UNIT 8 INTEGRAL CALCULUS

Structure

8.1 INTRODUCTION

In this unit, we shall introduce the notions of antiderivatives, indefinite integral and various methods and techniques of integration. The unit will also cover definite integrals which can be evaluated using these methods.

We know that one of the problems which motivated the concept of a derivative was a geometrical one – that of finding a tangent to a curve at a point. The concept of integration was also similarly motivated by a geometrical problem – that of finding the areas of plane regions enclosed by curves. Some recently discovered Egyptian manuscripts reveal that the formulas for finding the areas of triangles and rectangles were known even in 1800 BC. Using these formulas, one could also find the area of any figure bounded by straight line segments. But no method for finding the area of figures bounded by curves had evolved till much later.

In the third century BC, Archimedes was successful in rigorously proving the formula for the area of a circle. His solution contained the seeds of the present day integral calculus. But it was only later, in the seventeenth century, that Newton and Leibniz were able to generalize. Archimedes' method and also to establish the link between differential and integral calculus. The definition of the definite integral of a function, which we shall give in this unit, was first given by Riemann in 1854. We will also acquaint you with various application of integration.

Objectives

After studying this unit, you should be able to

- compute the antiderivative of a given function,
- define the indefinite integral of a function,
- evaluate certain standard integrals by finding the antiderivatives of the integrals,
- compute integrals of various elementary and trigonometric functions,
- integrate rational functions of a variable by using the method of partial fractions,
- evaluate the integrals of some specified types of irrational functions,
- define the definite integral of a given function as a limit of a sum,
- state the fundamental theorems of calculus,
- learn the different properties of definite integral,
- use the fundamental theorems to calculate the definite integral of an integrable function, and
- use the definite integrals to evaluate areas of figures bounded by curves.

8.2 ANTIDERIVATIVES

In Unit 7, we have been occupied with the problem of finding the derivative of a given function. Some of the important applications of the calculus lead to the inverse problem, namely, given the derivative of a function, is it possible to find the function? This process is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative**. The importance of the antiderivative results partly from the fact that scientific laws often specify the rates of change of quantities. The quantities themselves are then found by antidifferentiation.

To get started, suppose we are given that $f'(x) = 9$, can we find $f(x)$? It is easy to see that one such function f is given by $f(x) = 9x$, since the derivative of $9x$ is 9.

Before making any definite decision, consider the functions

$$9x + 4, 9x - 10, 9x + \sqrt{3}$$

Each of these functions has 9 as its derivative. Thus, not only can $f(x)$ be $9x$, but it can also be $9x + 4$ or $9x - 10, 9x + \sqrt{3}$. Not enough information is given to help us determine which is the correct answer.

Let us look at each of these possible functions a bit more carefully. We notice that each of these functions differs from another only by a constant. Therefore, we can say that if $f'(x) = 9$, then $f(x)$ must be of the form $f(x) = 9x + c$, where c is a constant. We call $9x + c$ the antiderivative of 9.

More generally, we have the following definition.

Definition

Suppose f is a given function. Then a function F is called an antiderivative of f, if $F'(x) = f(x)\ \forall x$.

We now state an important theorem without giving its proof.

Theorem 1

If F_1 and F_2 are two antiderivatives of the same function, then F_1 and F_2 differ by a constant, that is

$$F_1(x) = F_2(x) + c$$

Remark

From above Theorem, it follows that we can find all the antiderivatives of a given function, once we know one antiderivative of it. For instance, in the above example, since one antiderivative of 9 is $9x$, all antiderivative of 9 have the form $9x + c$, where c is a constant. Let us do one example.

Example 8.1

Find all the antiderivatives of $4x$.

Solution

We have to look for a function F such that $F'(x) = 4x$. Now, an antiderivative of $4x$ is $2x^2$. Thus, by Theorem 1, all antiderivatives of $4x$ are given by $2x^2 + c$, where c is a constant.

SAQ 1

Find all the antiderivatives of each of the following function

(i) $f(x) = 10x$

(ii) $f(x) = 11x^{10}$

(iii) $f(x) = -5x$

8.3 BASIC DEFINITIONS

We have seen, that the antiderivative of a function is not unique. More precisely, we have seen that if a function F is an antiderivative of a function f, then $F + c$ is also an antiderivative of f, where c is any arbitrary constant. Now we shall introduce a notation here : we shall use the symbol $\int f(x)\,dx$ to denote the class of all antiderivatives of f. We call it the indefinite integral or just the integral of f. Thus, if $F(x)$ is an antiderivative of $f(x)$, then we can write $\int f(x)\,dx = F(x) + c$.

Here c is called the constant of integration. The function $f(x)$ is called the integrand, $f(x)\,dx$ is called the element of integration and the symbol $\int$ stands for the integral sign.

The indefinite integral $\int f(x)\,dx$ is a class of functions which differ from one another by constant. It is not a definite number; it is not even a definite function. We say that the indefinite integral is unique up to an arbitrary constant.

Thus, having defined an indefinite integral, let us get acquainted with the various techniques for evaluating integrals.

8.3.1 Standard Integrals

We give below some elementary standard integrals which can be obtained directly from our knowledge of derivatives.

Table 8.1

Sl. No.	Function	Integral
1	x^n	$\frac{x^{n+1}}{n+1} + c,\ n \neq -1$
2	$\sin x$	$-\cos x + c$
3	$\cos x$	$\sin x + c$
4	$\sec^2 x$	$\tan x + c$
5	$\mathrm{cosec}^2 x$	$-\cot x + c$
6	$\sec x \tan x$	$\sec x + c$
7	$\mathrm{cosec}\, x \cot x$	$-\mathrm{cosec}\, x + c$
8	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$ or $-\cos^{-1} x + c$
9	$\frac{1}{1+x^2}$	$\tan^{-1} x + c$ or $-\cot^{-1} x + c$
10	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + c$ or $-\mathrm{cosec}^{-1} x + c$
11	$\frac{1}{x}$	$\ln \lvert x \rvert + c$
12	e^x	$e^x + c$
13	a^x	$\frac{a^x}{\ln \lvert a \rvert} + c$

Now let us see how to evaluate some functions which are linear combination of the functions listed in Table 8.1.

8.3.2 Algebra of Integrals

You are familiar with the rule for differential of sum of functions, which says

$$\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

There is a similar rule for integration :

Rule 1

$$\int [a\,f(x) + b\,g(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

This rule follows from the two theorems.

Theorem 2

If f is an integrable function, then so is $K f(x)$ and $\int K\,f(x)\,dx = K\int f(x)\,dx$, where K is a constant.

Proof

Let $\int f(x)\,dx = F(x) + c$

Then by definition, $\frac{d}{dx}[F(x) + c] = f(x)$

$\therefore \qquad \frac{d}{dx}[K\{F(x) + c\}] = K\,f(x)$

Again, by definition of antiderivatives, we have

$$\int K\,f(x)\,dx = K[F(x) + c] = K\int f(x)\,dx$$

Theorem 3

If f and g are two integrable functions, then $f + g$ is integrable, and we have $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Proof

Let $\int f(x)\,dx = F(x) + c, \int g(x)\,dx = G(x) + c_1$

Then $\frac{d}{dx}[\{F(x) + c\} + \{G(x) + c_1\}] = f(x) + g(x)$

Thus, $\int [f(x) + g(x)]\,dx = [F(x) + c] + [G(x) + c_1]$

$$= \int f(x)\,dx + \int g(x)\,dx$$

Rule (1) may be extended to include a finite number of functions, that is, we can write

Rule 2

$$\int [K_1 f_1(x) + K_2 f_2(x) + \ldots + K_n f_n(x)]\,dx$$

$$= K_1 \int f_1(x)\,dx + K_2 \int f_2(x)\,dx + \ldots + K_n \int f_n(x)\,dx$$

We can make use of Rule (2) to evaluate certain integrals which are not listed in Table 8.1.

Example 8.2

Let us evaluate $\int (2 + 4x + 3\sin x + 4e^x)\,dx$

$$= 2\int dx + 4\int x\,dx + 3\int \sin x\,dx + 4\int e^x\,dx$$

$$= 2x + 2x^2 - 3\cos x + 4e^x + c$$

Example 8.3

Suppose we want to evaluate $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx$

Thus. $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx$

$$= \int \frac{1 - 2x + x^2}{x^{\frac{3}{2}}} dx$$

$$= \int x^{-\frac{3}{2}} dx - \int 2x^{-\frac{1}{2}} dx + \int x^{\frac{1}{2}} dx$$

$$= -2x^{-\frac{1}{2}} - 4x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c$$

And now some exercises for you.

SAQ 2

Write down the integrals of the following using Table 8.1 and Rule 2

(i) (a) x^8 (b) $x^{-\frac{5}{2}}$ (c) $4x^{-2}$ (d) 9

(ii) (a) $x^2 - x - 1$ (b) $\frac{1}{\sqrt{x}} - 3\sqrt{x}$ (c) $\left(x - \frac{1}{x}\right)^2$

(iii) (a) $e^x + e^{-x} + 4$ (b) $4\cos x - 3\sin x + e^x + x$ (c) $4\operatorname{sech}^2 x + e^x - 8x$

(iv) (a) $\frac{2}{\sqrt{1 - x^2}} + \frac{5}{x}$ (b) $\frac{2x^2 + 5}{x^2 + 1}$

(v) (a) $ax^3 + bx^2 + cx + d$ (b) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

(vi) (a) $\frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x}$ (b) $(2 + x)(3 - \sqrt{x})$

8.4 METHODS OF INTEGRATION

We have seen in Section 8.3 that the decomposition of an integrand into the sum of a number of integrands, with known integrals, is itself an important method of integration.

