be that the effect of gravity will not be considered.

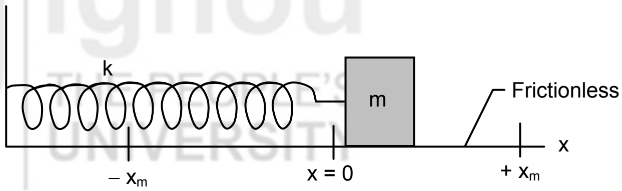


Figure 3.7

So we can directly write the restoring force on mass as $F = -kx$.

Applying Newton's second law we can write the equation of motion as,

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

or $$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 \qquad \ldots (3.14)$$

3.2.5 Energy Conservation in SHM

As the particle oscillates, its speed varies and so does its kinetic energy *K*. Where does the energy go when its speed is 0? If the oscillation is produced by a spring, the spring is compressed or stretched to some maximum amount when its speed is zero and then its energy is in the form of potential energy *U*.

Therefore the energy of a simple harmonic oscillator transfers back and forth between kinetic and potential energy, while the total mechanical energy *E* remains constant.

$$U(t) = \left(\frac{1}{2}\right)kx^2 = \left(\frac{1}{2}\right)kx_m^2\cos^2(\omega t + \phi)$$

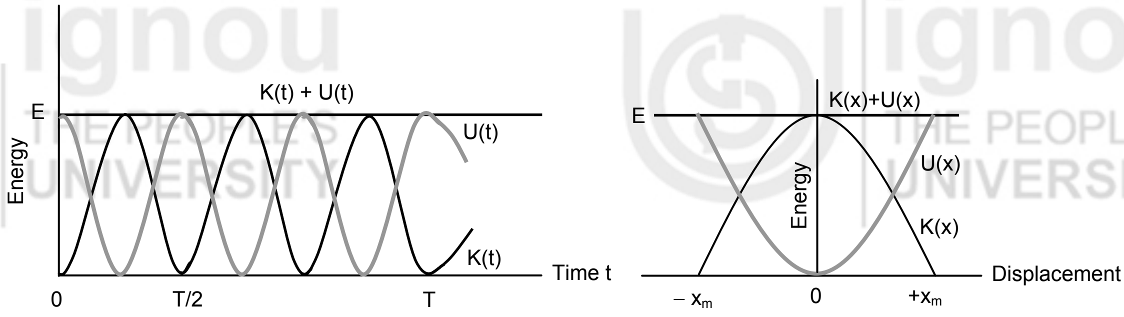
$$K(t) = \left(\frac{1}{2}\right)mv^2 = \left(\frac{1}{2}\right)m\,\omega^2 x_m^2\sin^2(\omega t + \phi)$$

Using $k = m\omega^2$ and $\cos^2\phi + \sin^2\phi = 1$,

$$E(\text{total}) = U(t) + K(t) = \left(\frac{1}{2}\right)kx_m^2 \qquad \ldots (3.15)$$

$$= \left(\frac{1}{2}\right)m\,\omega^2 x_m^2 \qquad \ldots (3.16)$$

The mechanical energy of a linear oscillator is constant and independent of time. The variation of potential energy and kinetic energy of a linear oscillator are shown as functions of time *t* and displacement *x* in Figures 3.8(a) and (b).



(a) (b)

Figure 3.8

Once again, we give a few solved examples so that you can understand these ideas better.

Example 3.6

A 3 kg object stretches a spring by 16 cm when it hangs vertically in equilibrium. The spring is then stretched further from equilibrium and the object released.

(i) What is the frequency of the motion?

(ii) What is the frequency if the 3 kg object is replaced by a 6 kg object?

Solution

In the equilibrium position the object is held by the spring force (F) which is proportional to the stretch of the springs δ and constant of proportionality (k) is called spring constant, i.e. $F = k\,\delta$ = weight attached to spring (mg).

$$3 \times 9.81 = k \times 0.16 \quad \text{(static deflection condition.)}$$

Therefore, $k = 184$ N/m

Frequency of oscillation is given by,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{184}{3}} = 1.25 \text{ Hz}$$

Now replace $m = 3$ kg by $m = 6$ kg,

$$f = \frac{1}{2\pi}\sqrt{\frac{184}{6}} = 0.884 \text{ Hz}$$

Example 3.7

A small body of mass 0.12 kg is undergoing SHM of amplitude 8.5 cm and period 0.20 s.

(i) What is the maximum force?

(ii) If the motion is due to a spring, what is k?

Solution

Maximum force of an oscillating mass = Mass × Maximum acceleration

$$F_{\max} = m \,.\, a_{\max} = m\,|\omega^2 x_m|$$

Now, $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$ rad/s

Therefore, $F_{\max} = (0.12)\,(0.085)\,(10\,\pi)^2 = 10$ N

If motion is due to a spring then restoring force $F_{\max} = k \,.\, x_m = m\,|\omega^2 x_m|$.

Therefore, $k = m\,\omega^2 = 0.12 \times (10\,\pi)^2 = 118.43$ N/m

Try the following exercises to apply these concepts.

SAQ 2

(a) A load is suspended from a vertically mounted spring. At rest it deflects the spring 12 mm. Calculate the number of complete oscillations/second. If the load's mass is 3 kg, what is the maximum force in the spring when it is displaced a further 25 mm below the rest position and then released?

(b) A small body of mass 0.12 kg is undergoing SHM of amplitude 8.5 cm and period 0.20 s (i) what is maximum force? (ii) if the motion is due to a spring, what is k?

(c) A tray of mass 12 kg is supported by two identical springs as shown in Figure 3.9. When the tray is pressed down slightly and released it executes SHM with a period of 1.5 sec.

(i) what is the force constant of each spring?

