UNIT 5 VECTOR ALGEBRA

Structure

5.1 INTRODUCTION

Vectors are used extensively in almost all branches of physics, mathematics and engineering. The usefulness of vectors in engineering mathematics results from the fact that many physical quantities – for example, velocity of a body, the forces acting on a body, linear and angular momentum of a body, magnetic and electrostatic forces, may be represented by vectors. In several respects, the rule of vector calculations are as simple as the rules governing the system of real numbers.

It is true that any problem that can be solved by the use of vectors can also be treated by non-vectorial methods, but vector analysis simplifies many calculations considerably. Further more, it is a way of visualizing physical and geometrical quantities and relations between them. For all these reasons, extensive use is made of vector notation in modern engineering literature. It is a very useful tool in the hands of scientists and engineers.

Many of you may have studied vectors at school level while some of you might not have done so. This unit has been developed keeping in view the interest of all the students.

We have listed important results of vector algebra. The proofs of some of elementary properties have been omitted. Interested learner may look up for the proofs in the books recommended for your further study, which are available at your study centers. We have begun this unit by giving basic definitions in Section 5.2. The operations on vectors, such as addition and subtraction of vectors, multiplication of a vector by a scalar, etc. have been taken up in Section 5.3. We have devoted Section 5.4 to the representation of the vectors in component forms. Sections 5.5 and 5.6 have been used to discuss vector products. Polar and axial vectors as well as different coordinate systems for space have been taken up in Sections 5.7 and 5.8. We have given the geometrical and physical interpretations, wherever possible. In the following units we shall take up differentiation and integration of vectors.

Objectives

After studying this unit you should be able to

- distinguish between scalars and vectors,
- define a null (or zero) vector, a unit vector, negative of a vector and equality of vectors,
- identify coinitial vectors, like and unlike vectors, free and unlike vectors, free and localized vectors, coplanar and co-linear vectors,
- add and subtract vectors, graphically and analytically,
- multiply a vector by a scalar,
- define the system of linearly independent and dependent vectors,
- compute scalar and vector products of two vectors and give their geometrical interpretation,
- compute the scalar triple products and vector triple products and give their geometrical interpretation,
- compute quadruple product of vectors, and
- solve problems on the application of vector algebra.

5.2 BASIC CONCEPTS

In physics, geometry or engineering mathematics, we come across physical quantities such as mass of a body, the charge of an electron, the specific heat of water, the resistance of a resistor, the diameter of a circle and the volume of a cube. With a suitable choice of units of measure, each of these quantities is described by a single number. Such a quantity is called a scalar. Length, temperature, time, density, frequency are some other familiar examples of scalars. Thus we may define a scalar as follows :

Definition

> *A physical quantity is called a scalar if it can be completely specified by a single number (with a suitable choice of units of measure).*

On the other hand physical quantities like velocity, acceleration, displacement, momentum, force, electric field intensity, etc. are entities which cannot be specified by a single number. They require for their complete characterization the specification of a direction as well as a magnitude.

Figure 5.1 shows the force of attraction for the earth's motion around the sun. The instantaneous velocity of the earth may be represented by an arrow of suitable length and direction. This illustrates that velocity is characterized by a magnitude and a direction.

Figure 5.1 : Force and Velocity

Figure 5.2 shows a displacement (without rotation) of a triangle in the plane. We may represent this displacement graphically by a directed line segment whose *initial point* is the original position of a point P of the given triangle and where *terminal point* is the new position Q of that point after displacement.

Figure 5.2 : Displacement

Definition

> *Quantities, which are specified by a magnitude and a direction are called vector quantities*.

We know that a directed line segment is a line segment with an arrow-head showing direction (Figure 5.3).

Figure 5.3 : Directed Line Segment *AB*

A directed line segment is characterized by

Length

Length of directed line segment AB is the length of line segment AB.

Support

The support of a directed line segment AB is the line P of infinite length of which AB is a portion. It is also called *line of action* of that directed line segment.

Sense

The sense of a directed line segment AB is from its *tail or the initial point A* to its *head or the terminal point B.*

Thus we may also define a vector as follows.

A directed line segment is called a vector. Its length is called the length or Euclidean norm or magnitude of the vector and its direction is called the direction of the vector.

In Figure 5.3, the direction of directed line AB is from A to B. *The two end points of a directed line segments are not interchangeable* and you must think of directed line segments AB and BA as different.

From the definition of a vector, we see that a vector may be translated (displaced without rotation) or, in other words, its initial point may be chosen in an arbitrary fashion. Once we choose a certain point as the initial point of a given vector, the terminal point of the vector is uniquely determined.

For the sake of completeness, we mention that in physics there are situations where we want to impose the restrictions on the initial point of a vector. For example, in mechanics a force acting on a rigid body may be applied to any point of the body. This suggests the concept of *sliding vector,* which is defined as "*a vector where initial point can be any point on a straight line which is parallel to the vector*." However, a force acting on an elastic body is a vector where initial point cannot be changed at all. In fact, it we choose another point of application of the force, the effect of the force will in general be different. This suggests the notion of a *bounded-vecto*r, whose definition is '*a vector having a certain fixed initial point* (*or point of application*)'.

When there is no restriction to choose the initial point of a vector, it is called a *free vector.* When there is restriction on the choice of a certain point as the initial point of a vector, then it is called a *localized vector*.

Throughout the block, we shall denote vectors by bold face letters, e.g., $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{F}$, etc. You may also denote vectors by drawing arrows above them. e.g., $\vec{v}$, $\vec{a}$, $\vec{F}$ or by drawing a line (straight or curly) below them, e.g., $\underline{v}$, $\underline{a}$, $\underline{F}$ or $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{F}$. The magnitude of vector $\boldsymbol{v}$ is denoted by $|\boldsymbol{v}|$, called modulus of $\boldsymbol{v}$, or by v (a light letter in italics). In diagrams we shall show vectors as straight lines with arrowheads on them (as in Figure 5.3) and denote directed line segment AB as $\boldsymbol{AB}$.

Before taking up the algebra of vectors, we state some definitions related to vectors.

Zero Vector or Null Vector

A vector whose length is zero is called a Null or Zero Vector and is denoted by $\mathbf{0}$.

Clearly, a null vector has no direction.

Thus $\boldsymbol{a} = \boldsymbol{AB}$ is a null vector if and only if $|\boldsymbol{a}| = 0$,

i.e., if and only if $|\boldsymbol{AB}| = \mathbf{0}$, i.e., if and only if A and B coincide.

Any non-zero vector is called a ***proper vector*** or ***simply vector***.

Unit Vector

A vector whose length (modulus or magnitude) is unity is called a unit vector.

Generally, a unit vector is denoted by a single letter with *a* cap '^' over it.

Thus $\hat{\boldsymbol{a}}$ denotes a unit vector.

Again $$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}.$$

Co-initial Vectors

All vectors having the same initial point are called co-initial vectors.

Hence $\boldsymbol{AB}, \boldsymbol{AC}, \boldsymbol{AD}$ are all co-initial vectors.

Like and Unlike Vectors

Vectors are said to be like if they have the same direction and unlike if they have opposite directions (*Figure 5.4*).

(a) Like Vectors **(b) Unlike Vectors** **(c) Vectors having Different Directions**

Figure 5.4

Both unlike and like vectors have the same line of action or have the lines of action parallel to one another; such vectors are also called *Collinear* or *Parallel Vectors*.

Coplanar Vectors

Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

Negative of a Vector

A vector whose magnitude is the same as that of the given vector ***a*** *but has the direction opposite to that of* ***a*** *is called the negative of* ***a*** *and is denoted by* **– *a*.**

Thus, if ***AB*** represents the vector ***a***, then ***BA*** represents the vector – ***a*.**

It is evident that | ***a*** | = | – ***a*** |.

Equal Vectors

Two vectors are said to be equal if they have

(i) the same length

(ii) the same or parallel supports

(iii) the same sense.

In Figure 5.5, the three vectors ***a***, ***b***, ***c*** represented by the directed line segments ***AB***, ***CD***, ***EF*** respectively have different initial and terminal points but the same length, the same or parallel supports and the same sense and hence are equal.