We now give two general methods of integration, namely,

(i) Integration by substitution,

(ii) Integration by parts.

The method of substitution consists in expressing the integral $\int f(x)\, dx$ in terms of another simpler integral, $\int F(t)\, dt$, say, where the variables x and t are connected by some suitable relation $x = \phi(t)$.

The method of integration by parts enables one to express the given integral of a product of two functions in terms of another, whose integration may be simpler.

8.4.1 Integration by Substitution

Consider the following integral

$$\int f'[g(x)]\, g'(x)\, dx \qquad \ldots(1)$$

Since $\quad \frac{d}{dx} f[g(x)] = f'[g(x)]\, g'(x)$ (by Chain rule)

$\therefore \quad \int f'[g(x)]\, g'(x)\, dx = f[g(x)] + c$

In Eq. (1), if we substitute $g(x) = t$

Then $\quad g'(x) = \frac{dt}{dx}$

i.e. $\quad g'(x)\, dx = dt$

Hence $\quad f'[g(x)]\, g'(x)\, dx = f'(t)\, dt$

$$\therefore \quad \int f'[g(x)]\, g'(x)\, dx = \int f'(t)\, dt$$

$$= f(t) + c$$

$$= f[g(x)] + c$$

Let us now illustrate this technique with examples.

Example 8.4

Find $\int (x^2 + 1)^3\, 2x\, dx$

Solution

Let $\quad t = x^2 + 1$

$dt = 2x\, dx$

Therefore, $\quad \int (x^2 + 1)^3\, 2x\, dx$

$$= \int t^3\, dt$$

$$= \frac{t^4}{4} + c = \frac{1}{4}(x^2 + 1)^4 + c, \text{ since } t = x^2 + 1.$$

Example 8.5

Find $\int x^3 e^{x^4}\, dx$.

Solution

Let $\quad t = x^4$

Then $\quad dt = 4x^3\, dx$

Therefore, $\quad \int x^3 e^{x^4}\, dx = \frac{1}{4} \int 4x^3 e^{x^4}\, dx$

$$= \frac{1}{4} \int e^t\, dt$$

$$= \frac{1}{4}\left[e^t + c\right]$$

$$= \frac{1}{4}\left[e^{x^4} + c\right], \text{ since } t = x^4$$

Some Typical Examples of Substitution

We now consider the integral $\int f(x)\,dx$, where the integrand $f(x)$ is in some typical form and the integral can be obtained easily by the method of substitution.

Various forms of integral can be obtained easily by the method of substitution. Various forms of integrals considered are as follows :

(a) $\int f(ax+b)\,dx$

To integrate $f(ax+b)$, put $ax + b = t$

Therefore $a\,dx = dt$ or $dx = \frac{1}{a}\,dt$

Thus $\int f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt$

which can be evaluated, once the right hand side is known, for example, to find $\int \cos(ax+b)\,dx$, we put $ax + b = t$ and $a\,dx = dt$ or $dx = \frac{1}{a}\,dt$.

Then $\int \cos(ax+b)\,dx = \frac{1}{a}\int \cos t\,dt = \frac{1}{a}\sin t + c$

or $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

Similarly, we have the following results

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c, n \neq -1$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c \text{ etc.}$$

You can make direct use of the above results in solving exercises.

(b) $\int f(x^n)\,x^{n-1}\,dx$

To integrate $f(x^n)\,x^{n-1}$, we let $x^n = t$.

Then $nx^{n-1}\,dx = dt$

and $\int f(x^n)\,x^{n-1}\,dx = \frac{1}{n}\int f(t)\,dt$

which can be found out once the right hand side is known.

For example, to find $\int x^2 \sin x^3\, dx$, put $x^3 = t$; then $3x^2\, dx = dt$, that is $x^2\, dx = \frac{1}{3}\, dt$.

Then, $$\int x^2 \sin x^3\, dx = \frac{1}{3}\int \sin t\, dt$$
$$= -\frac{1}{3}\cos t + c$$
$$= -\frac{1}{3}\cos x^3 + c$$

(c) $\int \{f(x)\}^n f'(x)\, dx,\ n \neq -1$

Putting $f(x) = t$; we see that $f'(x)\, dx = dt$ and

$$\int \{f(x)\}^n f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} + c$$
$$= \frac{\{f(x)\}^{n+1}}{n+1} + c$$

For example, $\int \cos^2 x \sin x dx$

$$= -\int t^2\, dt, \text{ where } t = \cos x \text{ (and hence } -dt = \sin x\, dx)$$

Therefore, $$\int \cos^2 x \sin x dx = -\frac{1}{3}t^3 + c$$
$$= -\frac{1}{3}(\cos x)^3 + c$$

(d) $\int \frac{f'(x)}{f(x)}\, dx$

Putting $f(x) = t$, we have $f'(x)\, dx = dt$

and $$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c$$

i.e. the integral of a function in which the numerator is the differential co-efficient of the denominator, is equal to the logarithm of the denominator (plus a constant).

For example, applying this result, we have

$$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{(-\sin x)}{\cos x}\, dx = c - \ln \cos x$$

(Since $f(x) = \cos x$, in this case.)

Therefore, $\int \tan x\, dx = c - \ln|\cos x|$

Similarly, you can obtain the following integrals

$$\int \cot x\, dx = \ln|\sin x| + c$$

$$\int \sec x \, dx = \ln |(\sec x + \tan x)| + c$$

$$\int \operatorname{cosec} x \, dx = \ln \left| \tan\left(\frac{x}{2}\right) \right| + c$$

Remember that logarithm of a quantity exists only when the quantity is positive. Thus, while making use of these formulas, make sure that the integrand to be integrated is positive in the domain under consideration.

(e) $\int f(a^2 \pm x^2)\, dx$

Under this category we now give some results obtained by putting $x = a\,t$ (and hence $dx = a\,dt$) and using some standard integrals.

Now, $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \qquad \left(\because \frac{d}{dt}(\tan^{-1} t) = \frac{1}{1+t^2}\right)$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \left(\because \frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1+t^2}}\right)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \qquad \left(\because \frac{d}{dt}(\sec^{-1} t) = \frac{1}{t\sqrt{t^2+1}}\right).$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) + c \left(\because \frac{d}{dt}\ln = (t + \sqrt{1+t^2} = \frac{1}{\sqrt{1+t^2}}\right)$$

We usually use all these integrals given under (a) – (e) directly whenever required without actually proving them. Sometimes it may happen that two or more substitutions have to be used in succession. We now illustrate this point with the help of the following example.

Example 8.6

Calculate

(i) $\displaystyle\int \frac{2x}{1+x^2}\, dx$

(ii) $\displaystyle\int \sin^3 x \cos^2 x \, dx$

Solution

(i) $\displaystyle\int \frac{2x}{1+x^2}\, dx$

Put $\quad 1 + x^2 = t$

Then $\quad 2\,x\,dx = dt$

$$\therefore \quad \int \frac{2x}{1+x^2}\, dx = \int \frac{1}{t}\, dt$$

$$= \ln t + c$$

$$= \ln(1 + x^2) + c$$

(ii) $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$

$$= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx$$

Put $\cos x = t$

Then $-\sin x \, dx = dt$

$\therefore$ $\int \sin^3 x \cos^2 x \, dx = -\int t^2 (1 - t^2) \, dt$

$$= -\int (t^2 - t^4) \, dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + c$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + c$$

$$= \frac{(\cos x)^5}{5} - \frac{(\cos x)^3}{3} + c$$

So far we have developed the method of integration by substitution, by turning the chain rule into an integration formula. Let us do the same for the product rule. We know that the derivative of the product of two functions $f(x)$ and $g(x)$ is given by

$$\frac{d}{dx}[f(x)\, g(x)] = g(x)\, f'(x) + f(x)\, g'(x),$$

where the dashes denote differentiation w. r. t. x. Corresponding to this formula, we have a rule called integration by parts.

8.4.2 Integration by Parts

Let us now discuss the method of integration by parts in detail. We begin by taking two functions $f(x)$ and $g(x)$. Let $G(x)$ be an antiderivative of $g(x)$, that is,

$$\int g(x)\, dx = G(x) \text{ or } G'(x) = g(x)$$

Then, by the product rule for differentiation, we have

$$\frac{d}{dx}[f(x)\, G(x)] = f(x)\, G'(x) + f'(x)\, G(x) = f(x)\, g(x) + f'(x)\, G(x)$$

Integrating both sides, we get

$$f(x)\, G(x) = \int f(x)\, g(x)\, dx + \int f'(x)\, G(x)\, dx$$

or $$\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx$$

Thus, $$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int f'(x) \{\int g(x)\, dx\}\, dx \quad \ldots (8.1)$$

The integration done by using the Eq. (8.1) is called integration by parts. In other words, it can be stated as follows :

The integral of the product of two functions

= first function × integral of the second function

– integral of (differential coefficient of the first × integral of the second).

We now illustrate this method through some examples.

Example 8.7

Integrate $x\,e^x$ with respect to x.

Solution

We use integration by parts.

Step 1

Take $f(x) = x$ and $g(x) = e^x$.

Then $f'(x) = 1$ and $\int e^x\,dx = e^x$

Step 2

From Eq. (8.1), we have

$$\int x\,e^x\,dx = x\,e^x - \int 1\,.\,e^x\,dx + c$$

or

$$\int x\,e^x\,dx = x\,e^x - e^x + c$$

Sometimes we need to integrate by parts more than once. We now illustrate it through the following example.