(ii) when a block of mass M is placed on the tray, the period of SHM changes to 3.0 sec. what is the mass of the block

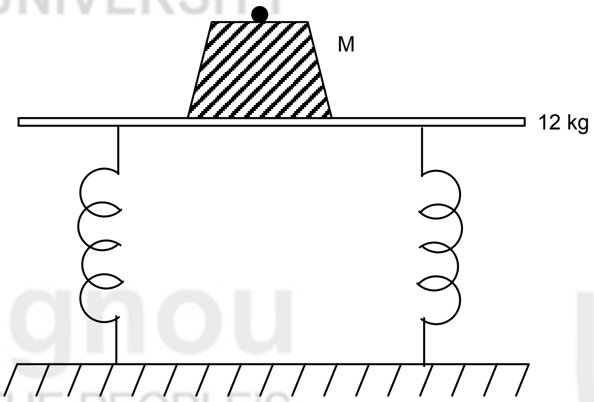


Figure 3.9

(d) A mass of 6 kg is suspended from a vertically mounted spring. The static deflection is 6 mm. A further load of 12 kg is hung from the spring, pulled 10 mm below the equilibrium position and released. Find the period of the resulting oscillation.

(e) An instrument is spring mounted to the body of a rocket, the spring axis lying along the rocket axis. The natural frequency of the instrument upon its mount is 40 Hz. If the rocket is accelerated smoothly to an acceleration 10 times that of g in such a way that the instrument is not set into vibration, find the static deflection of the instrument on its mounting during the acceleration.

(f) A small motor of mass 20 kg is symmetrically mounted on 4 equal springs, each with a spring constant of 25 N/cm. Estimate the frequency and period of vibration of the motor.

(g) A trolley of mass 3.0 kg is connected to two identical springs each of force constant 600 Nm^{-1} as shown in Figure 3.10. If the trolley is displaced from its equilibrium position by 5.0 cm and released, what is

(i) the period of ensuing oscillations

(ii) the maximum speed of the trolley

(iii) how much is the total energy dissipated as heat by the time the trolley comes to rest due to damping forces?

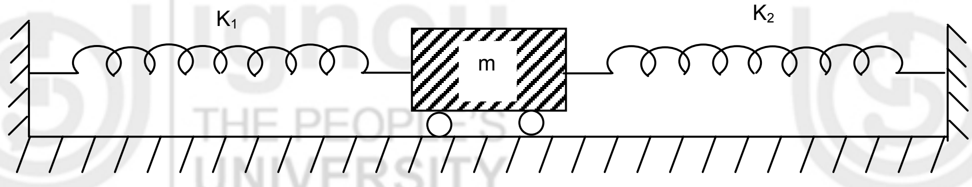


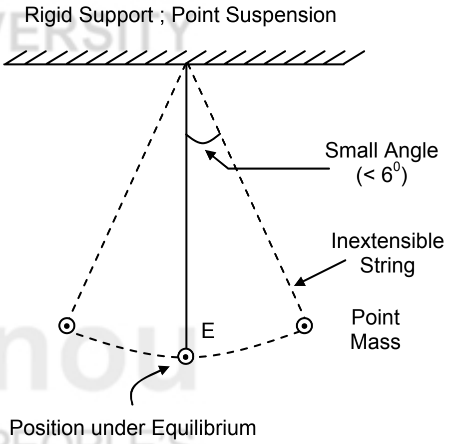
Figure 3.10

Apart from a spring-mass system, several other physical systems can be made to undergo simple harmonic oscillations. We now discuss some of these.

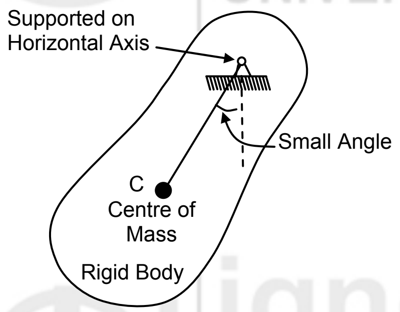
3.3 PENDULUMS

A pendulum is a type of simple harmonic oscillator in which the restoring force comes from gravitational force acting on the suspended mass and not because of elastic properties of a compressed or stretched spring.

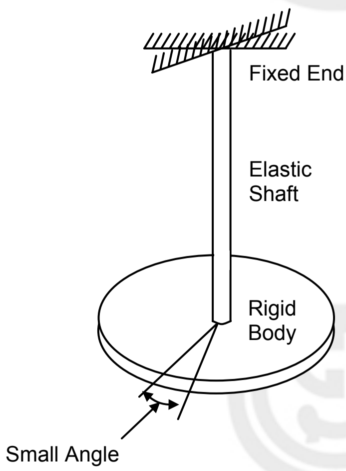
We now discuss three different types of pendulums : the **simple pendulum**, the **compound pendulum** and the **torsional pendulum** (Figure 3.11).



(a) Simple Pendulum



(b) Compound Pendulum



(c) Torsional Pendulum

Figure 3.11 : Different Types of Pendulum

3.3.1 Simple Pendulum

It consists of a mass suspended at the end of string which at its upper end in suspended from a rigid support as shown in Figure 3.11(a). It is assumed that the mass is concentrated at the end and the string is massless and inextensible. The suspended point mass is called the bob of the pendulum. The bob is free to swing back and forth in a plane about its equilibrium position shown in Figure 3.11(a). The initial disturbance is given by displacing the bob through a small angle θ and then releasing it to perform a periodic motion.

Let us now analyse the motion of this pendulum (Figure 3.12). The simple pendulum generates the simple harmonic motion under the action of the following factors :

(h) The initial disturbing force is applied by hand to bring it to one of its extreme positions and then it is 'let go'.

(ii) The restoring force is the component of the gravitational force on the bob, i.e. its weight along the motion of the bob.

(iii) When the initial disturbing force is removed, the restoring force is a function of the angular location of the bob; maximum at the extremities and zero at the equilibrium position.