We write ***a*** = ***b*** = ***c*** or ***AB*** = ***CD*** = ***EF***

Figure 5.5 : Equal Vectors

Thus equal vectors may be represented by directed segment of equal length in the same sense along the same or parallel supports.

We may remark here that if ***AB*** = ***CD*** and ***AB*** and ***CD*** do not lie along the same line, then it is evident that ***ABCD*** is a parallelogram.

Further, two vectors cannot be equal if

(i) they have different magnitudes, or

(ii) they have inclined supports, or

(iii) they have different senses.

Let us consider an example.

Example 5.1

ABCDEF is a regular hexagon. If $\boldsymbol{P} = \boldsymbol{AB}$, $\boldsymbol{Q} = \boldsymbol{BC}$ and $\boldsymbol{R} = \boldsymbol{CD}$, name the vectors represented by $\boldsymbol{AF}$, $\boldsymbol{ED}$ and $\boldsymbol{FE}$.

Solution

Since *ABCDEF* is a regular hexagon,

Figure 5.6 : A Regular Hexagon

$\therefore$ $AB = BC = CD = AF = ED = FE$

Also $AB \parallel ED$,

$BC \parallel FE$,

and $CD \parallel AF$.

Further sense of *AB* is the same as that of *ED*, sense of *BC* is the same as that of *FE* and sense of *CD* is the same as that of *AF*.

$\therefore$ $\boldsymbol{AB} = \boldsymbol{ED}$, $\boldsymbol{BC} = \boldsymbol{FE}$ and $\boldsymbol{CD} = \boldsymbol{AF}$

Thus $\boldsymbol{ED} = \boldsymbol{P}$, $\boldsymbol{FE} = \boldsymbol{Q}$ and $\boldsymbol{AF} = \boldsymbol{R}$.

So far we have introduced vectors geometrically (using the notion of a directed line segment). We have defined vectors without referring them to any coordinate system. A point in three-dimensional space is a geometric object, but if we introduce a coordinate system, we may describe it by (or even regard it as) an ordered triple of numbers (called its coordinates). Similarly if we use a coordinate system, we may describe vectors in algebraic terms. This alternative way to represent the vectors is also termed as analytic, approach. In the next section, we shall discuss how to express a vector analytically, i.e. in terms of its components. We shall also give a new and precise way of defining vectors, which will be of practical use to you in the study of your other courses also.

5.3 COMPONENTS OF A VECTOR

Let us introduce a coordinate system in space whose axes are three mutually perpendicular straight lines. On all the three axes, we choose the same scale. Then the three unit points on the axes, whose coordinates are (1, 0, 0), (0, 1, 0) and (0, 0, 1), have the same distance from the *origin,* the point of intersection of the axes. The rectangular coordinate system thus obtained is called a ***Cartesian coordinate system*** in space (refer Figure 5.7(a)).

(a) Cartesian Coordinate System **(b) Components of a Vector**

Figure 5.7

We now introduce a vector ***a*** obtained by directing a line segment PQ such that P is the initial point and Q is the terminal point (Figure 5.7(b)). Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the point P and Q respectively. Then the numbers

$$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1, \; a_3 = z_2 - z_1 \qquad \ldots (5.1)$$

are called the *components* of the vector ***a*** with respect to that coordinate system.

By definition, the length or magnitude $|\boldsymbol{a}|$ of the vector ***a*** is the distance $\overline{\boldsymbol{PQ}}$ and from equation (5.1) and the theorem of Pythagoras, it follows that

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad \ldots (5.2)$$

For instance, the vector ***a*** with initial point ***P*** : (3, 1, 4) and terminal point ***Q*** : (1, − 2, 4) has the components $a_1 = 1 - 3 = -2,\ a_2 = -2 - 1 = -3,\ a_3 = 4 - 4 = 0$ and the length

$$|\boldsymbol{a}| = \sqrt{(-2)^2 + (-3)^2 + 0^2} = \sqrt{13}\,.$$

Conversely, if ***a*** has the components – 2, – 3 and 0 and if we choose the initial point of ***a*** as (– 1, 5, 8) then the corresponding terminal point is (– 1 – 2, 5 –3, 8 + 0), i.e., (–3, 2, 8).

You may observe from Eq. (5.1) that if we choose the initial point of a vector to be the origin, then its components are equal to the coordinates of the terminal point and the vector is then called the ***position vector*** of the terminal point (with respect to our coordinate system) and is usually denoted by ***r*** (refer Figure 5.8).

Figure 5.8 : Position Vector *r* of a Point *A* (*x*, *y*, *z*)

We see that $\boldsymbol{r} = \overline{\boldsymbol{OA}}$

$$= \overline{\boldsymbol{OM}} + \overline{\boldsymbol{MA_0}} + \overline{\boldsymbol{A_0}}\ \overline{\boldsymbol{A}}$$

$$= x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$

where, $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are unit vectors parallel to the axes of x, y, z respectively.

The vector $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are mutually perpendicular.

From Eq. (5.1), we can also see that the components a_1, a_2, a_3 of the vector ***a*** are independent of the choice of initial point of ***a***.

This is because if we translate (displace without rotation) ***a*** then corresponding coordinates of *P* and *Q* are altered by the same amount.

Hence, given a fixed Cartesian coordinate system, each vector is uniquely determined by the ordered triple of its components w.r.t. that coordinate system.

We may introduce the ***null vector*** or ***zero vector*** **O** as the vector with components 0, 0, 0.

Further two vectors ***a*** and ***b*** are *equal* if and only if corresponding components of these vectors are equal. Consequently, a vector equation.

$$\boldsymbol{a} = \boldsymbol{b}$$

is equivalent to three equations for the components of ***a*** and ***b***, i.e.,

$$a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,$$

where the components a_1, a_2, a_3 of ***a*** and b_1, b_2, b_3 of ***b*** refer to the same Cartesian coordinate system.

From Eq. (5.1), you may note that a_1, a_2, a_3 are the projections of ***a*** on coordinates axes as

$$\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$$

In Figure 5.7(b), if α, β, γ are the angles which PQ makes with the three axes, then

$$a_1 = |a| \cos\alpha,\ a_2 = |a| \cos\beta,\ a_3 = |a| \cos\gamma$$

The three angles α, β, γ which any vector makes with the three coordinate axes are called ***direction angles*** and the cosines of these angles are called ***direction cosines.*** Let us now consider an example.

Example 5.2

If cos α, cos β, cos γ are the direction cosines of *r*, show that

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \cos\alpha\ \hat{\boldsymbol{i}} + \cos\beta\ \hat{\boldsymbol{j}} + \cos\gamma\ \hat{\boldsymbol{k}}$$

Solution

Let the position vector of a point with coordinates (*x*, *y*, *z*) be ***r***, so that *x*, *y*, *z* are the projections of *r* on the coordinate axes.

∵ We have $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$

Also $x = |\boldsymbol{r}|\cos\alpha,\ y = |\boldsymbol{r}|\cos\beta,\ z = |\boldsymbol{r}|\cos\gamma$

∵ $\boldsymbol{r} = |\boldsymbol{r}|\cos\alpha\ \hat{\boldsymbol{i}} + |\boldsymbol{r}|\cos\beta\ \hat{\boldsymbol{j}} + |\boldsymbol{r}|\cos\gamma\ \hat{\boldsymbol{k}}$

⇒ $\dfrac{\boldsymbol{r}}{|\boldsymbol{r}|} = \hat{\boldsymbol{r}} = \cos\alpha\ \hat{\boldsymbol{i}} + \cos\beta\ \hat{\boldsymbol{j}} + \cos\gamma\ \hat{\boldsymbol{k}}$

You may now try the following exercises.

SAQ 1

(a) If $P(3, -2, 1)$ and $Q(1, 2, -4)$ are the initial and terminal points respectively of a vector $\boldsymbol{a}$, find the components of $\boldsymbol{a}$ and $|\boldsymbol{a}|$.

(b) If $\frac{1}{2}, 1, \frac{3}{2}$ are the components of a vector $\boldsymbol{a}$ and $P = \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$ is a particular initial point of $\boldsymbol{a}$, find the corresponding terminal point and the length of $\boldsymbol{a}$.

We now introduce algebraic operations for vectors in the next section. These operations help us to do various calculations with the vectors.

5.4 OPERATIONS ON VECTORS

By 'algebraic operations on vectors', we mean various ways of combining vectors and scalars, satisfying different laws, called **laws of calculations**. Let us take this one by one.