Example 8.8

Find $\int x^2 \cos x\,dx$

Solution

$$\int x^2 \cos x\,dx = x^2 \int \cos x\,dx - \int 2x \left\{\int \cos x\,dx\right\} dx$$

$$= x^2 \sin x - 2\int x \sin x\,dx + c_1 \qquad \ldots (8.2)$$

where c_1 is a constant of integration.

Integrating $\int x \sin x\,dx$, again by parts, we get

$$\int x \sin x\,dx = x \int \sin x\,dx - \int 1\,.\left\{\int \sin x\,dx\right\} dx$$

$$= -\,x\cos x + \int \cos x\,dx$$

$$= -\,x\cos x + \sin x + c_2 \qquad \ldots (8.3)$$

where c_2 being the constant of integration. From Eqs. (8.2) and (8.3), we get

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\,(-\,x\cos x + \sin x + c_2) + c_1,$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + c,$$

where we have written c for $c_1 - 2c_2$.

We now consider some examples of integrals which occur quite frequently and can be integrated by parts.

Example 8.9

Find $\int e^{ax} \cos bx\,dx$

Solution

Step 1

Choose $f(x) = e^{ax}$ and $g(x) = \cos bx$; then integrating by parts gives

$$\int e^{ax} \cos bx \, dx = e^{ax} \frac{\sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} dx + c_1 \qquad \ldots (8.4)$$

Step 2

Integrating $\int e^{ax} \sin bx \, dx$ by parts again, we get

$$\int e^{ax} \sin bx \, dx = e^{ax} \frac{(-\cos bx)}{b} - \int a e^{ax} \frac{(-\cos bx)}{b} dx + c_2$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx + c_2$$

Note that the second term on the right hand side is nothing but a constant multiple of the given integral.

Step 3

Substituting the value of $\int e^{ax} \sin bx \, dx$, in Eq. (8.4), we have

$$\int e^{ax} \cos bx \, dx = e^{ax} \frac{\sin bx}{b} - \frac{a}{b}\left[-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx + c_2\right] + c_1$$

$$= e^{ax} \frac{\sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx + c_3 \qquad \ldots (8.5)$$

where $\qquad c_3 = c_1 - \frac{a}{b} c_2$

Step 4

Transposing the last term from the right of Eq. (8.5) to left, we get

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + c_3$$

Dividing by $\left(1 + \frac{a^2}{b^2}\right)$, we finally get

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + c\,,$$

where $c = \frac{c_3}{a^2 + b^2}$, as the required integral.

Similarly, the integral of the type $\int e^{ax} \sin bx \, dx$ can be obtained.

And now some exercises for you.

SAQ 3

(a) Evaluate the following integrals :

(i) $\int \frac{dx}{9x^2 - 12x + 8}$

(ii) $\int \frac{x\,dx}{x^4 + x^2 + 1}$

(iii) $\int (\tan x)^5 \sec^2 x\,dx$

(iv) $\int \frac{dx}{e^x + 1}$

(v) $\int \frac{\cot x}{\ln \sin x}\,dx$

(vi) $\int \frac{1}{e^x - 1}\,dx$

(vii) $\int \frac{dx}{(e^x + e^{-x})^2}$

(viii) $\int x \sec^2 x^2\,dx$

(ix) $\int \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}}\,dx$

(x) $\int \frac{(1 + \ln x)^3}{x}\,dx$

(xi) $\int \frac{(\operatorname{cosec}^2 x)}{(1 + \cot x)}\,dx$

(b) Evaluate

(i) $\int x^2 \ln x\,dx$

(ii) $\int x \operatorname{cosec}^2 x\,dx$

(iii) $\int e^{3x} \cos 4x\,dx$

(iv) $\int \sin^{-1} x\,dx$

(v) $\int x \tan^{-1} x\,dx$

(vi) $\int \frac{x \sin^{-1} x}{\sqrt{(1 - x^2)}}\,dx$

(vii) $\int \frac{x\,e^x}{(1 + x^2)}\,dx$

8.5 INTEGRATION OF RATIONAL FUNCTIONS

We know, by now, that it is easy to integrate any polynomial function, that is, a function f given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$. In this section, we shall see how a rational function can be integrated.

Definition

A function R is called a rational function if it is given by $R(x) = \dfrac{Q(x)}{P(x)}$, *where*

Q (x) and P (x) are polynomials. It is defined for all x for which P (x) ≠ 0. If the degree of Q (x) is less than the degree of P (x), we say that R (x) is a proper rational function. Otherwise, it is called an improper rational function.

Thus $f(x) = \dfrac{x+1}{x^2+x+2}$ is a proper rational function, and $g(x) = \dfrac{x^3+x+5}{x-2}$ is an improper one. But $g(x)$ can also be written as $g(x) = (x^2 + 3x + 6) + \dfrac{17}{x-2}$ (by long division).

Here we have expressed $g(x)$, which is an improper rational function, as the sum of a polynomial and a proper rational function. This can be done for any improper rational function.

8.5.1 Some Simple Rational Functions

Now we shall consider some simple types of proper rational functions, like $\dfrac{1}{x-a}$, $\dfrac{1}{(x-b)^k}$ and $\dfrac{x-m}{ax^2+bx+c}$. We shall illustrate the method of integrating these functions through some examples.

Example 8.10

Consider the function $f(x) = \dfrac{1}{(x+2)^4}$.

Solution

To integrate this function we shall use the method of substitution.

Thus, if we put $u = x + 2$ or $\dfrac{du}{dx} = 1$, and we can write

$$\int \frac{1}{(x+2)^4}\,dx = \int \frac{1}{u^4}\,du = \frac{u^{-3}}{-3} + c = -\frac{1}{3(x+2)^3} + c.$$

Example 8.11

Consider the function $f(x) = \dfrac{2x+3}{x^2-4x+5}$.

Solution

This has a quadratic polynomial in the denominator. Now

$$\int \frac{2x+3}{x^2-4x+5}\,dx = \int \frac{2x-4}{x^2-4x+5}\,dx + \int \frac{7}{x^2-4x+5}\,dx.$$

Perhaps you are wondering why we have split the integral into two parts.

The reason for this break-up is that now the integrand in the first integral on the right is of the form $\dfrac{g'(x)}{g(x)}$; and we know that $\int \dfrac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c$.

Thus $\int \frac{2x-4}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + c_1$.

To evaluate the second integral on the right, we write

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x^2-4x+4)+1}\,dx = \int \frac{1}{(x-2)^2+1}\,dx.$$

Now, if we put $u = x-2, \frac{du}{dx} = 1$ and

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1} u + c_2 = \tan^{-1}(x-2) + c_2$$

This implies, $\int \frac{2x+3}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + 7\tan^{-1}(x-2) + c$.

8.5.2 Partial Fraction Decomposition

You must have studied the factorisation of polynomials. For example, we know that

$$x^2 - 5x + 6 = (x-2)(x-3)$$

Here $(x-2)$ and $(x-3)$ are two linear factors of x^2-5x+6.

You must have also come across polynomial like x^2+x+1, which cannot be factorised into real factors. Thus, it is not always possible to factorise a given polynomial into linear factors. But any polynomial can, in principle, be factorised into linear and quadratic factors. We shall not prove this statement here. It is a consequence of the fundamental theorem of algebra. The actual factorization of a polynomial may not be very easy to carry out. But, whenever we can factorise the denominator of a proper rational function, we can integrate it by employing the method of partial fractions. The following examples will illustrate this method.

Example 8.12

Let us evaluate $\int \frac{5x-1}{x^2-1}\,dx$.

Solution

Here the integrand $\frac{5x-1}{x^2-1}$ is a proper rational function.

Its denominator x^2-1 can be factored into linear factors as :

$$x^2 - 1 = (x-1)(x+1)$$

This suggests that we can write the decomposition of $\frac{5x-1}{x^2-1}$ into partial fraction as :

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

If we multiply both sides by $(x-1)(x+1)$, we get

$$5x - 1 = A(x+1) + B(x-1).$$

That is $5x - 1 = (A+B)x + (A-B)$

By equalling the coefficients of x, we get $A + B = 5$. Equating the constant terms on both sides, we get $A - B = -1$.

Solving these two equations in A and B, we get $A = 2$ and $B = 3$.

Thus, $$\frac{5x-1}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$$

Integrating both sides of this equation, we obtain

$$\int \frac{5x-1}{x^2-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{x+1}\,dx$$

$$= 2\ln|x-1| + 3\ln|x+1| + c$$

Let us go to our next example now.

Example 8.13

Evaluate $\int \frac{x}{x^3 - 3x + 2}\,dx$.

Solution

Take a look at the denominator of the integrand in $\int \frac{x}{x^3 - 3x + 2}\,dx$.

It factors into $(x-1)^2\,(x+2)$. The linear factor $(x - 1)$ is repeated twice in the decomposition of $x^3 - 3x + 2$. In this case, we write

$$\frac{x}{x^3 - 3x + 2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

From this point, we proceed as before to find A, B and C. We get

$$x = A\,(x-1)^2 + B\,(x+2)\,(x-1) + C\,(x+2)$$

We put $x = 1$, and $x = -2$, and get $C = \frac{1}{3}$ and $A = -\frac{2}{9}$. Then to find B, let us put any other convenient value, say $x = 0$.

This gives us $0 = A - 2B + 2C$

or $$0 = -\frac{2}{9} - 2B + \frac{2}{3}$$

This implies $B = \frac{2}{9}$

Thus $$\int \frac{x}{x^3 - 3x + 2}\,dx = -\frac{2}{9}\int \frac{1}{x+2}\,dx + \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx$$

$$= -\frac{2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\frac{1}{(x-1)} + c$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3\,(x-1)} + c$$

In our next example, we shall consider the case when the denominator of the integrand contains an irreducible quadratic factors (i.e. a quadratic factor which cannot be further factored into linear factors).