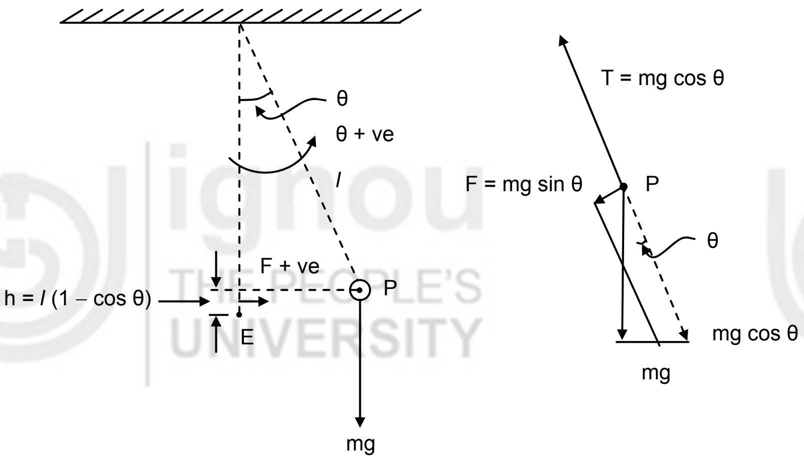
As shown in the free body diagram of bob *P* in Figure 3.12, the restoring force *F* is given by, $mg \sin \theta \approx mg\, \theta$ for θ (< 6°) directed towards the equilibrium position. Since the positive direction is denoted away from the equilibrium position,

$$F = -\, mg\, \theta$$

According to the Newton's law of motion,

$$F = ma$$

$$-\, mg\, \theta = m\,(l\, \ddot{\theta}) = ml \left(\frac{d^2 \theta}{dt^2}\right)$$



(a) **(b)**

Figure 3.12

Cancelling *m* from both sides and rearranging,

$$\left(\frac{d^2 \theta}{dt^2}\right) + \left(\frac{g}{l}\right) \theta = 0 \qquad \ldots (3.17)$$

This is the equation of motion of simple pendulum, which is similar to Eq. (3.2), i.e. equation of simple harmonic motion.

Here, $$\omega_n = \sqrt{\frac{g}{l}} \qquad \ldots (3.18)$$

Frequency $$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \qquad \ldots (3.19)$$

Time period $$T = \frac{1}{f} = 2\pi \sqrt{\frac{l}{g}} \qquad \ldots (3.20)$$

Some of the salient features of the periodic motion of a simple pendulum are :

(i) The velocity of the bob is maximum at the equilibrium position and zero at the extremities.

(ii) The acceleration is zero at the equilibrium position and increases with θ but it is always directed towards the equilibrium position. It is maximum at the extreme positions.

(iii) The frequency and time period of the motion are independent of the mass of the bob.

(iv) The frequency and time period are independent of the amplitude θ of motion if θ is small ($< 6^{o}$). Their dependence upon the length of the string *l* and the acceleration due to gravity *g* are given by Eqs. (3.19) and (3.20).

(v) The motion of the bob would go on for ever if there were no frictional forces, such as the aerodynamic force on the bob, which damp the motion to bring it to rest eventually.

3.3.2 Compound Pendulum

It consists of a rigid body that oscillates about a horizontal axis through the body at some point other than the center of mass. The moment of inertia of the entire rigid body comes into play to establish the frequency of the periodic motion.

A compound pendulum supported about a horizontal axis at a distance *r* above the center of mass *C*, oscillating to produce a periodic motion, is shown in Figure 3.13.

The restoring moment due to the weight of the pendulum about the pivot *O* is given by,

$$M = (-\, mg \sin\theta\, r)$$

considering the moment to be positive along θ increasing.

By Euler's law, the equation of motion is

$$M = I\ddot{\theta} = I\left(\frac{d^2\theta}{dt^2}\right)$$

where *I* is the moment of inertia about the pivotal axis.

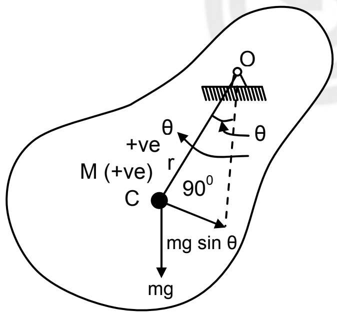


Figure 3.13

It follows that

$$-\, mgr \sin\theta = I\left(\frac{d^2\theta}{dt^2}\right)$$

which for small θ, becomes

$$\left(\frac{d^2\theta}{dt^2}\right) + \left(\frac{mgr}{I}\right)\theta = 0 \qquad \ldots (3.21)$$

which is similar to Eq. (3.2), i.e. Simple harmonic motion equation.

Therefore $$\omega_n = \sqrt{\frac{mgr}{I}} \qquad \ldots (3.22)$$

3.3.3 Torsional Pendulum

The torsional pendulum, shown in Figure 3.11(c), consists of a rigid body suspended by a vertical elastic shaft which when twisted develops a restoring moment. Analysis similar to that for the compound pendulum can be done for this pendulum also.

There are other types of pendulums also such as the conical pendulum, 'Foucault's pendulum', double and multiple bob pendulums. But we will not discuss them here. We now apply the concepts discussed so far to a few problems.

Example 3.8

A pendulum of length l is displaced by an angle θ, and is observed to have a period of 4 seconds. The string is then cut in half, and displaced to the same angle θ. How does this affect the period of oscillation?

Solution

From equation of simple pendulum,

$$T = (2\pi)\sqrt{\frac{l}{g}}$$

Therefore, it is clear from above relationship that if we reduce the length of string by a factor of 2 we reduce the time period by a factor of $\sqrt{2}$.

Example 3.9

A pendulum is commonly used to calculate the acceleration due to gravity at various points on the earth. Often areas with low acceleration indicate a cavity in the earth in the area, many times filled with petroleum. An oil prospector uses a pendulum of length 1 meter, and observes it to oscillate with a period of 2 seconds. What is the acceleration due to gravity at this point?

Solution

Again we use the equation $T = (2\pi)\sqrt{\frac{l}{g}}$

or $$g = \frac{(4\pi^2 l)}{T^2}$$

$$= \frac{(4\pi^2)}{2^2} = 9.87 \text{ m/s}^2$$

This value indicates a region of high density near the point of measurement – probably not a good place to drill for oil.

Example 3.10

A disc of radius 10 cm is suspended from a point on its circumference. Determine its frequency of oscillation.

Solution

The moment of inertia about the pivotal axis is,

$$I = \frac{mr^2}{2} + mr^2 = \frac{(3mr^2)}{2}$$

$$= \frac{3}{2} m \times 0.1^2 = 0.015\, m \text{ kg m}^2$$

where m is the mass of the disc.