5.4.1 Addition of Vectors

The motivation for addition of two vectors is provided by displacements. Also, experiments show that the resultant of two forces can be determined by the familiar parallelogram law or by the triangle law. We now define addition of vectors and their properties.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors. Let the vector $\boldsymbol{a}$ be the directed line segment $\boldsymbol{AB}$ and the vector $\boldsymbol{b}$ be the directed segment $\boldsymbol{BC}$ (so that the terminal point B of $\boldsymbol{a}$ is the initial point of $\boldsymbol{b}$) (Figure 5.9). Then the directed line segment $\boldsymbol{AC}$ (i.e., $\boldsymbol{AC}$) represents the sum (or resultant) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and is written as $\boldsymbol{a} + \boldsymbol{b}$.

Figure 5.9 : Addition of Two Vectors

Thus $\boldsymbol{AC} = \boldsymbol{AB} + \boldsymbol{BC} = \boldsymbol{a} + \boldsymbol{b}$.

The Method of drawing a triangle in order to define the vector sum ($\boldsymbol{a} + \boldsymbol{b}$) is called *triangle law of addition of two vectors,* which states as follows :

If two vectors are represented by two sides of a triangle, taken in order, then their sum (or resultant) is represented by the third side of the triangle taken in the reverse order.

Since any side of a triangle is less than the sum of the other two sides of the triangle; hence modulus of $\boldsymbol{AC}$ is less than the sum of module of $\boldsymbol{AB}$ and $\boldsymbol{BC}$.

It may be noted that the vector sum does not depend upon the choice of initial point of the vector as can be seen from Figure 5.10.

If in some fixed coordinate system, $\boldsymbol{a}$ has the components a_1, a_2, a_3 and $\boldsymbol{b}$ has the components b_1, b_2, b_3, then the components c_1, c_2, c_3 of the sum vector $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ are obtained by the addition of corresponding components of $\boldsymbol{a}$ and $\boldsymbol{b}$; thus

$$c_1 = a_1 + b_1,\ c_2 = a_2 + b_2,\ c_3 = a_3 + a_3 \qquad \ldots (5.3)$$

Figure 5.10 : Vector Sum is Independent of Initial Point

This fact is represented in Figure 5.11 in the case of plane. In space the situation is similar.

Figure 5.11 : Vector Addition in Terms of Components in a Plane

5.4.2 Properties of Vector Addition

From the definition of vector addition and using Eq. (5.3), it can be shown that vector addition has the following properties.

Vector Addition is Commutative

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two vectors, then

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

Let $\boldsymbol{OA} = \boldsymbol{a}$ and $\boldsymbol{AB} = \boldsymbol{b}$ (Figure 5.12)

$\because$ $\boldsymbol{OB} = \boldsymbol{OA} + \boldsymbol{AB}$

$$= \boldsymbol{a} + \boldsymbol{b} \quad \ldots (5.4)$$

Let us complete the parallelogram *OABC*. Then $\boldsymbol{OC} = \boldsymbol{AB} = \boldsymbol{b}$ and $\boldsymbol{CB} = \boldsymbol{OA} = \boldsymbol{a}$

$\therefore$ $\boldsymbol{OB} = \boldsymbol{OC} + \boldsymbol{CB}$

$$= \boldsymbol{b} + \boldsymbol{a} \quad \ldots (5.5)$$

From Eqs. (5.4) and (5.5), we have

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

Figure 5.12 : Commutative Vector Addition

Vector Addition is Associative

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are any three vectors, then

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$$

Above equation can be easily verified from Figure 5.13.

Note that the sum of three vectors ***a***, ***b***, ***c*** is independent of the order in which they are added and is written as

$$\mathbf{a} + \mathbf{b} + \mathbf{c}$$

Figure 5.13 : Associative Law of Vector Addition

Existence of Additive Identity

For any vector ***a***,

$$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a} = \mathbf{0} + \boldsymbol{a},$$

where **0** is a null (or zero) vector.

Thus 0 is called additive identity of vector addition.

Existence of Additive Inverse

For any vector ***a***, there exists another vector – ***a*** such that

$$\boldsymbol{a} + (-\,\boldsymbol{a}) = \mathbf{0},$$

where – ***a*** denotes the vector having the length | ***a*** | and the direction opposite to that of ***a***.

In view of the above property, the vector (– ***a***) is called the additive inverse of vector ***a***.

Let us take an example from geometry to illustrate the use of vector addition.

Example 5.3

Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Figure 5.14 : Meridians of a Triangle

Solution

In Δ *ABC*, *AD*, *BE*, and *CF* are the median.

Now $\quad \boldsymbol{AD} + \boldsymbol{BE} + \boldsymbol{CF}$

$$= (\boldsymbol{AB} + \boldsymbol{BD}) + (\boldsymbol{BC} + \boldsymbol{CE}) + (\boldsymbol{CA} + \boldsymbol{AF})$$

$$= \boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CA} + \frac{1}{2}\,\boldsymbol{BC} + \frac{1}{2}\,\boldsymbol{CA} + \frac{1}{2}\,\boldsymbol{AB}$$

$$= \frac{3}{2}\,(\boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CA}) \qquad \ldots (5.6)$$

By triangle law of addition,

$$\boldsymbol{AB} + \boldsymbol{BC} = \boldsymbol{AC}$$

From Eqs. (5.6) and (5.7), we get

$$\boldsymbol{AD} + \boldsymbol{BE} + \boldsymbol{CF}$$

$$= \frac{3}{2}\,(\boldsymbol{AC} + \boldsymbol{CA})$$

$$= \frac{3}{2}\,[\boldsymbol{AC} + (-\,\boldsymbol{AC})]$$

$$= \frac{3}{2}\,(0) = \boldsymbol{0}$$

You may now attempt the following exercise.

SAQ 2

Show that the sum of vectors represented by the sides $\boldsymbol{AB}$ and $\boldsymbol{DC}$ of any quadrilateral *ABCD* is equal to the sum of the vectors represented by the diagonals $\boldsymbol{AC}$ and $\boldsymbol{DB}$.

It may be mentioned that associative law of vector addition is true even if $\boldsymbol{PQRS}$, as shown in Figure 5.13, is a skew quadrilateral (i.e., when $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-coplanar vectors).

Further, instead of $\boldsymbol{a} + \boldsymbol{a}$, we also write $\mathbf{2}\boldsymbol{a}$. This and the notion $-\boldsymbol{a}$ suggests that we define the second algebraic operation for vectors viz., the multiplication of a vector by an arbitrary real number (called a scalar).

5.4.3 Multiplication of Vectors by Scalars

Let ***a*** be any vector and *m* be any given scalar. Then the vector *m* ***a*** (product of vector ***a*** and scalar *m*) is a vector whose

(i) magnitude $| m \boldsymbol{a} | = | m | . | \boldsymbol{a} |$

$$= m \, | \boldsymbol{a} | \text{ if } m \geq 0$$

$$= - \, m \, | \boldsymbol{a} | \text{ if } m < 0$$

(ii) support is the same or parallel to that of support of ***a*** if $\boldsymbol{a} \neq 0$ and $m > 0$, and

(iii) $m\boldsymbol{a}$ has the direction of vector ***a*** if $\boldsymbol{a} \neq 0$ and $m > 0$ and $m\boldsymbol{a}$ has the direction opposite to vector ***a*** if $\boldsymbol{a} \neq 0$ and $m < 0$.

Further, if $\boldsymbol{a} = 0$ or $m = 0$ (or both), then $m\boldsymbol{a} = \mathbf{0}$. Geometrically, we can represent $m\boldsymbol{a}$ as follows :

Let $\boldsymbol{AB} = \boldsymbol{a}$, then $\boldsymbol{AC} = m\boldsymbol{a}$ if $m > 0$. Here we choose the point *C* on *A****B*** on the same side of *A* as ***B*** (see Figure 5 .15(a)).

Now if $m < 0$ and $\boldsymbol{AB} = \boldsymbol{a}$, then $\boldsymbol{AC} = m\boldsymbol{a}$ where we have choosen the point *C* on *AB* on the side of *A* opposite to that of ***B*** (Figure 5.15(b)).