Example 8.14

Evaluate $\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx$

We factorise $x^4 - 2x^3 + x^2 - 2x$ as $x(x-2)(x^2+1)$.

Then we write $\frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{Cx + D}{x^2+1}$

Thus

$$6x^3 - 11x^2 + 5x - 4 = A(x-2)(x^2+1) + Bx(x^2+1) + (Cx + D)\, x(x-2)$$

Next, we substitute $x = 0$, and $x = 2$, to get $A = 2$ and $B = 1$. Then we put $x = 1$ and $x = -1$ (some convenient values) to get $C = 3$ and $D = -1$.

Thus, $\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx = 2\int \frac{1}{x}\, dx + \int \frac{1}{x-2}\, dx + \int \frac{3x-1}{x^2+1}\, dx$

$$= 2 \ln|x| + \ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}\, dx - \int \frac{dx}{x^2+1}$$

$$= 2 \ln|x| + \ln|x-2| + \frac{3}{2} \ln|x^2+1| - \tan^{-1} x + c$$

Thus, you see, once we decompose integrand, which is a proper rational function, into partial fractions, then the given integral can be written as the sum of some integrals of the type discussed in previous examples.

All the functions which we integrated till now were proper rational functions. Now we shall take up an example of an improper rational function.

Example 8.15

Let us evaluate $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx$.

Solution

Since the integrand is an improper rational function, we shall first write it as the sum of a polynomial and a proper rational function.

Now $\frac{x^3 + 2x}{x^2 - x - 2} = (x+1) + \frac{5x+2}{x^2 - x - 2}$

Therefore, $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \int (x+1)\, dx + \int \frac{5x+2}{x^2 - x - 2}\, dx$

$$= \int x\, dx + \int dx + 4\int \frac{dx}{x-2} + \int \frac{dx}{x+1}$$

Hence $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \frac{x^2}{2} + x + 4 \ln|x-2| + \ln|x+1| + c$

Try to do the following exercises now.

SAQ 4

Evaluate

(i) $\int \frac{x}{x^2 - 2x - 3}\, dx$

(ii) $\int \frac{3x - 13}{x^2 + 3x - 10}\, dx$

(iii) $\int \frac{6x^2 + 22x - 23}{(2x - 1)(x^2 + x - 6)}\, dx$

(iv) $\int \frac{x^2 + x - 1}{(x - 1)(x^2 - x + 1)}\, dx$

(v) $\int \frac{x^2}{x^2 - a^2}\, dx$

(vi) $\int \frac{x^2 + 4}{x^2 + 2x + 3}\, dx$

8.6 INTEGRATION OF IRRATIONAL FUNCTIONS

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{Q(x)}{P(x)}$. In this section, we shall give you some tips for evaluating some particular types of irrational functions. In most cases, our endeavour will be to arrive at a rational function through an appropriate substitution. This rational function can then be easily evaluated by using the techniques developed in Section 8.5.

Integration of Functions Containing only Fractional Powers of x

In this case, we put $x = t^n$, where n is the lowest common multiple (l. c. m.) of the denominators of powers of x. This substitution reduces the function to a rational function of t.

Look at the following example.

Example 8.16

Let us evaluate $\int \frac{2x^{1/2} + 3x^{1/3}}{1 + x^{1/3}}\, dx$

Solution

We put $x = t^6$, as 6 is the l. c. m. of 2 and 3. We get

$$\int \frac{2x^{1/2} + 3x^{1/3}}{1 + x^{1/3}}\, dx = \int \frac{2t^3 + 3t^2}{1 + t^2}\, 6t^5\, dt$$

$$= 6 \int \frac{2t^8 + 3t^7}{1 + t^2}\, dt = 6 \int \left[2t^6 + 3t^5 - 2t^4 - 3t^3 + 2t^2 + 3t - 2 - \frac{3t - 2}{1 + t^2} \right] dt$$

$$= 6\left[\frac{2}{7}t^7 + \frac{1}{2}t^6 - \frac{2}{5}t^5 - \frac{3}{4}t^4 + \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t - \frac{3}{2}\ln(1+t^2) + 2\tan^{-1} t\right] + c$$

$$= \frac{12}{7}x^{7/6} + 3x - \frac{12}{5}x^{5/6} - \frac{9}{2}x^{2/3} + 4x^{1/2} + 9x^{1/3} - 12x^{1/6} - 9\ln|1 + x^{1/3}| + 12\tan^{-1}(x^{1/6}) + c$$

Integrals of the Types

(i) $\int \sqrt{x^2 - a^2}\, dx$, (ii) $\int \sqrt{x^2 + a^2}\, dx$

(iii) $\int \sqrt{a^2 - x^2}\, dx$, (iv) $\int \sqrt{ax^2 + bx + c}\, dx$

(v) $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

Now, let us evaluate the above integrals.

(i) Let $I = \int \sqrt{x^2 - a^2}\, dx$

Integrating by parts taking 1 as the second function, we have

$$I = x\sqrt{x^2 - a^2} - \int x \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= x\sqrt{x^2 - a^2} - I - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$\therefore \quad 2I = x\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\left(\log x + \sqrt{x^2 - a^2}\right) + c_1, \text{ where } c_1 = \frac{c}{2}$$

Similarly

(ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + c$, and

(iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(iv) $\int \sqrt{ax^2 + bx + c}\, dx$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Put $\quad x + \frac{b}{2a} = t, \frac{c}{a} - \frac{b^2}{4a^2} = k^2$

Then the integral is reduced to any of the forms (i), (ii) or (iii).

(v) $\int (px + q) \sqrt{ax^2 + bx + c}\, dx$

Choose constants A and B such that

$$px + q = A\left[\frac{d}{dx}(ax^2 + bx + c)\right] + B$$

$$= A\,(2ax + b) + B$$

i.e. $\quad 2aA = p,\ Ab + B = q$

Thus the integral is reduced to

$$A \int (2ax + b)\sqrt{ax^2 + bx + c}\, dx + B \int \sqrt{ax^2 + bx + c}\, dx$$

$$= A I_1 + B I_2$$

$$I_1 = \int (2ax + b)\sqrt{ax^2 + bx + c}\, dx$$

Put $ax^2 + bx + c = t$

$$(2ax + b)\, dx = dt$$

i.e. $\quad I_1 = \frac{2}{3}(ax^2 + bx + c)^{3/2} + c_2$

Similarly, $I_2 = \int \sqrt{ax^2 + bx + c}\, dx$ which can be worked out as in (iv).

$\therefore$ (v) can be determined.

Example 8.17

Evaluate $\int \sqrt{4 - x^2}\, dx$.

Solution

$$\int \sqrt{4 - x^2}\, dx = \int \sqrt{2^2 - x^2}\, dx$$

$$= \frac{1}{2} x \sqrt{4 - x^2} + 2 \sin^{-1} \frac{x}{2} + c.$$

Example 8.18

Evaluate $\int \sqrt{x^2 + 2x + 5}\, dx$

Solution

$$\int \sqrt{x^2 + 2x + 5}\, dx = \int \sqrt{(x + 1)^2 + 4}\, dx$$

Put $\quad x + 1 = t$, then $dx = dt$

$\therefore \quad \int \sqrt{x^2 + 2x + 5}\, dx = \int \sqrt{t^2 + 4}\, dt = \int \sqrt{t^2 + 2^2}\, dt$

$$= \frac{1}{2} t \sqrt{t^2 + 4} + \frac{1}{2} . 4 \log\left(t + \sqrt{t^2 + 4}\right) + c$$

$$= \frac{1}{2}(x + 1)\sqrt{x^2 + 2x + 5} + 2 \log\left[(x + 1) + \sqrt{x^2 + 2x + 5}\right] + c$$

Example 8.19

Evaluate $\int x\sqrt{1+x-x^2}\,dx$

Solution

Let $x = A\left[\dfrac{d}{dx}(1+x-x^2)\right] + B$

$= A(1-2x) + B$

$\therefore \quad A = -\dfrac{1}{2},\ B = \dfrac{1}{2}$

Thus $\int x\sqrt{1+x-x^2}\,dx = -\dfrac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx$

$+\dfrac{1}{2}\int \sqrt{1+x-x^2}\,dx$

$= -\dfrac{1}{2}I_1 + \dfrac{1}{2}I_2$

$I_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx$

Put $1+x-x^2 = t$

Then $(1-2x)\,dx = dt$

$\therefore \quad I_1 = \int t^{\frac{1}{2}}\,dt = \dfrac{2}{3}t^{\frac{3}{2}} + c_1$

$= \dfrac{2}{3}(1+x-x^2)^{\frac{3}{2}} + c_1 \qquad \ldots (1)$

$I_2 = \int \sqrt{1+x-x^2}\,dx = \int \sqrt{\dfrac{5}{4} - \left(x-\dfrac{1}{2}\right)^2}\,dx$

Put $x - \dfrac{1}{2} = t$, then $dx = dt$

$\therefore \quad I_2 = \int \sqrt{\dfrac{5}{4} - t^2}\,dt$

$= \dfrac{1}{2}t\sqrt{\dfrac{5}{4} - t^2} + \dfrac{1}{2}\cdot\dfrac{5}{4}\sin^{-1}\dfrac{2t}{\sqrt{5}} + c_2$

$= \dfrac{1}{4}(2x-1)\sqrt{1+x-x^2} + \dfrac{5}{8}\sin^{-1}\dfrac{2x-1}{\sqrt{5}} + c_2$

Hence $\int x\sqrt{1+x-x^2}\,dx$

$= -\dfrac{1}{3}(1+x-x^2)^{\frac{3}{2}} + \dfrac{1}{8}(2x-1)\sqrt{1+x-x^2} + \dfrac{5}{16}\sin^{-1}\dfrac{2x-1}{\sqrt{5}} + c$

SAQ 5

Integrate the following functions :

(i) $\sqrt{x^2 + 4x + 6}$

(ii) $(x + 1)\sqrt{2x^2 + 3}$

(iii) $\sqrt{1 + 3x - x^2}$

8.7 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

If the integrand is a rational function of sin x or cos x or both, it can be reduced to a rational function by substituting $t = \tan \frac{x}{2}$.