The frequency is given by

$$f = \frac{1}{(2\pi)} \frac{(m \times 9.81 \times 0.1)}{0.15 \text{ m}}$$

$$f = 1.287 \text{ s}^{-1}$$

and the time period is,

$$T = \frac{1}{f} = \frac{1}{1.287} = 0.777 \text{ s}.$$

You can check your understanding with the following exercises.

SAQ 3

(a) Select the most suitable statements out of the given choices *a*, *b*, *c* and d in the following questions.

(i) A periodic motion of a system occurs

(a) if the initial state of the system is in equilibrium.

(b) if the restoring forces or moments come into play as soon as the system is disturbed.

(c) if and only if the system undergoes simple harmonic motions.

(d) if the initial states as well as the disturbed states are in equilibrium.

(ii) The resisting force in an oscillatory system tends to

(a) reduce the time period.

(b) oppose the restoring force proportionately.

(c) reduce the amplitude.

(d) reduce the amplitude with time.

(iii) A simple harmonic oscillator has an amplitude r and time period T.

The time required by it to travel for $x = r$ to $x = \frac{r}{2}$ is

(a) $\frac{T}{6}$

(b) $\frac{T}{4}$

(c) $\frac{T}{3}$

(d) $\frac{T}{2}$

(iv) A particle executes SHM given by

$$y = 0.02 \sin 100\, t$$

The amplitude and frequency are

(a) 0.02 and 100

(b) 0.02 and $\frac{50}{\pi}$

(c) 0.01 and 50

(d) $\frac{1}{0.02}$ and $\frac{50}{\pi}$

(v) A particle is vibrating in a SHM with an amplitude of 4 cm. At what displacement from the equilibrium position is its energy half potential and half kinetic?

(a) 1 cm

(b) $\sqrt{2}$ cm

(c) 3 cm

(d) $2\sqrt{2}$ cm

(b) Determine the following parameters for a simple pendulum at the mean surface of the earth :

(i) Length of a 1 second pendulum.

(ii) Period of a 1m pendulum.

If these pendulums are taken to the mean surface of the moon where $g = 1.67$ m/s^2, determine the corresponding periods.

(c) A simple pendulum was observed to perform 40 oscillations in 100 s of amplitude 4^o. Find

(i) the length of the pendulum,

(ii) the maximum linear acceleration of the pendulum bob,

(iii) the maximum velocity of the bob, and

(iv) the maximum angular velocity of the pendulum.

(d) A simple pendulum is formed by a bob of mass 2 kg at the end of a cord 600 mm long. How many complete oscillations will it make per second? The same pendulum is suspended inside a train accelerating smoothly along the level at 3 m/s^2. If the pendulum is not set oscillating find the angle the cord makes with the vertical.

(e) A simple pendulum swings 4 oscillations in the same time as another 0.48 m longer swings 3 oscillations. Determine their lengths.

(f) A man wants to measure the height of a building. He notes that a long pendulum extends from the celling almost to the floor and that its period is 24 sec. Determine

(i) the height of the tower, and

(ii) the period when the pendulum is taken to the moon where $g = 1.67$ ms^{-2}.

3.4 DAMPED SIMPLE HARMONIC MOTION

So far, we have discussed simple harmonic motion in which we have neglected **damping**, i.e. **resistance to motion** (aerodynamic drag, etc.). By damped oscillatory motion we mean motion in which the amplitude of the motion is reduced by an external force. The external force may be resistance due to friction between the medium and the oscillating object or friction at the support or friction.

The free undamped oscillations of a spring-mass system are represented by the following equation as seen earlier,

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right) x = 0 \qquad \ldots (3.23)$$

Now let us consider a case of a spring-mass system, which is attached to a vane moving in a viscous liquid at the other end as shown in Figure 3.14. The resistance to the motion is caused because of the drag force exerted by the liquid on the vane. This type of damping is called **viscous damping**. In this damping, the resisting force is directly proportional to magnitude of the velocity v of the vane (an assumption that is accurate if the vane moves slowly). Therefore damping force

$$F_d \propto \frac{dx}{dt}$$

or
$$F_d = c\left(\frac{dx}{dt}\right) \qquad \ldots (3.24)$$

where c (Ns/m) = damping coefficient.

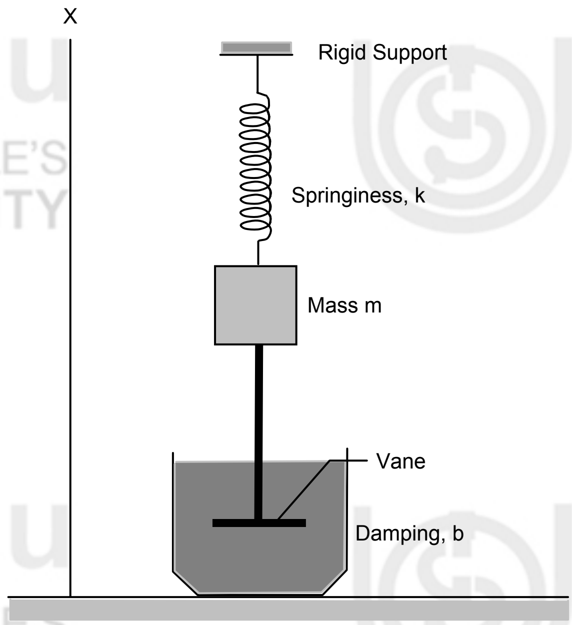


Figure 3.14

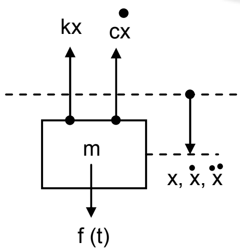


Figure 3.15

Now consider the free body diagram of the mass as shown in Figure 3.15.