(a)

(b)

Figure 5.15

Further, if ***a*** has the component a_1, a_2, a_3, then $m\boldsymbol{a}$ has the components ma_1, ma_2, ma_3, (w.r.t. the same coordinate system).

Multiplication of a vector by a scalar helps us to define linearly dependent and independent vectors, which we take up next.

Linearly Dependent and Independent Vectors

Two non-zero vectors ***a*** and ***b*** are said to be *linearly dependent* if there exists a scalar $t \ (\neq 0)$ such that

$$\boldsymbol{a} = t \, \boldsymbol{b}$$

Thus two vectors can be linearly dependent if and only if the vectors ***a*** and ***b*** are parallel.

Further, if $\boldsymbol{a} = \boldsymbol{AB}$ and $\boldsymbol{b} = \boldsymbol{BC}$, then ***a*** and ***b*** are linearly dependent if and only if *A*, *B*, *C* lie in a straight line.

If the vectors ***a*** and ***b*** are not linearly dependent, they are said to be ***linearly independent*** and in this case ***a*** and ***b*** are not parallel vectors.

From the definition, we have the following *properties of* ***multiplication of a vector*** *by a scalar*.

Figure 5.16

(i) $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$

Here $\boldsymbol{OA'} = m\boldsymbol{a}$ and $\boldsymbol{A'B'} = m\boldsymbol{b}$

$\because$ $\boldsymbol{OB'} = m\boldsymbol{a} + m\boldsymbol{b}$

Also $\boldsymbol{OB'} = m(\boldsymbol{a} + \boldsymbol{b})$

(ii) $(m + n)\,\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$ (Distributive law)

(iii) $m\,(n\,\boldsymbol{a}) = (mn)\,\boldsymbol{a} = mn\boldsymbol{a}$ (Associative Law)

(iv) $1\boldsymbol{a} = \boldsymbol{a}$ (Existence of multiplicative identity)

(v) $0\boldsymbol{a} = \boldsymbol{a}$

(vi) $(-1)\,\boldsymbol{a} = -\boldsymbol{a}$

Also if $\hat{\boldsymbol{a}}$ is the unit vector, then

$$\boldsymbol{a} = |\boldsymbol{a}|\,\hat{\boldsymbol{a}} \text{ and } \hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}.$$

Let us take up some examples to illustrate the above properties.

Example 5.4

If $\boldsymbol{a}$ is a non-zero vector, find a scalar λ such that $|\lambda\boldsymbol{a}| = 1$.

Solution

We have to determine λ such that

$$|\lambda\boldsymbol{a}| = 1$$

$$\Rightarrow \quad |\lambda|\,|\boldsymbol{a}| = 1$$

$$\Rightarrow \quad |\lambda| = \frac{1}{|\boldsymbol{a}|} \quad (\because \boldsymbol{a} \text{ is non-zero, } \therefore |\boldsymbol{a}| \neq 0)$$

$$\Rightarrow \quad \lambda = \pm\frac{1}{|\boldsymbol{a}|}$$

The + sign is to be taken when $\lambda > 0$ and the – sign is to be taken when $\lambda < 0$.

Example 5.5

Show that the vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ constitute a linearly dependent set.

Solution

Here $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

$$= 2(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) - 2\hat{\boldsymbol{k}}$$

$$= 2(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) - \frac{2}{3} . 3\hat{\boldsymbol{k}}$$

$$= 2\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$$

Hence the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as given here, constitute a linearly dependent set.

We can now define the difference of two vectors.

Difference of Two Vectors

The difference $\boldsymbol{a} - \boldsymbol{b}$ of two vectors is defined as the sum $\boldsymbol{a} + (-\boldsymbol{b})$, where **(– *b*)** is the negative of ***b*.** We can geometrically represent the difference $\boldsymbol{a} - \boldsymbol{b}$ as in Figure 5.17.

It is evident that

$$\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{0}$$

and $$\boldsymbol{a} - \boldsymbol{0} = \boldsymbol{a}$$

Figure 5.17 : Difference of Two Vectors

If two vectors are given in their component forms then to obtain their difference, subtract the vectors component wise.

For example, if

$$\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$$

and $$\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$$

then $$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\hat{\boldsymbol{i}} + (a_2 - b_2)\hat{\boldsymbol{j}} + (a_3 - b_3)\hat{\boldsymbol{k}}$$

Let us take a few examples.

Example 5.6

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

Solution

Let ***OA*** and ***AB*** be two unit vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$.

Then by triangle law of addition,

$$\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} = \boldsymbol{OB}$$

Figure 5.18

We are given $|\hat{a}| = 1, |\hat{b}| = 1, |\hat{a} + \hat{b}| = 1$

$\therefore \quad OA = AB = OB = 1$

Let $\boldsymbol{AC} = -\hat{b}$. Then $AC = |\boldsymbol{AC}| = |-\hat{b}| = |\hat{b}| = 1$

Since $OA = OB = AC$, then by geometry $\Delta\ BOC$ is a right-angled triangle, with $\angle BOC = \frac{\pi}{2}$.

Now $\quad \hat{a} - \hat{b} = \hat{a} + (-\hat{b}) = \boldsymbol{OA} + \boldsymbol{AC} = \boldsymbol{OC}$

$\therefore \quad |\hat{a} - \hat{b}| = |\boldsymbol{OC}| = OC$

Now $\quad BC^2 = OB^2 + OC^2 \Rightarrow OC = \sqrt{BC^2 - OB^2} = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$

Example 5.7

What is the geometrical significance of the relation $|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a} - \boldsymbol{b}|$?

Solution

Let $\boldsymbol{a} = \boldsymbol{AB}$ and $\boldsymbol{b} = \boldsymbol{AD}$

We complete the parallelogram *ABCD* having *AB* and *AD* as adjacent sides. Let us draw the two diagonals *AC* and *BD* also.

Figure 5.19

By definition,

$$\boldsymbol{AC} = \boldsymbol{AB} + \boldsymbol{BC}$$
$$= \boldsymbol{AB} + \boldsymbol{AD} \quad (\because\ \boldsymbol{BC} = \boldsymbol{AD})$$
$$= \boldsymbol{a} + \boldsymbol{b}$$

$\therefore \quad |\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{AC}|$

Again, $\quad \boldsymbol{DB} = \boldsymbol{DA} + \boldsymbol{AB}$

$$= -\boldsymbol{AD} + \boldsymbol{AB}$$

$= -\boldsymbol{b} + \boldsymbol{a}$

$= \boldsymbol{a} - \boldsymbol{b}$

$\therefore \quad |\boldsymbol{a} - \boldsymbol{b}| = |\boldsymbol{DB}|$

We are given

$|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a} - \boldsymbol{b}|$

$\therefore \quad |\boldsymbol{AC}| = |\boldsymbol{DB}|$

$\Rightarrow \quad \boldsymbol{AC} = \boldsymbol{BD}$

$\Rightarrow$ Diagonals of the parallelogram are equal.

$\Rightarrow$ Parallelogram *ABCD* is a rectangle and hence ***a*** is perpendicular to ***b***.

You may now attempt the following exercises.

SAQ 3

(a) Prove that

(i) $|\boldsymbol{a} + \boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$

(ii) $|\boldsymbol{a}| - |\boldsymbol{b}| \le |\boldsymbol{a} - \boldsymbol{b}|$

(b) Show that the vectors

$$\boldsymbol{a} = 3\sqrt{3}\,\hat{\boldsymbol{i}} - 3\,\hat{\boldsymbol{j}},\ \boldsymbol{b} = 6\,\hat{\boldsymbol{j}},\ \boldsymbol{c} = 3\sqrt{3}\,\hat{\boldsymbol{i}} + 3\,\hat{\boldsymbol{j}}$$

form the sides of an equilateral triangle.

(c) Show that the three points with position vectors $2\,\hat{\boldsymbol{i}} + 3\,\hat{\boldsymbol{j}},\ 3\,\hat{\boldsymbol{i}} + \frac{9}{4}\,\hat{\boldsymbol{j}}$ and $5\,\hat{\boldsymbol{i}} + 0.75\,\hat{\boldsymbol{j}}$ are collinear.