Then $$\frac{dt}{dx} = \sec^2 \frac{x}{2} \cdot \frac{1}{2} = \frac{1 + t^2}{2}$$

i.e. $$dx = \frac{2dt}{1 + t^2}$$

$$\sin x = \frac{2t}{1 + t^2}, \cos x = \frac{1 - t^2}{1 + t^2} \text{ as } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

and $$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}.$$

Example 8.20

Evaluate $\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\, dx$

Solution

Let $\tan \frac{x}{2} = t,$

Then $$\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\, dx = \int \frac{1 + \frac{2t}{1 + t^2}}{\frac{2t}{1 + t^2}\left[1 + \frac{1 - t^2}{1 + t^2}\right]} \cdot \frac{2\, dt}{1 + t^2}$$

$$= 2 \int \frac{1 + t^2 + 2t}{2t\,[1 + t^2 + 1 - t^2]}\, dt$$

$$= \frac{1}{2} \int \frac{1 + t^2 + 2t}{t}\, dt = \frac{1}{2} \int \left[\frac{1}{t} + t + 2\right] dt$$

$$= \frac{1}{2} \left[\log|t| + \frac{t^2}{2} + 2t + c\right]$$

$$= \frac{1}{2} \log\left|\tan \frac{x}{2}\right| + \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + c$$

SAQ 6

Integrate the following :

(i) $\dfrac{1}{5 + 4 \sin x}$

(ii) $\dfrac{1}{2 + \cos \theta}$

(iii) $\dfrac{1}{1 + \sin x + \cos x}$

8.8 DEFINITE INTEGRALS

We have studied indefinite integrals so far. Now, we define a definite integral and see how it can be used to find the area under certain curves.

8.8.1 Definite Integral as the Limit of a Sum

Let f be a continuous function defined on a closed interval $[a, b]$. Assume that all the values taken by the function are non-negative, i.e. the graph of the function is a curve above the x-axis.

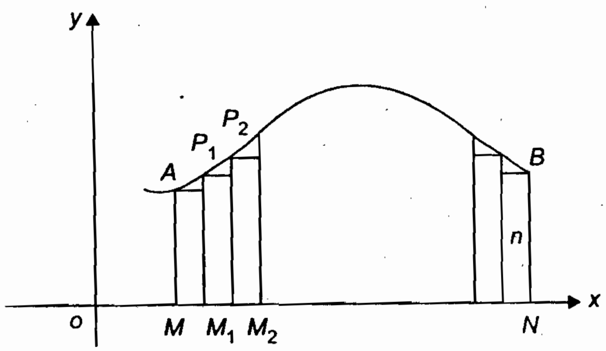

Figure 8.1

Consider the area of Figure 8.1. Let us find the area of this region.

Let AM and BN be the ordinates for $x = a$ and $x = b$. Divide MN into n equal parts of length h each and let $M_1 P_1, M_2 P_2, \ldots, M_{n-1} P_{n-1}$ be the ordinates at

$M_1, M_2, \ldots, M_{n-1}$, then $nh = b - a$, i.e. $h = \dfrac{b-a}{n}$.

Also abscissae of the point $A, P_1, P_2, \ldots, P_{n-1}, B$ are

$a, a + h, a + 2h, \ldots, a + \overline{n-1}\, h, b$.

$$\therefore \quad \begin{aligned} MA &= f(a) \\ M_1 P_1 &= f(a + h) \\ M_2 P_2 &= f(a + 2h) \\ &\vdots \\ M_{n-1} P_{n-1} &= f(a + \overline{n-1}\, h) \\ MB &= f(b) \end{aligned}$$

We consider the left end points of these sub regions and construct rectangles 1, 2, 3, . . . , n as shown in Figure 8.1.

Area of the first rectangle $= h f(a)$

Area of the second rectangle $= h f(a + h)$

Area of the third rectangle $= h f(a + 2h)$

. . . .

. . . .

. . . .

Area of the nth rectangle $= h\, f(a + \overline{n-1}\, h)$

$\therefore$ Sum of these areas $= h\, f(a) + h\, f(a + h) + \ldots + h\, f(a + \overline{n-1}\, h)$

We note that this area is approximately equal to the area of the region AMNB. Further as the number of sub-divisions increases, the estimation becomes better. Let the subdivisions become very large, i.e. $n \to \infty$, then $h = \dfrac{b-a}{n} \to 0$, which in turn implies that the area of the region AMNB

$$= \underset{h \to 0}{\text{Lt}}\; h\,[f(a) + f(a + h) + f(a + 2h) + \ldots + f(a + \overline{n-1}\, h)] \qquad \ldots (1)$$

The expression on the R. H. S of Eq. (1) is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)\, dx$, where a is called the lower limit and b is called the upper limit.

Thus, $\int_a^b f(x)\, dx = \underset{h \to 0}{\text{Lt}}\; [f(a) + f(a + 2h) + \ldots + f(a + \overline{n-1}\, h)]$, where $nh = b - a$.

Cor.

$\int_a^b f(x)\, dx$ = the area of the region below the curve $y = f(x)$ above the x-axis and bounded by the ordinates $x = a$ and $x = b$.

Remarks

For simplicity of the above concept, we have taken non-negative values of $f(x)$. In fact it makes sense for negative values of $f(x)$ as well.

Example 8.21

Evaluate $\int_a^b x^2\,dx$ as the limit of a sum.

Solution

$$\int_a^b x^2\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+\overline{n-1}\,h)]$$, where $nh = b - a$ and $f(x) = x^2$.

i.e. $$\int_a^b x^2\,dx = \lim_{h\to 0} h\left[a^2 + (a+h)^2 + (a+2h)^2 + \ldots + (a+\overline{n-1}\,h)^2\right]$$

$$= \lim_{h\to 0} h\left[(a^2 + a^2 + \ldots + a^2) + 2ah\,(1 + 2 + 3 + \ldots + \overline{n-1}) + (1^2 + 2^2 + 3^2 + \ldots + \overline{n-1}^2)\,h^2\right]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah\frac{(n-1)\,n}{2} + \frac{h^2\,(n-1)\,n\,[2(n-1)+1]}{6}\right]$$

$$= \lim_{h\to 0}\left[a^2\,nh + a\,(nh)\,(nh-h) + \frac{1}{6}\,nh\,(nh-h)\,(2nh-h)\right]$$

$$= \lim_{h\to 0}\left[a^2\,(b-a) + a\,(b-a)\,(b-a-h) + \frac{1}{6}\,(b-a)\,(b-a-h)\,(2\overline{b-a}-h)\right]$$

$$= a^2\,(b-a) + a\,(b-a)^2 + \frac{1}{6}\,(b-a)^2\;2\,(b-a)\,(b-a)$$

$$= (b-a)\,[a^2 + a\,(b-a) + \frac{1}{3}\,(b-a)^2] = \frac{1}{3}\,(b^3 - a^3)$$

Example 8.22

Evaluate $\int_0^2 e^x\,dx$ as a limit of a sum.

Solution

Here $b - a = 2 - 0 \therefore nh = 2$, i.e. $h = \frac{2}{n}$ and $f(x) = e^x$

$$\int_0^2 e^x\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+\overline{n-1}\,h)]$$

$$= \lim_{h\to 0} h\,[e^a + e^{a+h} + e^{a+2h} + \ldots + e\;a + (n-1)\,h]$$

$$= \lim_{h\to 0} h\,[e^0 + e^h + e^{2h} + \ldots + e^{\overline{n-1}\,h}]$$ as $a = 0$

$$= \lim_{h\to 0} h\left[\frac{e^{nh}-1}{e^h-1}\right]$$ using the formula for the sum of a G. P.

$$= \lim_{h\to 0} h\left[\frac{e^2-1}{e^h-1}\right]$$

$$= \frac{e^2-1}{\lim_{h\to 0}\frac{e^h-1}{h}} = e^2 - 1.$$

Example 8.23

Evaluate $\int_0^{\pi} \sin x \, dx$ as the limit of a sum.