Therefore, applying Newton's second law of motion, we get from $F = ma$

$$-c\left(\frac{dx}{dt}\right) - kx = m\left(\frac{d^2x}{dt^2}\right)$$

$$m\left(\frac{d^2x}{dt^2}\right) + c\left(\frac{dx}{dt}\right) + kx = 0$$

or
$$\left(\frac{d^2x}{dt^2}\right) + \left(\frac{c}{m}\right)\left(\frac{dx}{dt}\right) + \left(\frac{k}{m}\right)x = 0 \qquad \ldots (3.25)$$

where $\omega_n^{\,2} = k/m$.

Eq. (3.25) is also written as,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Now Eq. (3.25) is a linear differential equation of the second order and its solution can be written as,

$$x = e^{\omega t} \qquad \text{. . . (3.26)}$$

where e = base of natural logarithms = 2.718

t = time

and ω = a constant to be determined.

Differentiating Eq. (3.26) twice with respect to time, we have

$$\frac{dx}{dt} = \omega\, e^{\omega t}$$

$$\left(\frac{d^2x}{dt^2}\right) = \omega^2\, e^{\omega t}$$

Substituting these expressions in Eq. (3.25), we get

$$m\,\omega^2\, e^{\omega t} + c\,\omega\, e^{\omega t} + k\, e^{\omega t} = 0$$

$$(m\,\omega^2 + c\,\omega + k)\, e^{\omega t} = 0$$

$$m\,\omega^2 + c\,\omega + k = 0 \qquad \text{. . . (3.27)}$$

The above equation is called the Characteristic equation of the system. Eq. (3.27) is quadratic in ω, and in this case the two values of ω are given by

$$\omega_{1,2} = \frac{-c \pm \sqrt{c^2 - 4\,mk}}{2m} \qquad \text{. . . (3.28)}$$

Hence, $x = e^{\omega_1 t}$ and $x = e^{\omega_2 t}$ may both be the solutions of Eq. (3.28), and therefore, the most general solution may be written as

$$x = C_1\, e^{\omega_1 t} + C_2\, e^{\omega_2 t} \qquad \text{. . . (3.29)}$$

where C_1 and C_2 are the two arbitrary constants to be determined from the initial conditions.

Eq. (3.29) tells us how the system will behave when it is undergoing damped free oscillations. Although it suggests that the amplitude of the displacement varies exponentially with time but to get a more clear picture let us look at its various possible solutions.

Case 1

When $c = 2\sqrt{(mk)}$, then $\omega_{1,2} = -\frac{c}{(2m)} = -\sqrt{\left(\frac{k}{m}\right)}$, i.e. both the roots are real and equal and therefore solution will be

$$x = e^{-\left(\frac{c}{2m}\right)t}\,(A + B)$$

The **displacement decreases exponentially with time because of the exponential term raised to negative power**. This type of damping is called critical damping (shown in Figure 3.16) and value of c is denoted by c_c and called critical damping coefficient. So in a critically damped system mass does not oscillates when displaced and released from its mean position rather it comes back to its mean position in the shortest possible time.

Case 2

When $c > 2\sqrt{(mk)}$, then values of ω are

$$\Rightarrow \qquad \omega_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$= \frac{-c}{2m} \pm \beta$$

so that, $x = e^{\frac{-c}{2m}} (A \cosh \beta t + B \sinh \beta t)$ . . . (3.30)

Eq. (3.30) suggests that displacement decreases with time exponentially. It also involves cosh and sinh terms, which can further be written in terms of exponents as,

$$\sinh y = \frac{1}{2}(e^y - e^{-y}), \; \cosh x = \frac{1}{2}(e^y + e^{-y})$$

So, the solution is completely exponential in nature. Such type of system in which damping is greater than the critical damping is called **overdamped system** (shown in Figure 3.16).

Case 3

When $c < 2\sqrt{(mk)}$, i.e. damping coefficient is less than the critical damping coefficient, the values of ω are given as,

$$\Rightarrow \qquad \omega_{1,2} = \frac{-c}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\; i$$

so that $\Rightarrow$ $x = e^{\frac{-c}{2m}} (A \cos \omega_d^t + B \sin \omega_d^t)$

where $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$ . . . (3.31)

Eq. (3.31) suggests that the displacement amplitude decreases exponentially with time (negative power of exponential term) but there are sine and cosine terms also which means that the displacement amplitude is harmonic in nature with amplitude decreasing exponentially as shown in Figure 3.16. Such type of system is known as **underdamped system**.

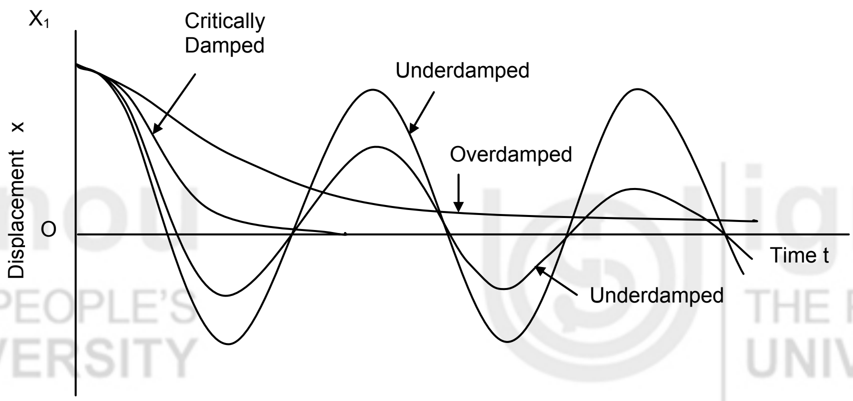


Figure 3.16

3.5 FORCED OSCILLATIONS

Forced oscillations are defined as those oscillations that are caused and maintained by periodic excitation. A person swinging in a swing without anyone pushing it is an example of free oscillation. However, if someone pushes the swing periodically, the swing is said to perform forced oscillations.

Consider the spring-mass system of Figure 3.17 in which a forcing function, i.e. a periodic force acting on the mass is also shown as $\boldsymbol{f(t) = F_0 \sin \omega_d t}$. The periodic force which is acting on the system is also harmonic in nature. Detailed analysis of such a system is also done in the same manner as done for free damped case, i.e. we will write the differential equation of motion for the above system and solve it to know how the system will behave when set into oscillations. Equation of motion for such a system is written as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_d t \qquad \ldots (3.32)$$

where (**.**) dot denotes differentiation with respect to time.