(d) Show that the set of vectors $5\boldsymbol{a} + 6\boldsymbol{b} + 7\boldsymbol{c},\ 7\boldsymbol{a} - 8\boldsymbol{b} + 9\boldsymbol{c},\ 3\boldsymbol{a} + 20\boldsymbol{b} + 5\boldsymbol{c}$ are coplanar.

So far we have defined addition and subtraction of vectors as well as multiplication of vectors by scalars. We shall now introduce multiplication of vectors by vectors.

5.5 PRODUCT OF TWO VECTORS

When one vector is multiplied with another vector, result can be a scalar or a vector. There are in general two different ways in which vectors can be multiplied. These are the *scalar* or *inner* or dot product which is a mere number (or scalar) having magnitude alone and the other is called *vector* or *cross* product, which is a vector having a definite direction. We shall now take up these two products one by one.

5.5.1 Scalar or Dot Product

The scalar or dot or inner product of two vectors ***a*** and ***b*** in the two/three-dimensional space is written as ***a . b*** (read as '***a*** dot ***b***') and is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{bmatrix} |\boldsymbol{a}|\,|\boldsymbol{b}| \cos \gamma, & \text{when } \boldsymbol{a} \neq \mathbf{0}, \boldsymbol{b} \neq \mathbf{0} \\ 0 & \text{when } \boldsymbol{a} = \mathbf{0} \text{ or } \boldsymbol{b} = \mathbf{0} \end{bmatrix} \quad \ldots (5.8)$$

Figure 5.20 : Angle between Vectors and their Dot Product

where γ $(0 \leq \gamma \leq \pi)$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (computed when the vectors have their initial points coinciding) (Refer Figure 5.20).

The value of the dot product is a scalar (a real number), and this motivates the term "scalar product". Since the cosine in Eq. (5.8) may be positive, zero or negative, the same is true for the dot product.

Angle γ in Eq. (5.8) lies between 0 and π and we know that $\cos \gamma = 0$, if and only if $\gamma = \pi/2$, we thus have the following important result :

Two non-zero vectors are orthogonal (perpendicular to each other) if and only if their dot product is zero.

If we put $\boldsymbol{b} = \boldsymbol{a}$ in Eq. (5.8), we have $\boldsymbol{a} \cdot \boldsymbol{a} = |\,\boldsymbol{a}^2\,| = |\vec{\boldsymbol{a}}|^2$ and this shows that the length (or Euclidean norm or modulus or magnitude) of a vector can be written in terms of scalar product as

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} \; (\geq 0) \quad \ldots (5.9)$$

From Eqs. (5.8) and (5.9), we obtain the angle γ between two non-zero vectors as

$$\cos \gamma = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}\,\sqrt{\boldsymbol{b} \cdot \boldsymbol{b}}}$$

The scalar product has the following properties :

Property 1

$(q_1 \boldsymbol{a} + q_2 \boldsymbol{b}) \cdot \boldsymbol{c} = q_1 \boldsymbol{a} \cdot \boldsymbol{c} + q_2 \boldsymbol{b} \cdot \boldsymbol{c}$ (linearity)

Property 2

$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (symmetry or commutative law)

Property 3

$$\left.\begin{array}{l} \boldsymbol{a} \cdot \boldsymbol{a} \geq 0 \\ \text{Also} \quad \boldsymbol{a} \cdot \boldsymbol{a} = 0 \text{ if and only if } \boldsymbol{a} = 0 \end{array}\right\} \text{(positive definiteness)}$$

Property 4

In Property 1, with $\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2 = 1$, we have

$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$ (distributivity)

Hence scalar product is commutative and distributive with respect to vector addition.

Property 5

From the definition of scalar product, we get

$| \boldsymbol{a} \,.\, \boldsymbol{b} | \leq | \boldsymbol{a} | \; | \boldsymbol{b} |$ ($\because$ $| \cos \gamma | \leq 1$) (Schwarz Inequality)

Property 6

Also using the definition and simplifying, we get

$| \boldsymbol{a} + \boldsymbol{b} |^2 + | \boldsymbol{a} - \boldsymbol{b} |^2 = 2 \, (| \boldsymbol{a} |^2 + | \boldsymbol{b} |^2)$ (parallelogram equality)

Property 7

Further, if $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are unit vectors forming an orthogonal triad, then, from definition of scalar product, we have

$$\begin{Bmatrix} \hat{\boldsymbol{i}} \,.\, \hat{\boldsymbol{i}} = 1, & \hat{\boldsymbol{j}} \,.\, \hat{\boldsymbol{j}} = 1, & \hat{\boldsymbol{k}} \,.\, \hat{\boldsymbol{k}} = 1 \\ \hat{\boldsymbol{i}} \,.\, \hat{\boldsymbol{j}} = 0, & \hat{\boldsymbol{j}} \,.\, \hat{\boldsymbol{k}} = 0, & \hat{\boldsymbol{k}} \,.\, \hat{\boldsymbol{i}} = 0 \end{Bmatrix}$$

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented in terms of components, say,

$$\boldsymbol{a} = a_1 \hat{\boldsymbol{i}} + a_2 \hat{\boldsymbol{j}} + a_3 \hat{\boldsymbol{k}} \text{ and } \boldsymbol{b} = b_1 \hat{\boldsymbol{i}} + b_2 \hat{\boldsymbol{j}} + b_3 \hat{\boldsymbol{k}}$$

then their scalar product is given by the formula

$$\boldsymbol{a} \,.\, \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(using Property 7)}$$

and

$$\cos \gamma = \left(\frac{\boldsymbol{a} \,.\, \boldsymbol{b}}{|\boldsymbol{a}| \, |\boldsymbol{b}|} \right) = \frac{(a_1 b_1 + a_2 b_2 + a_3 b_3)}{(\sqrt{a_1^2 + a_2^2 + a_3^2} \, \sqrt{b_1^2 + b_2^2 + b_3^2})}$$

where γ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Before we take up some applications of scalar products, we give below the geometrical interpretation of scalar product of two vectors.

Geometrical Interpretation of Dot Product

The scalar product of two vectors is the product of the modulus of either vector and the resolution (projection) of the other in its direction.

(a) Projection of *a* in the Direction of *b* **(b) Projection of *b* in the Direction of *a***

Figure 5.21

Let $\boldsymbol{OA} = \boldsymbol{a}, \boldsymbol{OB} = \boldsymbol{b}$ and $\angle \boldsymbol{BOA} = \gamma$

By definition, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}| \cos\gamma$ where $|\boldsymbol{a}| = \boldsymbol{OA}$ and $|\boldsymbol{b}| = \boldsymbol{OB}$

From the point B draw BC perpendicular on OA (Figure 5.21(b)) and from the point A, draw AD perpendicular on OB (Figure 5.21(a))

$\therefore$ $\quad OC$ = Projection of OB on $OA = OB \cos\gamma = |\boldsymbol{b}| \cos\gamma$

and $\quad OD$ = Projection of OA on $OB = OA \cos\gamma = |\boldsymbol{a}| \cos\gamma$

Thus $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}| \cos\gamma = |\boldsymbol{b}|\,(|\boldsymbol{b}| \cos\gamma) = |\boldsymbol{a}|$ (projection of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$).

Also $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}| \cos\gamma = |\boldsymbol{b}|\,(|\boldsymbol{a}| \cos\gamma) = |\boldsymbol{b}|$ (projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$).

Hence the result.

Remember that if $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are two vectors and* $\boldsymbol{a} \neq 0$, *then* $p = |\boldsymbol{b}| \cos\gamma = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$ *is called the component of* $\boldsymbol{b}$ *in the direction of* $\boldsymbol{a}$, *or the projection of* $\boldsymbol{b}$ *in the direction of* $\boldsymbol{a}$, *where* γ *is the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$. *If* $\boldsymbol{a} = \boldsymbol{0}$, *then* γ *is undefined and we set* $p = 0$.

It follows that $|p|$ is the length of the orthogonal projection of $\boldsymbol{b}$ on a straight line l in the direction of $\boldsymbol{a}$. Here p may be positive, zero or negative. See Figure 5.22

Figure 5.22 : Components of *b* in the Direction of *a*

In particular, if $\boldsymbol{a}$ is a unit vector, then we simply have

$$\boldsymbol{p} = \boldsymbol{a} \cdot \boldsymbol{b}$$

The following examples illustrate the application of dot product.