Solution

Here $\quad f(x) = \sin x, a = 0, b = \pi$

$$\therefore \quad h = \frac{b-a}{n} = \frac{\pi}{n}, \text{ i.e. } nh = \pi$$

$$f(a) = f(0) = \sin 0 = 0$$

$$f(a+h) = f(h) = \sin h$$

$$f(a+2h) = f(2h) = \sin 2h$$

. . . .

$$f(a + \overline{n-1}\,h) = \sin \overline{n-1}\,h$$

$$\therefore \quad \int_0^{\pi} \sin x \, dx = \underset{h \to 0}{\text{Lt}} \; h \left[0 + \sin h + \sin 2h + \ldots + \sin (\overline{n-1}\,h)\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h}{2 \sin \frac{h}{2}} \left[2 \sin h \sin \frac{h}{2} + 2 \sin 2h \sin \frac{h}{2} + \ldots + 2 \sin (nh - h) \sin \frac{h}{2}\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{\frac{\sin \frac{h}{2}}{\frac{h}{2}}} \left[\left(\cos \frac{h}{2} - \cos \frac{3h}{2}\right) + \left(\cos \frac{3h}{2} - \cos \frac{5h}{2}\right) + \ldots + \left[\cos \left(nh - \frac{3h}{2}\right) - \cos \left(nh - \frac{h}{2}\right)\right]\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{\frac{\sin \frac{h}{2}}{\frac{h}{2}}} \left[\cos \frac{h}{2} - \cos \left(nh - \frac{h}{2}\right)\right]$$

$$= 1 \,.\, \underset{h \to 0}{\text{Lt}} \left[\cos \frac{h}{2} - \cos \left(\pi - \frac{h}{2}\right)\right] \text{ as } nh = \pi \text{ and } \underset{h \to 0}{\text{Lt}} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 1$$

$$= \underset{h \to 0}{\text{Lt}} \left[\cos \frac{h}{2} + \cos \frac{h}{2}\right] = 1 + 1 = 2.$$

SAQ 7

Evaluate the following definite integrals as a limit of a sum :

(i) $\int_a^b e^x \, dx$

(ii) $\int_a^b \cos x \, dx$

(iii) $\int_1^2 (x^2 - 1) \, dx$

8.9 FUNDAMENTAL THEOREM OF CALCULUS

8.9.1 Area Function

We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. Let $x \in [a, b]$.

Then $\int_a^x f(x)\,dx$ represents the area of the shaded region in Figure 8.2.

(Here it is assumed that $f(x) > a$ for $x \in [a, b]$.)

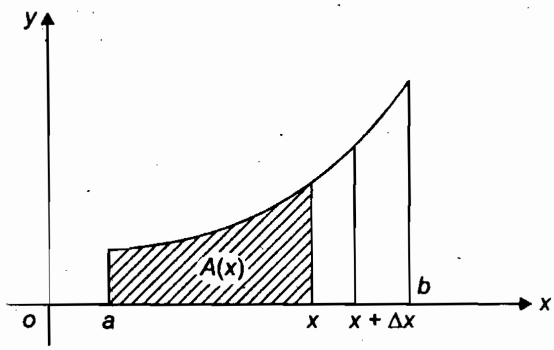

Figure 8.2

The area of this shaded region depends on x, i.e. in other words is a function of x. We denote it by $A(x)$

$$\therefore \qquad A(x) = \int_a^x f(x)\,dx$$

We will now state two fundamental theorems of integral calculus.

8.9.2 First Fundamental Theorem of Integral Calculus

Let the area function be defined by $A(x) = \int_a^x f(x)\,dx$ for all $x \geq a$, where the function f is continuous on $[a, b]$. Then $A'(x) = f(x)$ for all $x \in [a, b]$.

(We assume it without proof.)

8.9.3 Second Fundamental Theorem of Integral Calculus

Let f be a continuous function defined on an interval $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(We assume it without proof.)

Remarks

(i) $\int_a^b f(x)\,dx$ = (value of an antiderivative at the upper limit b) – (value of the same antiderivative at the lower limit a).

(ii) This theorem is very useful as it gives us a method of calculating a definite integral more easily without calculating the limit of a sum.

For convenience $F(b) - F(a)$ is denoted by $F(x)|_a^b$.

(iii) If we consider $F(x) + c$ to be an antiderivative value of $f(x)$ instead of $F(x)$, then

$$\int_a^b f(x)\,dx = [F(x) + c]_a^b = (F(b) + c) - (F(a) - c)$$
$$= F(b) - F(a)$$

Hence, there is no need to keep the integration constant c in definite integrals.

Example 8.24

Evaluate $\int_0^4 x^{\frac{3}{2}}\,dx$

Solution

$$\int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}x^{\frac{5}{2}}$$

$$\therefore \quad \int_0^4 x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_0^4 = \frac{2}{5}\left(4^{\frac{5}{2}} - 0\right)$$

$$= \frac{2}{5}.2^5 = \frac{64}{5}$$

Example 8.25

Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Solution

$$\int \frac{\tan^{-1}x}{1+x^2}\,dx = \frac{(\tan^{-1}x)^2}{2}$$

(Let $\tan^{-1}x = t$, then $\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int t\,dt = \frac{t^2}{2}$)

$$\therefore \quad \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \frac{1}{2}(\tan^{-1}x)^2\Big|_0^1$$

$$= \frac{1}{2}\left[(\tan^{-1}1)^2 - (\tan^{-1}0)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right]$$

$$= \frac{1}{2}\cdot\frac{\pi^2}{16} = \frac{\pi^2}{32}$$

Example 8.26

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx$$

Now $\int \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx = \tan^{-1} (\tan^2 x)$

(Put $\tan^2 x = t \therefore 2\tan x \sec^2 x dx = dt$ and $\int \frac{2\tan \sec^2 x}{\tan^2 x + 1} = \int \frac{dt}{t^2 + 1} = \tan^{-1} t$)

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^4 x + \cos^4 x} dx = \tan^{-1} (\tan^2 x) \Big|_0^{\frac{\pi}{2}}$$

$$= \tan^{-1}\left(\tan^2 \frac{\pi}{2}\right) - \tan^{-1} (\tan^2 0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

8.9.4 Evaluation of a Definite Integral by Substitution

When we use the method of substitution for evaluating an integral $\int_b^a f(x)\, dx$, we follow the following steps :

Step 1

Substitute $x = g(y)$.

Step 2

Integrate the new integrand with respect to y.

Step 3

Resubstitute the value of y in terms of x in the answer.

Step 4

Find the value of the answer in Step 3 at the given limits and find the difference.

In order to quicken this method we can proceed as follows :

After performing Step 2, there is no need for Step 3. Instead the integral will be kept in the new variable y and the limit of the integral will be accordingly changed.

Example 8.27

Evaluate $\int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\, dx$.

Solution

Let $t = x^5 + 1$, then $dt = 5x^4\, dx$.

When $x = 1, t = 1^5 + 1 = 2$ and when $x = -1$, $t = (-1)^5 + 1 = -1 + 1 = 0$.

$$\therefore \int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\, dx = \int_0^2 \sqrt{t}\, dt$$

$$= \frac{2}{3} t^{\frac{3}{2}} \Big|_0^2$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0\right) = \frac{4\sqrt{2}}{3}.$$

SAQ 8

(a) Evaluate

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$

(ii) $\int_0^2 \frac{6x+3}{x^2+4} \, dx$

(iii) $\int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos x \, dx$

(ii) $\int_0^\pi \frac{dx}{5 + 4 \cos x}$

(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{2 \cos x + 4 \sin x}$

8.10 PROPERTIES OF DEFINITE INTEGRALS

We consider below some important properties of the definite integral. These will be useful in evaluating the definite integrals more easily.

Property 1

$$\int_a^b f(x)\, dx = - \int_b^a f(x)\, dx$$

Property 2

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad \text{for } a < c < b$$

Property 3

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

Property 4

$$\int_0^{2a} f(x)\, dx = 2 \int_0^a f(x)\, dx \quad \text{if } f(2a-x) = f(x)$$

$$= 0 \qquad \text{if } f(2a-x) = -f(x)$$

Property 5

$$\int_{-a}^a f(x)\, dx = 2 \int_0^a f(x)\, dx \quad \text{if } f \text{ is an even function.}$$

$$= 0 \qquad \text{if } f \text{ is an odd function.}$$

We give proof of these properties.

Property 1

Let F be an antiderivative of f.

Then $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

Property 3

Let $t = a - x$

Then $dt = -dx$

When $x = 0, t = a$ and when $x = a, t = 0$

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt$$

$$= +\int_0^a f(a-t)\,dt \text{ by Property 1}$$

$$= +\int_0^a f(a-x)\,dx \text{ by changing the variable } t \text{ to } x.$$

Property 4

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \text{ by Property 2}$$

Put $t = 2a - x$ in the second integral

Then $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt$

$$= \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

$$\therefore \quad \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= 2\int_0^a f(x)\,dx \text{ or } 0$$

according as $f(2a-x) = f(x)$

or $f(2a-x) = -f(x)$

Property 2 and Property 5 are left as exercises.

Example 8.28

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

Solution

Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \qquad \ldots (1)$$

by Property 3

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \ldots (2)$$

Adding Eqs. (1) and (2), we have

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} dx = x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \qquad I = \dfrac{\pi}{4}$

Example 8.29

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\, dx$

Solution

$\cos^2 x$ is an even function.

$\therefore$ by Property 5

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\, dx = 2\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1 + \cos 2x}{2}\, dx$$

$$= \frac{2}{2}\left| x + \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin 0\right) = \frac{\pi}{4} + \frac{1}{2}$$

Example 8.30

Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x\, dx$

Solution

Let $\qquad I = \int_0^{\frac{\pi}{2}} \log \sin x\, dx$

Then $\qquad I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$$= \int_0^{\frac{\pi}{2}} \log \cos x\, dx$$

$\therefore \qquad 2I = \int_0^{\frac{\pi}{2}} \{\log \sin x + \log \cos x\}\, dx$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cos x}{2}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{\sin 2x}{2}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \log 2 \,.\, x \Big|_0^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \frac{\pi}{2} \log 2 \qquad \ldots (1)$$

To evaluate $\int_0^{\frac{\pi}{2}} \log \sin 2x\, dx$,

Putting $2x = t$, we have $\int_0^{\frac{\pi}{2}} \log \sin 2x\, dx = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt$

$$= \frac{1}{2} \int_0^{\pi} \log \sin x\, dx$$

$$= \frac{1}{2} \,.\, 2 \int_0^{\frac{\pi}{2}} \log \sin x\, dx \text{, as } \sin(\pi - x) = \sin x$$

$$= I \qquad \ldots (2)$$

From Eqs. (1) and (2), we have

$\therefore \qquad 2I = I - \frac{\pi}{2} \log 2$

i.e. $\qquad I = -\frac{\pi}{2} \log 2$

SAQ 9

Evaluate

(i) $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$

(iii) $\int_0^{\frac{\pi}{2}} \frac{x\, dx}{\sin x + \cos x}$

(iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x\, dx$

(v) $\int_0^1 \frac{\log(1 + x)}{1 + x^2}\, dx$

8.11 APPLICATIONS

We have seen that the area below (or above) the curve $y = f(x)$, above (or below) the x-axis and between the ordinates $x = a$ and $x = b$ is represented by the definite integral

$$\int_a^b f(x)\,dx = \int_a^b y\,dx$$

Likewise the area enclosed between the graph of the curve $x = F(y)$, y-axis and the lines $y = c$, $y = d$ is given by

$$\int_c^d F(y)\,dy = \int_c^d x\,dy$$

Example 8.31

Draw a rough sketch of the curve $y = \sqrt{3x + 4}$ and find the area under the curve, above the x-axis and between $x = 0$, $x = 4$.