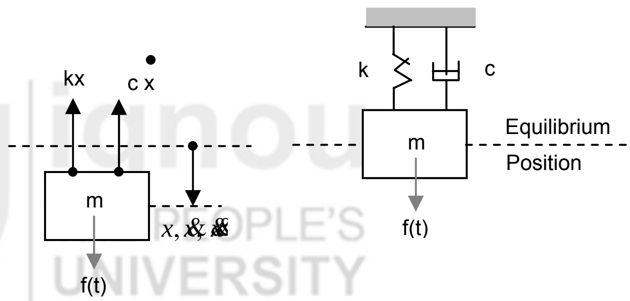


Figure 3.17

But few points which need to mention here are as follows. Firstly, there are two angular frequencies associated with this system :

(i) the natural angular frequency ω_n of the system which is the angular frequency at which it would oscillate if it were suddenly disturbed and left to oscillate freely, and

(ii) the angular frequency ω_d of the external driving force causing the driven or forced oscillations.

Secondly, it is worth mentioning that the above forced oscillating system will oscillate at frequency ω_d of the excitation frequency, and its displacement $x(t)$ is given by

$$x(t) = x_m \cos(\omega_d t + \phi) \qquad \ldots (3.33)$$

where x_m is the amplitude of the oscillations.

3.5.1 Resonance

Resonance is a condition of an oscillating system when excitation frequency ω_d becomes equal to the natural frequency of the oscillating system ω_n

i.e. $$\omega_d = \omega_n \qquad \ldots (3.34)$$

In this condition, velocity amplitude of the oscillation will be maximum and Eq. (3.34) is also approximately the condition at which the displacement amplitude x_m of the oscillations is greatest. Therefore considering the same example of the swing, if we push the swing at its natural angular frequency then displacement and velocity amplitude of the oscillation will increase to large values.

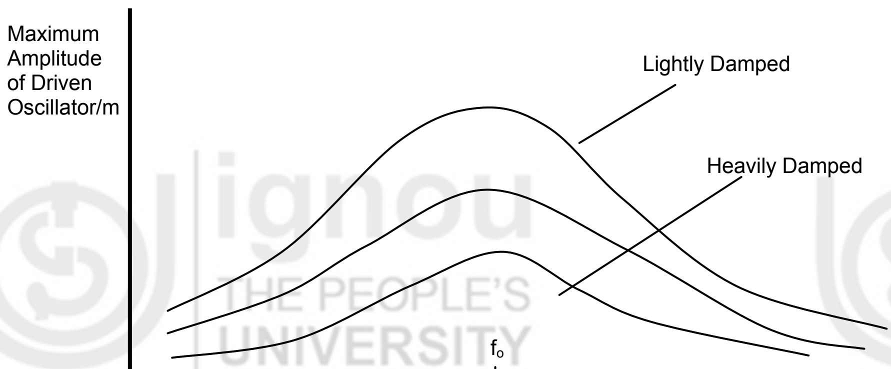


Figure 3.18

Figure 3.18 shows amplitude of forced oscillations when the system is lightly damped and heavily damped. The peak of the curves show the point of resonance, i.e. when the excitation frequency matches with the natural frequency of the system.

3.6 SUMMARY

- **Simple Harmonic Motion :** A type of periodic motion which is repeated at regular intervals such that the acceleration is proportional to the displacement from the mean position and is directed towards it, i.e. $a \propto -x$.

- In SHM $x(t) = x_m \cos(\omega t + \phi)$; x is the amplitude, $(\omega t + \phi)$ is the phase of the motion, ϕ is the phase constant (phase angle) and $x(0) = x_m \cos\phi$.

- **Frequency :** It is the number of oscillations per second. It is denoted by *f*. In the system it is measured in Hz (hertz). 1 Hz = 1 oscillation per second = 1 s^{-1}.

- **Period :** It is the time required to complete one oscillation. It is denoted by *T* and is equal to 1/*f*.

- Velocity in SHM $v(t) = -\omega x_m \sin(\omega t + \phi)$

  Acceleration in SHM $\quad a(t) = -\omega^2 x_m \cos(\omega t + \phi)$,

  and $\quad |v_{max}| = \omega x_m, |a_{max}| = \omega^2 x_m$

- A spring-mass system is a linear oscillator that moves according to $F = kx$. For such a system $\omega = \sqrt{\left(\frac{k}{m}\right)}$ and $T = 2\pi\sqrt{\left(\frac{m}{k}\right)}$.

- Total mechanical energy in SHM can be written as :

$$E = \left(\frac{1}{2}\right) mv_x^2 + \left(\frac{1}{2}\right) kx^2 = \text{constant.}$$

- **Pendulums :** For a simple pendulum, $T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$

  For a compound pendulum, $T = 2\pi\sqrt{\left(\frac{mgr}{l}\right)}$.

- **Damped Harmonic Motion :** In practical oscillating systems motion has resistance from various sources which decrease its amplitude with time. This type of motion is called damped motion. If a viscous type of damping is assumed then damping force is equal to

$$F_d = c\,\frac{dx}{dt}$$

  Such type of a spring-mass oscillator will oscillate at a frequency less than its natural frequency.

- The behaviour of the damped system, say in the case of a free vibration, will depend upon the amount of damping present in the system. Accordingly we have 3 cases : overdamped, critically damped and underdamped. Only underdamped system will have oscillatory response when disturbed from its mean position and let go. The other two types of systems will not oscillate if disturbed from their equilibrium position and left free. They will slowly come back to their mean position.
- **Forced Oscillations and Resonance :** If an external driving force also acts on an oscillating system then the system will be associated with 2 frequencies, one natural frequency and other the forced frequency and system will vibrate with the forced frequency. If the forced frequency becomes equal to natural frequency of the system then amplitude reaches very high levels. This condition is called Resonance, which is of lot of practical importance as regards design and operation of various machinery.