Example 5.8

Give a representation of *work done by a force* in terms of scalar product.

Solution

Consider a particle P on which a constant force $\boldsymbol{a}$ acts. Let the particle be given a displacement $\boldsymbol{d}$ by the application of this force. Then the work done W by $\boldsymbol{a}$ in this displacement is defined as the product of $|\boldsymbol{d}|$ and the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{d}$, i.e.,

$$W = |\boldsymbol{a}|\,(|\boldsymbol{d}| \cos\alpha) = \boldsymbol{a} \cdot \boldsymbol{d},$$

where α is the angle between $\boldsymbol{a}$ and $\boldsymbol{d}$.

Figure 5.23 : Word Done by a Force

Example 5.9

Find a representation of the straight line l_1 through the point P in the xy-plane and perpendicular to the line l_2 represented by $x - 2y + 2 = 0$.

Solution

Any straight line l_1 in the xy-plane can be represented in the form $a_1 x + b_1 y = c$. If $c = 0$, then l_1 passes through the origin. If $c \neq 0$, then $a_1 x + b_1 y = 0$ represents a line l_1' through the origin and parallel to l_1. The position vector of a point on the line l_1' is $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

If we introduce the vector $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$, then, by the definition of dot product, we can give a representation of l_1' as

$$\boldsymbol{a} \,.\, \boldsymbol{r} = 0$$

Certainly $\boldsymbol{a} \neq \boldsymbol{0}$ and vector $\boldsymbol{a}$ is perpendicular to $\boldsymbol{r}$ and, therefore, perpendicular to the line l_1'. It is called ***normal** vector to* l_1'.

Since l_1 and l_1' are parallel lines, thus $\boldsymbol{a}$ is also normal to line l_1.

Hence two lines l_1 and l_2 are perpendicular or orthogonal if and only if their normal vectors say $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, i.e. $\boldsymbol{a} \,.\, \boldsymbol{b} = 0$ (Figure 5.24). You may note that this implies that the slopes of the lines are negative reciprocals.

Figure 5.24 : Normal Lines

For the given line l_2 the form of the vector is $\boldsymbol{b} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and a vector perpendicular to $\boldsymbol{b}$ is $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$. Hence the representation of line l_1 must be of the form $2x + y = c$. The value of c can be obtained by substituting the coordinate of P in this representation.

Example 5.10

Find the projection of $\boldsymbol{b}$ on the line of $\boldsymbol{a}$ if $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

Solution

Here $\boldsymbol{a} \,.\, \boldsymbol{b} = 1.2 + 1.4 + 1.5 = 2 + 4 + 5 = 11$

Also $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\therefore$ Projection of $\boldsymbol{b}$ on the line of $\boldsymbol{a}$

$$= \frac{\boldsymbol{a} \,.\, \boldsymbol{b}}{|\boldsymbol{a}|} = \frac{11}{\sqrt{3}} = \frac{11\sqrt{3}}{3}$$

Example 5.11

Show by vector method that the diagonals of a rhombus are at right angles.

Solution

Let *ABCD* be a rhombus.

Let *A* be taken on the origin.

Let $\boldsymbol{b}$ and $\boldsymbol{d}$ be the position vectors of vertices *B* and *D* respectively referred to the origin *A* (refer Figure 5.25).

Then the position vector of *C* (using triangle or parallelogram law of addition) is $\boldsymbol{b} + \boldsymbol{d}$.

Also $\boldsymbol{BD} = \boldsymbol{d} - \boldsymbol{b}$

Now *ABCD* is a rhombus.

$\therefore$ $AB = AD \Rightarrow AB^2 = AD^2$

$\Rightarrow$ $\boldsymbol{b}^2 = \boldsymbol{d}^2$

Figure 5.25

Now $\boldsymbol{AC} = \boldsymbol{b} + \boldsymbol{d}$

$$\therefore \quad \boldsymbol{AC} \,.\, \boldsymbol{BD} = (\boldsymbol{b} + \boldsymbol{d}) \,.\, (\boldsymbol{d} - \boldsymbol{b})$$

$$= (\boldsymbol{d} + \boldsymbol{b}) \,.\, (\boldsymbol{d} - \boldsymbol{b})$$

$$= \boldsymbol{d}^2 - \boldsymbol{b}^2$$

$$= 0$$

Hence $\boldsymbol{AC}$ is perpendicular to $\boldsymbol{BD}$

Thus diagonals *AC* and *BD* of the rhombus *ABCD* are at right angles.

You may try the following exercises.

SAQ 4

(a) Prove by vector method that the angle in a semi-circle is a right angle.

(b) A particle is acted on by constant forces $-3\hat{i} + 2\hat{j} + 5\hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$ and is displaced from the point (2, – 1, –3) to the point (4, –3, 7). Find the total work done by the forces.

(c) Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by $\boldsymbol{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\boldsymbol{b} = -2\hat{i} + 2\hat{j} + 4\hat{k}$. Find

(i) magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the angle between them

(ii) projection of the vector $\left(\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ onto $\boldsymbol{a}$

(iii) Which of the following vectors are perpendicular to $\boldsymbol{a}$?

$$\boldsymbol{c} = -\hat{i} - 4\hat{j} + 2\hat{k},\ \boldsymbol{d} = -3\hat{i} + \hat{k},\ \boldsymbol{e} = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

Dot multiplication of two vectors gives the product as a scalar. Various applications suggest another kind of multiplication of vectors such that the product is again a vector. We next take up such **vector product** or **cross product** between two vectors. Cross products play an important role in the study of electricity and magnetism.

5.5.2 Vector Product of Two Vectors

The vector ***product or the cross product*** *of two vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, denoted by* $\boldsymbol{a} \times \boldsymbol{b}$*, is defined as*

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta\ \hat{n} \qquad \ldots (5.10)$$

where $|\boldsymbol{a}|$ *and* $|\boldsymbol{b}|$ *are the magnitudes of the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *respectively, ? is the angle between vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and* $\hat{\boldsymbol{n}}$ *is a unit vector perpendicular to both* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and is such that* $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ *in this order, form a right-handed triple or right-hand triod (see Figure 5.26).*

Figure 5.26 : Vector Product

The term *right-handed* comes from the fact that the vectors $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ in this order, assume the same sort of orientation as the thumb, index finger, and middle finger of right hand when these are held as shown in Figure 5.27(a). Let us now look at the screw shown in Figure 5.27(b). You may observe that if $\boldsymbol{a}$ is rotated in the direction of $\boldsymbol{b}$ through an angle $\theta\,(< \pi)$ then $\hat{\boldsymbol{n}}$ advances in a direction pointing towards the reader or away from the reader according as the screw is right-handed or left-handed. On the same account an ordered vector triad $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ **is right-handed** or **left-handed.**

(a) Right-handed Triple of Vectors $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ **(b) Right-handed Screw**

Figure 5.27

You may note here that

(i) $0 \le \theta \le \pi$

(ii) $\hat{\boldsymbol{n}}$ is perpendicular to the plane which contains $\boldsymbol{a}$ and $\boldsymbol{b}$ both

(iii) vector product of two vectors is always a vector quantity

(iv) The sine of the angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $\sin\theta = \dfrac{|\boldsymbol{a} \times \boldsymbol{b}|}{(|\boldsymbol{a}|\,.\,|\boldsymbol{b}|)}$.

Vector product $\boldsymbol{a} \times \boldsymbol{b}$ is also called the **cross product** because of the notation used and is read as $\boldsymbol{a}$ cross $\boldsymbol{b}$**.**

From the definition of vector product, you know that $\boldsymbol{b} \times \boldsymbol{a}$ is a vector of magnitude $|\boldsymbol{a}|\;|\boldsymbol{b}|\sin\theta$ and is normal to $\boldsymbol{a}$ and $\boldsymbol{b}$ and in a direction such that $\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{a}$ from a right-handed system (Figure 5.28). This is possible only if $\boldsymbol{b} \times \boldsymbol{a}$ is opposite to the direction of $\boldsymbol{a} \times \boldsymbol{b}$. Since $\boldsymbol{b} \times \boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b}$ have equal magnitudes, thus

$$\boldsymbol{b} \times \boldsymbol{a} = -\,\boldsymbol{a} \times \boldsymbol{b},$$

Figure 5.28

which shows that the cross-product of vectors is not commutative.