Solution

$$y = \sqrt{3x + 4}$$

$\therefore$ Its domain consists of those x for which $3x + 4 \geq 0$, i.e. $x \geq -\frac{4}{3}$.

We construct the table of values as under

x	$-\frac{4}{3}$	-1	0	1	2	3	4
y	0	1	2	$\sqrt{7}$	$\sqrt{10}$	$\sqrt{13}$	4

A portion of the rough sketch of curve is shown in Figure 8.3.

Required area is the shaded area $= \int_0^4 f(x)\,dx$.

$$= \int_0^4 \sqrt{3x + 4}\,dx$$

$$= \frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2}\cdot 3}\Bigg|_0^4 = \frac{2}{9}\left(16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{2}{9}\left[(4^2)^{\frac{3}{2}} - (2^2)^{\frac{3}{2}}\right]$$

$$= \frac{2}{9}[4^3 - 2^3] = \frac{2}{9}[64 - 8] = \frac{112}{9} \text{ sq. units}$$

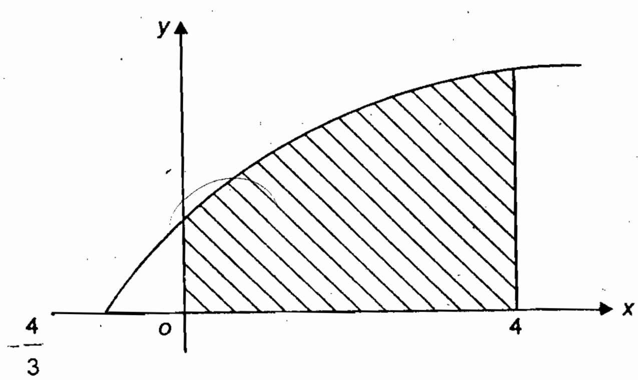

Figure 8.3

Example 8.32

Make a rough sketch of the graph of the function $y = 3 \sin x$, $0 \le x \le \pi$ and determine the area enclosed by the curve and the x-axis.

Solution

We construct the table of values as under

X	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
Y	0	$\frac{3}{2}$	$\frac{3\sqrt{3}}{2}$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{2}$	0

A rough sketch of the curve is shown in Figure 8.4.

$$\text{Required Area} = \int_0^{\pi} f(x)\, dx$$

$$= \int_0^{\pi} 3 \sin x\, dx$$

$$= [3(-\cos x)]\Big|_0^{\pi} = -3[\cos \pi - \cos 0^\circ]$$

$$= -3(-1-1) = 6 \text{ sq. units}$$

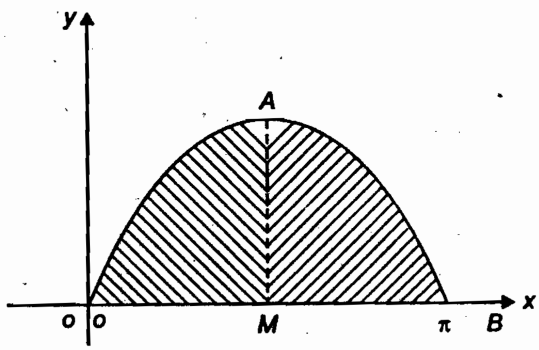

Figure 8.4

Note : Since the curve is symmetrical about the line $x = \frac{\pi}{2}$.

$\therefore$ Required Area = 2 Area OAM

$$= 2\int_0^{\frac{\pi}{2}} f(x)\, dx = 2\int_0^{\frac{\pi}{2}} 3 \sin x\, dx$$

$$= -6 \cos x \Big|_0^{\frac{\pi}{2}} = -6\left(\cos\frac{\pi}{2} - \cos 0^\circ\right)$$

$$= -6(0-1) = 6 \text{ sq. units}$$

Remark

In case of symmetrical closed area, find the area of the smaller part and multiply the result by the number of symmetrical parts.

Example 8.33

Find the area enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and above the line $\frac{x}{a} + \frac{y}{b} = 1$ which lies in the first quadrant.

Solution

The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

and the line is $\frac{x}{a}+\frac{y}{b}=1$ ... (2)

Line (2) meets the curve (1) in $A\ (a, 0)$ and $B\ (0, b)$. The required area is shown in Figure 8.5.

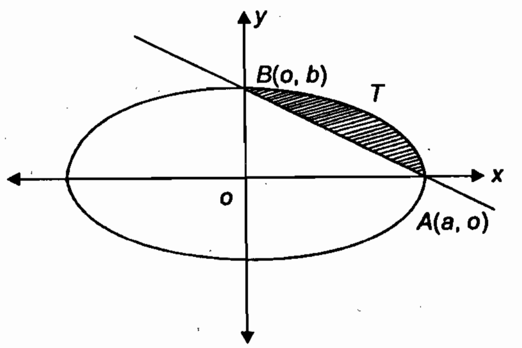

Figure 8.5

For the ellipse

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

i.e., $$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

i.e. for the first quadrant

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

Shaded Area = Area OATB – Area of the triangle OAB

Area of the triangle OAB $=\frac{1}{2}\text{OA}\,.\,\text{OB}=\frac{1}{2}ab$

Area OATB = Area bounded by the ellipse, x-axis in the first quadrant.

$$=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]\Bigg|_0^a$$

$$=\frac{b}{2a}[0+a^2\sin^{-1}1-(0+a^2\sin^{-1}0)]$$

$$=\frac{b}{2a}\left[a^2\frac{\pi}{2}-a^2\,.\,0\right]=\frac{\pi ab}{4}$$

Required area $=\frac{\pi ab}{4}-\frac{1}{2}ab=\frac{(\pi-2)\,ab}{4}$ sq. units.

xample 8.34

Find the area of the region bounded by the parabola $y=x^2+2$ and the lines $y=x, x=0, x=3$.

Solution

$y = x$ is the equation of a straight line lying below the parabola and the line $x = 3$ meets the parabola at (3, 11). The line $y = x$ meets the line $x = 3$ at (3, 3). The region whose area is required is shaded and shown in Figure 8.6.

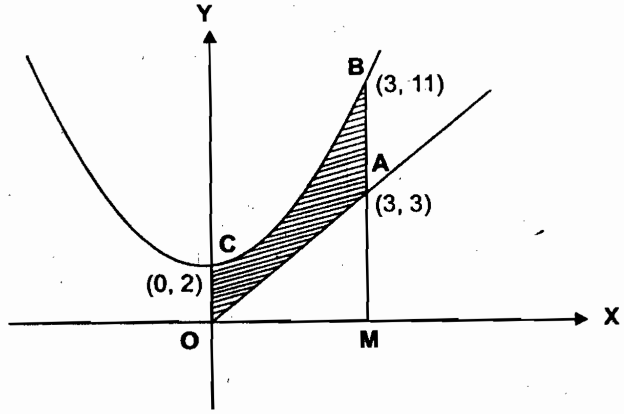

Figure 8.6

Required Area = Area bounded by the parabola, x-axis and the ordinates $x = 0, x = 3$ – (Area bounded by the line $y = x$, x-axis and the ordinates $x = 0, x = 3$).

$$= \int_0^3 (x^2 + 2)\, dx - \int_0^3 x\, dx$$

$$= \left(\frac{x^3}{3} + 2x\right)\Big|_0^3 - \frac{x^2}{2}\Big|_0^3$$

$$= 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

Note : Area bounded by the line $y = x$, x-axis and the ordinates at $x = 0$, and $x = 3$ is also the area of the triangle $\text{OAM} = \frac{1}{2}\text{OM.AM} = \frac{1}{2}.3.3 = \frac{9}{2}$.

SAQ 10

Find the area of the regions

(i) bounded by $y^2 = 9x$, $x = 2$ and $x = 4$ and the x-axis in the first quadrant.

(ii) bounded by $x^2 = y - 3$, $y = 4$, $y = 6$ and the y-axis in the first quadrant.

(iii) bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(iv) bounded by the circle $x^2 + y^2 = 4$, the line $x = \sqrt{3}\, y$, x-axis lying in the first quadrant.