3.7 KEY WORDS

Periodic Motion : Motions, processes or phenomena which repeat themselves are called periodic.

Simple Harmonic Motion : Periodic motion for which the displacement can be mathematically represented by the function

$$f(t) = A\cos\left(\frac{2\pi t}{T} + \phi\right)$$

where A and ϕ are constants is called simple harmonic motion.

Period : In a periodic motion the smallest interval of time after which the process repeats itself is called its period. Usually, the period is denoted by the symbol T and is measured in second.

Frequency : The reciprocal of T gives the number of periodic motions that occur per second. This quantity is called the frequency of the periodic motion. Frequency is represented by the symbol υ or f and is measured in units called hertz (1 Hz = 1 cycle/sec).

Vibration : One vibration is the to and fro motion of a particle between any two consecutive passages in the same direction.

Simple Pendulum : A simple pendulum is a heavy point mass suspended by a weightless, inextensible and a perfectly flexible string from a rigid support about which it can vibrate freely.

Second's Pendulum : A second's pendulum is a pendulum whose time period is two seconds.

Free Oscillation : The oscillation produced by an oscillator of frequency equal to its natural frequency such that no external periodic force acts in it are called free oscillation.

Forced Oscillation : The oscillation produced by an oscillator under the effect of an external periodic forces of the frequency other than the natural frequency of the oscillation are called forced oscillation.

Damped Oscillation : The periodic oscillation of gradually decreasing amplitude produced by an oscillation due to the presence of resistive forces are called damped oscillation.

Resonance : The phenomenon of setting an oscillator into oscillation on its natural frequency by bringing near it another oscillating oscillator of same frequency is called resonance.

3.8 ANSWERS TO SAQs

SAQ 1

(a) For a SHM, $x = a \sin \omega t$

We know, velocity, $v = \omega \sqrt{A^2 - x^2}$ and acceleration $a = -\omega^2 x$

Here, $a = -1 \text{ ms}^{-2}, \quad x = 0.250 \text{ m}$

$\Rightarrow \quad \omega^2 = \frac{1}{0.25}$

$\Rightarrow \quad \omega = 2 \text{ rad/sec}$

Time period $T = \frac{2\pi}{\omega}$

$= \frac{2\pi}{2} = 3.142 \text{ sec}$

Again, $v = \omega \sqrt{A^2 - x^2}$

$\Rightarrow \quad 3 = 2\sqrt{A^2 - 0.375^2}$

$\Rightarrow \quad A^2 = 1.5^2 + 0.375^2$

$\Rightarrow \quad A = 1.55 \text{ m}$

(b) We know,

$$v = \omega \sqrt{A^2 - x^2}$$

Maximum velocity, $v_{max} = \omega A$ and maximum acceleration, $a_{max} = \omega^2 A$.

Now, $\omega = \frac{2\pi}{T}$

$= \frac{2\pi}{2} = \pi \text{ rad/sec}$

$A = 100 \text{ cm} = 1 \text{ m}$

$v_{max} = \omega A$

$= \pi \times 1 = 3.14 \text{ m/s}$

$a_{max} = \omega^2 A$

$= \pi^2 \times 1 = 9.87 \text{ m/s}^2$

(c) Phase difference is $\frac{\pi}{2}$.

(d) Here, $a = -9x$

We know for SHM

$$a = -\omega^2 x$$

So, $\omega^2 = 9$

$\omega = 3$ rad/sec

Time period $T = \frac{2\pi}{\omega}$

$$= \frac{2\pi}{3} = 2.09 \text{ sec}$$

Frequency $f = \frac{1}{T}$

$$= \frac{1}{2.09} = 0.478 \text{ cps}$$

We know $x = A \sin \omega t$, $v = \omega \sqrt{A^2 - x^2}$, and $a = -\omega^2 x$.

Here, $x = 2 \sin (3 \times 2.15) = 1.99$ in

$$v = 3\sqrt{2^2 - 1.99^2} = 0.35 \text{ in/s}$$

$$a = -\omega^2 x$$

$\Rightarrow$ $a = -3^2 \times 1.99 = -17.73 \text{ in/s}^2$

(e) False.

SAQ 2

(a) We know $\omega^2 = \frac{k}{m}$

In this case $mg = k\delta$

where k is the spring constant, and δ is the deflection.

So, $\omega^2 = \frac{k}{m} = \frac{g}{s}$

$\Rightarrow$ $\omega^2 = \frac{9.81}{0.012}$

$\Rightarrow$ $\omega = 28.577$ rad/sec

Now, $\omega = 2\pi f$

$\Rightarrow$ $f = \frac{\omega}{2\pi}$

$$= \frac{28.577}{2\pi} = 4.5 \text{ Hz}$$

(b) We know, acceleration $= -\omega^2 x$

So, magnitude of acceleration $|a| = \omega^2 x$

So, $|a| = \left(\frac{2\pi}{0.2}\right)^2 \times 0.085 = 83.89 \text{ ms}^{-2}$

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$
$$= 0.12 \times 83.89 = 10 \text{ N}$$

We know, $\omega^2 = \frac{k}{m}$

$\Rightarrow$ $$k = m\,\omega^2$$

$$= 0.12 \times \left(\frac{2\pi}{0.2}\right)^2 = 118.43 \text{ N/m}$$

(c) When two springs each of spring constant k are joined in parallel to each other, then the attached mass m executes SHM of time period T given by

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

$\therefore$ $$k = \frac{2\pi^2 m}{T^2}$$

(i) Here, $m = 12$ kg; $T = 1.4$ sec.

$$k = \frac{2\pi^2 \times 12}{1.5^2} = 105.3 \text{ Nm}^{-1}$$

(ii) When a block of mass M is placed on tray, then

$$m = M + 12, \quad T = 3.0 \text{ sec},$$

Now, $$T = 2\pi\sqrt{\frac{M+12}{2k}}$$

$\Rightarrow$ $$M = \frac{T^2 k}{2\pi^2} - 12$$

$$= \frac{3.0^2 \times 105.2}{2\pi^2} - 12 = 48 - 12 = 36 \text{ kg}$$

(d) We know, $mg = k\delta$

$\Rightarrow$ $$k = \frac{mg}{\delta}$$

$$= \frac{6 \times 9.8}{6 \times 10^{-3}} = 9800 \text{ N/m}$$

A further load of 12 kg is hangs from the spring, so total load is 18 kg.