Hence the order of the factors in a vector product is of great importance and must be carefully observed.

It may be observed that the unit vector normal to both ***a*** and ***b***, namely, $\hat{\boldsymbol{n}}$ is given by

$$\hat{n} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$$

If the vectors ***a*** and ***b*** are parallel (or collinear) then θ = 0 or 180º $\Rightarrow$ sin θ = 0

Hence $\boldsymbol{a} \times \boldsymbol{b} = 0$ is the condition for the two vectors ***a*** and ***b*** to be parallel.

In particular $\boldsymbol{a} \times \boldsymbol{a} = 0$.

You know that $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ represent unit vectors along the axes of a Cartesian coordinate system. Also since $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ in this order, form a right-handed system of mutually perpendicular vectors, therefore, $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}$ is a vector having modulus as unity and direction parallel to $\hat{\boldsymbol{k}}$.

Thus $\qquad \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}}$

Similarly $\qquad \hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}}$

and $\qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} = -\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}}$

Also $\qquad \hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = 0$

From the definition of cross product, it follows that for any constant λ

$$(\lambda\, \boldsymbol{a}) \times \boldsymbol{b} = \lambda\, (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (\lambda\, \boldsymbol{b})$$

Further, cross-multiplication is distributive w.r. to vector addition, i.e.,

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

$$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$$

Cross multiplication has a very unusual and important property, namely, *Cross multiplication is not associative*, i.e., in general

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$$

For instance, $\qquad \hat{\boldsymbol{i}} \times (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}) = \hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}$,

whereas $\qquad (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}}) \times \hat{\boldsymbol{j}} = 0 \times \hat{\boldsymbol{j}} = \boldsymbol{0}$

From the definition of cross-product and dot product of two vectors, we have

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2\, |\boldsymbol{b}|^2 \sin^2 \theta$$

$$= |\boldsymbol{a}|^2\, |\boldsymbol{b}|^2 - (1 - \cos^2 \theta)$$

$$= |\boldsymbol{a}|^2\, |\boldsymbol{b}|^2 - [|\boldsymbol{a}|\,|\boldsymbol{b}| \cos \theta]^2$$

$$= (\boldsymbol{a} \,.\, \boldsymbol{a})\, (\boldsymbol{b} \,.\, \boldsymbol{b}) - (\boldsymbol{a} \,.\, \boldsymbol{b})^2$$

From the above identity, we obtain a useful formula for the modulus of a vector product as

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(\boldsymbol{a} \,.\, \boldsymbol{a})\,(\boldsymbol{b} \,.\, \boldsymbol{b}) - (\boldsymbol{a} \,.\, \boldsymbol{b})^2}$$

Before taking up geometrical interpretation of vector product we list all the properties of vector product discussed above for the ready reference.

PC 1

$$\boldsymbol{a} \times \boldsymbol{b} = -\,\boldsymbol{b} \times \boldsymbol{a}$$

i.e., vector product is not commutative.

PC 2

$$\boldsymbol{a} \times \boldsymbol{b} = (\,|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\ 0)\ \hat{\boldsymbol{n}} = 0, \text{if } \boldsymbol{a} \text{ is Parallel to } \boldsymbol{b}.$$

Vector product of parallel vectors is zero. In particular, $\boldsymbol{a} \times \boldsymbol{a} = 0$ and $\boldsymbol{a} \times 0 = 0$.

PC 3

If m and n are scalars then $(m\boldsymbol{a} \times n\boldsymbol{b}) = mn\,(\boldsymbol{a} \times \boldsymbol{b})$

PC 4

For a right handed triad of unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$

$$\hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = 0$$

$$\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}},\ \hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{i}},\ \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$$

$$\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}} = -\,\hat{\boldsymbol{k}},\ \hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}} = -\,\hat{\boldsymbol{i}},\ \hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}} = -\,\hat{\boldsymbol{j}}$$

PC 5

Vector product is distributive w.r.t vector addition, i.e.,

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

PC 6

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$$

i.e. vector product is not associative.

We next take up geometrical interpretation of vector product.

Geometrical Interpretation of Vector Product

Consider a parallelogram $OACB$ with $\boldsymbol{a}$ and $\boldsymbol{b}$ as adjacent sides.

Let $\quad \boldsymbol{a} = \boldsymbol{OA},\ \boldsymbol{b} = \boldsymbol{OB}$

Figure 5.29

and let BN be perpendicular to OA from $\boldsymbol{B}$

Let $\angle \boldsymbol{BON} = \theta$

$\therefore \quad \boldsymbol{BN} = \boldsymbol{OB} \sin\theta$

$= |\boldsymbol{b}| \sin\theta$

Now $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\ |\boldsymbol{b}|\,.\sin\theta$

$= \boldsymbol{OA}\,.\,\boldsymbol{BN}$ = Area of the parallelogram $OACB$.

Thus, the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the area of the parallelogram whose adjacent sides are the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The sign may also be assigned to the area. When a person travels along the boundary and the area lies to his left side, the area is positive and if the area lies to his right side, then the area is negative.

Thus $|\boldsymbol{a} \times \boldsymbol{b}|$ = vector area of the parallelogram $OACB$

and $|\boldsymbol{b} \times \boldsymbol{a}|$ = vector area of the parallelogram $OBCA$

We shall now represent a vector product in terms of the components of its factors with respect to a Cartesian coordinate system. In this connection it is important to note that there are two types of such systems, depending on the orientation of the axes, namely, right-handed and left-handed.

Vector Product in Terms of Components

We shall first define right-handed and left-handed triples. We call a Cartesian coordinate system to be *right-handed* if the unit vector $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ in the directions of positive x, y, z-axes form a right-handed triple (Figure 5.30(a); and it is called *left-handed*, if vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ form a left-handed triple. In applications, we usually consider a right-handed system.

Let us consider a right-handed Cartesian coordinate system and let a_1, a_2, a_3 and b_1, b_2, b_3 be the components of two vector $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, so that

$$\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}} \text{ and } \boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$$

$$\therefore \quad \boldsymbol{a} \times \boldsymbol{b} = (a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}) \times (b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}})$$

$$= a_1 b_1 (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}}) + a_1 b_2 (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}) + a_1 b_3 (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}})$$

$$+ a_2 b_1 (\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}}) + a_2 b_2 (\hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}}) + a_2 b_3 (\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}})$$

$$+ a_3 b_1 (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}}) + a_3 b_2 (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}}) + a_3 b_3 (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}})$$

$$= 0 + a_1 b_2 \hat{\boldsymbol{k}} - a_1 b_3 \hat{\boldsymbol{j}}$$

$$- a_2 b_1 \hat{\boldsymbol{k}} + 0 + a_2 b_3 \hat{\boldsymbol{i}} + a_3 b_1 \hat{\boldsymbol{j}} - a_3 b_2 \hat{\boldsymbol{i}} + 0$$

$$\Rightarrow \quad \boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\,\hat{\boldsymbol{i}} + (a_3 b_1 - a_1 b_3)\,\hat{\boldsymbol{j}} + (a_1 b_2 - a_2 b_1)\,\hat{\boldsymbol{k}},$$

or in terms of second-order determinants,

$$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{i}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\boldsymbol{j}} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(a) Right-handed System **(b) Left-handed System**

Figure 5.30

This enables us to obtain the cross product by expanding the third order determinant viz.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

by its first row. Remember it is not an ordinary determinant as the elements of the first row are vectors and it must be expanded only by the first row.

In a left-handed Cartesian coordinate system, $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = -\hat{\boldsymbol{k}}$ (Figure 5.30(b)), and other similar expression lead to

$$\boldsymbol{a} \times \boldsymbol{b} = -\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Linear dependence or independence of two vectors can also be tested by their cross product. We say that '*Two vectors form a linearly dependent set if and only if their vector product is zero*'.

Cross product has many applications.

We shall now take up a few examples to illustrate them.

Example 5.12

Express Velocity of a rotating body as a vector product.

Solution

A rotation of a rigid body B in space can be simply and uniquely described by a vector ω. The direction of ω is that of axis of rotation of the body and such that the rotation appears clockwise, if one looks from the initial point of ω to its terminal point. The magnitude of ω is equal to the *angular speed* ω (> 0) of the rotation.