(v) bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

(vi) enclosed between the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

8.12 SUMMARY

The main points covered in this unit are

- Given the derivative of a function, the process to find the function is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative.**
- The indefinite integral $\int f(x)\,dx$ denotes the class of all antiderivatives of $f(x)$.
- (a) $\int K f(x)\,dx = K \int f(x)\,dx$

  (b) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- By the method of substitution :

  (a) $\int f(x)\,dx = \int f[\phi(t)]\,\phi'(t)\,dt$

  (b) $\int [f(x)]^n . f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}$, where $n+1 \neq 0$

  (c) $\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)|$.
- By the method of integration by parts :

  Integral of product of two functions = first function × integral of second function – integral of (derivative of first function × integral of second function).
- A rational function f of x is given by $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial in x. It is called **proper** if the degree of $P(x)$ is less than the degree of $Q(x)$. Otherwise it is called **improper.**
- To integrate a proper rational function, we decompose the denominator into either linear or quadratic factors.
- Some rules to integrate irrational functions are

  (a) To integrate functions containing only fractional powers of x, put $x = t^n$, where n is l. c. m. of denominators of powers of x.

  (b) Rational functions of $\sin x$ or $\cos x$ or both can be reduced to a rational function of t by substituting $\tan \dfrac{x}{2} = t$ and then can be integrated.
- If f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx$ represents the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$.
- $\int_a^b f(x)\,dx = \lim\limits_{h\to 0} h \sum\limits_{i=1}^{n} f[a + (i-1)h]$, where $h = \dfrac{b-a}{n}$.
- Fundamental theorem of calculus

  (i) If f is continuous on $[a, b]$ then for all $x \in [a, b]$ if $A(x) \int_a^x f(x)\,dx$ then $A'(x) = f(x)$ for all $x \in [a, b]$.

(ii) If f is continuous function on $[a, b]$ and F is an antiderivative of f then $\int_a^b f(x)\,dx = F(b) - F(a)$.

- Area bounded by a curve $y = f(x)$, x-axis and the lines $x = a, x = b$ is $\int_a^b f(x)\,dx = \int_a^b y\,dx$.
- Area bounded by a curve $x = g(y)$, y-axis and the lines $y = c, y = d$ is $\int_c^d g(y)\,dy = \int_c^d x\,dy$

8.13 ANSWERS TO SAQs

SAQ 1

(i) $5x^2 + c$

(ii) $x^{11} + c$

(iii) $\frac{-5x^2}{2} + c$

SAQ 2

(i) (a) $\frac{x^9}{9} + c$

(b) $-\frac{2}{3} x^{-3/2} + c$

(c) $-4x^{-1} + c$

(d) $9x + c$

(ii) (a) $\frac{x^3}{3} - \frac{x^2}{2} - x + c$

(b) $2x^{1/2} - 2x^{3/2} + c$

(c) $\frac{x^3}{3} - 2x - \frac{1}{x} + c$

(iii) (a) $e^x - e^{-x} + 4x + c$

(b) $4\sin x + 3\cos x + e^x + \frac{x^2}{2} + c$

(c) $4\tanh x + e^x - 4x^2 + c$

(iv) (a) $2\sin^{-1} x + 5\ln|x| + c$

(b) $\int \frac{2(x^2+1)+3}{x^2+1}\,dx$

$= 2\int dx + 3\int \frac{1}{x^2+1}\,dx$

$= 2x + 3\tan^{-1} x + c$

(v) (a) $\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + c_1$

(b) $\frac{x^2}{2} - 2x + \ln|x| + c$

(vi) (a) $\int \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x + \cos^2 x) - 2 \sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{1 - 2\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{1}{\sin^2 x} dx + \int \frac{1}{\cos^2 x} dx - 2 \int dx = -\cot x + \tan x - 2x + c$

(b) $6x - \frac{4}{3} x^{\frac{3}{2}} + \frac{3}{2} x^2 - \frac{2}{5} x^{\frac{5}{2}} + c$

SAQ 3

(a) (i) $\frac{1}{6} \tan^{-1} \frac{3x - 2}{3} + c$

(ii) $\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + c$

(iii) $\frac{1}{6} \tan^6 x + c$

(iv) $x - \ln|(1 + e^x)| + c$

(v) $\ln|(\ln|\sin x|)| + c$

(vi) $\log|(e^x - 1)| - x + c$

(vii) $-\frac{1}{2} \frac{1}{(e^{2x} + 1)} + c$

(viii) $\frac{1}{2} \tan x^2 + c$

(ix) Put $\sin^{-1} x = t$

So $\frac{1}{\sqrt{1 - x^2}} dx = dt$ and $\int \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}} dx = \int t^2 \, dt$

$= \frac{t^3}{3} + c = \frac{1}{3} (\sin^{-1} x)^3 + c$

(x) $\frac{1}{4} (1 + \log x)^4 + c$

(xi) $-\ln|(1 + \cot x)| + c$

(b) (i) $(3 \ln|x| - 1) \frac{x^3}{9} + c$

(ii) $\ln|\sin x)| - x \cot x + c$

(iii) $\frac{e^{3x} (4 \sin 4x + 3 \cos 4x)}{25} + c$

(iv) $x \sin^{-1} x + \sqrt{1 - x^2} + c$

(v) $\int x \tan^{-1} x \, dx = (\tan^{-1} x) \, . \, \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{1 + x^2} dx$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

$$= \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + c$$

(vi) $- \phi \cos \phi + \sin \phi + c$, where $\phi = \sin^{-1} x$

(vii) $\frac{e^x}{1 + x}$

SAQ 4

(i) $\frac{3}{4} \ln |x - 3| + \frac{1}{4} \ln |x + 1| + c$

(ii) $4 \ln |x + 5| - \ln |x - 2| + c$

(iii) $\frac{6x^2 + 22x - 23}{(2x - 1)(x^2 + x - 6)} = \frac{6x^2 + 22x - 23}{(2x - 1)(x + 3)(x - 2)} = \frac{A}{2x - 1} + \frac{B}{x + 3} + \frac{C}{x - 2}$

$$6x^2 + 22x - 23 = A (x + 3)(x - 2) + B (x - 2)(2x - 1) + C (2x - 1)(x + 3)$$

$$x = 2 \Rightarrow C = 3$$

$$x = -3 \Rightarrow B = -1$$

$$x = \frac{1}{2} \Rightarrow A = 1$$

$$\therefore \int \frac{6x^2 + 22x - 23}{(2x - 1)(x^2 + x - 6)} dx = \frac{1}{2} \ln |2x - 1| - \ln |x + 3| + 3 \ln |x - 2| + c$$

(iv) $\frac{x^2 + x - 1}{(x - 1)(x^2 - x + 1)} dx = \frac{A}{x - 1} + \frac{Bx + C}{x^2 - x + 1}$

$$\therefore \; x^2 + x - 1 = A (x^2 - x + 1) + (Bx + C)(x - 1)$$

$$x = 1 \Rightarrow A = 1$$

$\therefore$ We have

$$x^2 + x - 1 = x^2 - x - 1 + Bx^2 + (C - B) x - C$$

Thus $\quad 1 = 1 + B$

$\therefore \quad B = 0$

Also $\quad -1 = 1 - C$

$\therefore \quad C = 2$

$$\therefore \int \frac{x^2+x-1}{(x-1)(x^2-x+1)}\,dx = \int \frac{dx}{x-1} + 2\int \frac{dx}{x^2-x+1}$$

$$= \ln|x-1| + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c$$

(v) $$\int \frac{x^2}{x^2-a^2}\,dx = \int \frac{x^2-a^2+a^2}{x^2-a^2}\,dx$$

$$= \int dx + \frac{a^2}{x^2-a^2}\,dx$$

$$= \int dx + \frac{a^2}{2a}\int\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}dx$$

$$= x + \frac{a}{2}\ln\left|\frac{x-a}{x+a}\right| + c$$

(vi) $$\int \frac{x^2+4}{x^2+2x+3}\,dx = \int\left\{1 - \frac{2x-1}{x^2+2x+3}\right\}dx$$

$$= \int dx - \int \frac{2x-1}{x^2+2x+3}\,dx$$

$$= \int dx - \int \frac{2x-2}{x^2+2x+3}\,dx + \frac{3}{x^2+2x+3}\,dx$$

$$= x - \ln|x^2+2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + c$$

SAQ 5

(i) $\frac{x+2}{2}\sqrt{x^2+4x+6} + \log\left(x+2+\sqrt{x^2+4x+6}\right) + c$

(ii) $\frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3\sqrt{2}}{4}\log\left(x+\sqrt{x^2+\frac{3}{2}}\right) + c$

(iii) $\frac{2x-3}{4}\sqrt{1+3x+x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + c$

SAQ 6

(i) $\frac{2}{3}\tan^{-1}\frac{\left(5\tan\frac{x}{2}+4\right)}{3} + c$

(ii) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + c$

(iii) $\log\left(1+\tan\frac{x}{2}\right) + c$

SAQ 7

(i) $c^b - e^a$

(ii) $\sin b - \sin a$

(iii) $\frac{4}{3}$

SAQ 8

(a) (i) 2

(ii) $3 \log 2 + \frac{3\pi}{8}$

(iii) 0

(b) (i) $\frac{2}{3}$

(ii) $\frac{\pi}{3}$

(iii) $\frac{1}{\sqrt{5}} \log\left(\frac{3+\sqrt{5}}{2}\right)$

SAQ 9

(i) $\frac{\pi}{8} \log 2$

(ii) $\frac{\pi}{4}$

(iii) $\frac{\pi}{2\sqrt{2}} \log (1 + \sqrt{2})$

(iv) $\frac{3\pi}{8}$

(v) $\frac{\pi}{8} \log 2$

SAQ 10

(i) $16 - 4\sqrt{2}$

(ii) $\frac{2}{3}(3\sqrt{3} - 1)$

(iii) $12\,\pi$

(iv) $\frac{\pi}{3}$

(v) $\frac{9}{8}$

(vi) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$