Now, $$\omega^2 = \frac{k}{m} = \frac{9800}{18}$$

$$\omega = 23.33 \text{ rad/sec.}$$

Time period, $$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{23.33} = 0.2692 \text{ sec}$$

(e) 1.55 Hz.

(f) We know, for parallel connection of four spring, the equivalent spring constant be $4k$.

So, $k_p = 4k$

$= 4 \times 25 = 100$ N/cm $= 10000$ N/m

Now, $\omega = \sqrt{\frac{k_p}{m}}$

$= \sqrt{\frac{1000}{20}} = 22.36$ rad/sec

Frequency $f = \frac{\omega}{2\pi}$

$= \frac{22.36}{2\pi} = 3.55$ Hz

Time period $T = \frac{1}{f} = \frac{1}{3.55} = 0.28$ sec

(g) (i) When two springs of force constants k_1 and k_2 are attached to a mass *m* as shown in the figure, the time period of the motion is given by

$$T = 2\pi \sqrt{\frac{m}{k_1 + k_2}}$$

Here, $m = 3.0$ kg, $k_1 = k_2 = 600 \text{ Nm}^{-1}$.

$$T = 2\pi \sqrt{\frac{3.0}{600 + 600}} = 0.314 \text{ sec}$$

(ii) The maximum speed of the trolley is given by

$$v_{max} = r\,\omega = \frac{2\pi}{T}\, r$$

Here, $r = 5.0$ cm $= 0.05$ m

$\therefore$ $v_{max} = \frac{2\pi}{0.314} \times 0.05 = 1$ m/s

(iii) Total energy of the system

$$E = \frac{1}{2}\, m\, r^2\, \omega^2$$

$$= \frac{1}{2}\, m r^2 \left(\frac{2\pi}{T}\right)^2$$

$$= \frac{2\pi^2}{T^2} \cdot m r^2$$

$$= \frac{2\pi^2}{(0.314)^2} \times 3.0 \times (0.05)^2 = 1.5 \text{ j}$$

When the trolley comes to rest due to the damping forces, the total energy will be dissipated as heat.

SAQ 3

(a) (i) (b)

(ii) (d)

(iii) (a)

(iv) (b)

(v) (d)

(b) (i) We know, $T = 2\pi\sqrt{\frac{l}{g}}$

Here, $\Rightarrow \quad 1 = 2\pi\sqrt{\frac{l}{g}}$

$\Rightarrow \quad l = \frac{g}{4\pi^2}$

$l = \frac{9.81}{4\pi^2} = 0.248$ m

Again, $T = 2\pi\sqrt{\frac{l}{g}}$

$= 2\pi\sqrt{\frac{1}{9.81}} = 2.01$ sec $\quad [\therefore \; l = 1 \text{ m}]$

(ii) On the surface of the moon $g = 1.67$ m/s^2.

So, $T = 2\pi\sqrt{\frac{l}{g}}$

$\Rightarrow \quad 1 = 2\pi\sqrt{\frac{l}{1.67}}$

$\Rightarrow \quad l = \frac{1.67}{4\pi^2} = 0.0423$ m

Again, $T = 2\pi\sqrt{\frac{l}{g}}$

$= 2\pi\sqrt{\frac{1}{1.67}} = 4.862$ sec

(c) (i) For a simple pendulum

Here, $f = \frac{40}{100} = 0.40$ Hz

Now, $0.40 = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

$\Rightarrow \quad l = \frac{g}{(0.40 \times 2 \times \pi)^2} = 1.55$ m

(iii) We know, the maximum linear acceleration of the pendulum bob = $g\theta$ [θ = maximum amplitude].

$= 9.81 \times \sin 4^{\circ} = 0.684 \text{ ms}^{-2}$

(d) $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

Here, $l = 600$ mm $= 0.6$ m

Now, $$f = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.6}} = 0.643 \text{ Hz}$$

From the given figure we know ϕ

$$\tan\phi = \frac{3}{9.81}$$

$$\phi = 17^{o}$$

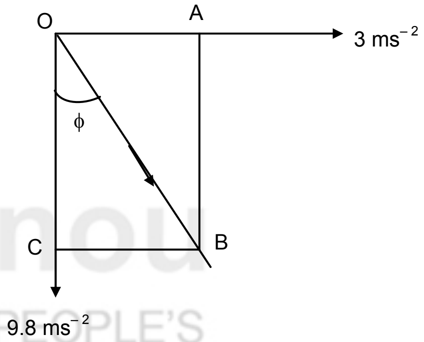


Figure 3.19

(e) Let the time is t, so $f_1 = \frac{4}{t}$ and $f_2 = \frac{3}{t}$.

Now, we know, $\frac{f_1}{f_2} = \sqrt{\frac{l_2}{l_1}}$

So, $$\frac{\frac{4}{t}}{\frac{3}{t}} = \sqrt{\frac{l + 0.48}{l}}$$

or $$\frac{4}{3} = \sqrt{\frac{l + 0.48}{l}}$$

$\Rightarrow$ $$\frac{16}{9} = \frac{l + 0.48}{l}$$

$\Rightarrow$ $$16l = 9l + 9 \times 0.48$$

$\Rightarrow$ $$l = 0.617 \text{ m and } l + 0.48 = 1.097 \text{ m}$$

So, $$l_1 = 0.617 \text{ m}$$

and $$l_2 = 1.097 \text{ m}$$

(f) (i) We know that

$$T = 2\pi\sqrt{\frac{L}{g}}$$

or $$L = \frac{gT^2}{4\pi^2}$$

$$= \frac{9.81 \times 24^2}{4\pi^2} = 143.12 \text{ m}$$

Therefore, height of the tower = 143.13 m.

(ii) Again $$T = 2\pi\sqrt{\frac{143.13}{1.67}} = 58.17 \text{ seconds}.$$