Now let P be any point of body B and let d be its distance from the axis. Then P has the speed $\omega\, d$. Let $\boldsymbol{r}$ be the position vector of P referred to a coordinate system with origin O on the axis of rotation. Then $d = |\,\boldsymbol{r}\,| \sin\gamma$, where γ is the angle between ω and $\boldsymbol{r}$. Therefore,

$$\omega d = |\omega||\boldsymbol{r}| \sin\gamma = |\omega \times \boldsymbol{r}|$$

Figure 5.31 : Rotation of a Rigid Body

The angular speed rotation of a body is the linear speed of a joint of the body divided by its distance from the axes of rotation.

From the above result and the definition of vector product, we see that the velocity $\boldsymbol{V}$ of P can be represented in the form

$$\boldsymbol{V} = \omega \times \boldsymbol{r}$$

This formula is useful for de termining velocity $\boldsymbol{V}$ at any point P of the body B.

Example 5.13

Find the moment of a force about a point in terms of vector product.

Solution

In mechanics the moment of a force $\boldsymbol{p}$ about a point Q is defined as the product $m = |\,\boldsymbol{p}\,|\, d$, where d is the perpendicular distance between Q and the line of action l of $\boldsymbol{p}$ (Refer Figure 5.32).

Figure 5.32 : Moment of a Force

If $\boldsymbol{r}$ is the vector from Q to any point A on l, then

$$\boldsymbol{d} = |\,\boldsymbol{r}\,| \sin \gamma$$

and $$m = |\,\boldsymbol{r}\,|\ |\,\boldsymbol{p}\,| \sin \gamma,$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{p}$.

Therefore, $$m = |\,\boldsymbol{p}\,|\ |\,\boldsymbol{r}\,| \sin \gamma$$

$$= |\,\boldsymbol{r} \times \boldsymbol{p}\,|$$

The vector $\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{p}$ is called the ***moment vector*** or ***vector moment*** of $\boldsymbol{p}$ about Q. Its magnitude is m and its direction is that of the axis of the rotation about Q which $\boldsymbol{p}$ has the tendency to produce.

Example 5.14

Express force on a charged particle and an element of current carrying conductor placed in a magnetic field in terms of cross product.

Solution

The magnitude of the force on a point charge q moving with velocity $\boldsymbol{V}$ in a magnetic field $\boldsymbol{B}$ is proportional to $|\boldsymbol{V}|$ times the perpendicular component of $\boldsymbol{B}$. Thus it can be expressed as a vector product as

$$\boldsymbol{F} = q\,(\vec{\boldsymbol{V}} \times \vec{\boldsymbol{B}})$$

If I is the current through the conductor, then the force on an element $d\boldsymbol{I}$ of a current carrying conductor in a magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = d\boldsymbol{I} \times \boldsymbol{B}$$

Example 5.15

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the position vectors of A, B, C in a $\Delta\, ABC$, show that the vector area of the triangle is

$$\frac{1}{2}[\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}]$$

Hence deduce the condition that the three points A, B, C may be collinear.

Solution

Let O be the origin of reference with respect to which $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the position vectors of A, B, C in $\Delta\, ABC$ (Figure 5.33).

Figure 5.33

$\therefore$ $OA = \boldsymbol{a}$, $OB = \boldsymbol{b}$, $OC = \boldsymbol{c}$

Now $BC = OC - OB = \boldsymbol{c} - \boldsymbol{b}$

$BA = OA - OB = \boldsymbol{a} - \boldsymbol{b}$

$\therefore$ Vector area of the triangle ABC

$$= \frac{1}{2}\,\boldsymbol{BC} \times \boldsymbol{BA}$$

$$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})$$

$$= \frac{1}{2}[\boldsymbol{c} \times \boldsymbol{a} - \boldsymbol{c} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{b}]$$

$$= \frac{1}{2}[\boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{b}]$$

Now if the points *A*, *B*, *C* are collinear, then the vector area of the $\Delta\, ABC$ must be zero, i.e.

$$\frac{1}{2}[\boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{b}] = 0$$

$$\Rightarrow \qquad \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a} = 0,$$

which is the required condition.

You may now attempt the following exercises.

SAQ 5

(a) Consider a force $\boldsymbol{F} = (-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}})$ newtons acting at a point $\boldsymbol{P} = (7\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})\,\boldsymbol{m}$. What is the torque about the origin?

(**Hint :** Torque τ is a measure of the ability of an applied force to produce a twist or to rotate a body $\Rightarrow \tau = \boldsymbol{r} \times \boldsymbol{F}$.)

(b) Show that for any two vectors ***a*** and ***b***

$$(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b}) = 2(\boldsymbol{a} \times \boldsymbol{b}),$$

and give its geometrical interpretation.

(c) Find the values of a for which the vectors

$$3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}} \text{ and } \hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$$

are (i) perpendicular (ii) parallel.

(d) A force represented by $5\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ is acting at a point $9\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Find its moment about the point $3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

In physics, repeated products of more than two vectors occur very often. For example, the electromotive force $d\boldsymbol{E}$ induced in an element of a conducting wire $d\boldsymbol{I}$ moving with velocity ***V*** through a magnetic field ***B*** is represented by

$$dE = (\boldsymbol{V} \times \boldsymbol{B}) \,.\, d\boldsymbol{I}$$

It is real economy in thinking to represent the result in the compact vector form that removes the necessity of carrying factors such as sine of the angle between *B* and *V* and the cosine of the angle between the normal to their plane and the vector $d\boldsymbol{I}$. These are taken into account by the given product of the three vectors.

Let us now discuss the product of three or more vectors in the next section.

5.6 MULTIPLE PRODUCTS OF VECTORS

Some of the commonly occurring multiple products in physical and engineering applications are the scalar and vector products of three vectors.

We know that if $\boldsymbol{b}$ and $\boldsymbol{c}$ are two vectors, then $\boldsymbol{b} \times \boldsymbol{c}$ is a vector perpendicular to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$. If $\boldsymbol{a}$ is a third vector, then scalar (or dot) product of $\boldsymbol{a}$ with $(\boldsymbol{b} \times \boldsymbol{c})$ is a scalar. This is called *scalar triple product* or *mixed triple product*. The cross (or vector) product of $\boldsymbol{a}$ with $(\boldsymbol{b} \times \boldsymbol{c})$ yields a vector and is called the vector triple product. Let us discuss the two types of products one by one.

5.6.1 Scalar Triple Product

The scalar triple product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, denoted by $\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$, is defined as

$$\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) = |\, \boldsymbol{a} \,|\; |\, \boldsymbol{b} \times \boldsymbol{c} \,| \cos \beta,$$

where β is the angle between $\boldsymbol{a}$ and the vector $(\boldsymbol{b} \times \boldsymbol{c})$

Here $\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$ is evidently a scalar. The scalar triple product $\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$ can be interpreted as the component of $\boldsymbol{a}$ along the vector $(\boldsymbol{b} \times \boldsymbol{c})$.

Note that $(\boldsymbol{a} \,.\, \boldsymbol{b})\, \boldsymbol{c}$ merely represents a vector in the direction of $\boldsymbol{c}$, whose modulus is $(\boldsymbol{a} \,.\, \boldsymbol{b})$ times the modulus of $\boldsymbol{c}$, whereas $(\boldsymbol{a} \,.\, \boldsymbol{b}) \times \boldsymbol{c}$ is meaningless since $\boldsymbol{a} \,.\, \boldsymbol{b}$ is not a vector but a scalar.

The different ways the scalar triple product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is written are as follows :

$$\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) \text{ or } [\boldsymbol{a}\; \boldsymbol{b}\; \boldsymbol{c}] \text{ or } [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] \text{ or } (\boldsymbol{a}\; \boldsymbol{b}\; \boldsymbol{c})$$

The absolute value of the scalar triple product has a geometrical meaning too. Let us see what it is?

Geometrical Interpretation of the Scalar Triple Product

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ by any three vectors. Consider a parallelopiped with its three coterminus edges having the magnitude and directions as of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively (Figure 5.34).

Figure 5.34 : Geometrical Interpretation of a Scalar Triple Product

Let θ be the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ and β be the angle between $\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$, the unit vector normal to $\boldsymbol{b}$ and $\boldsymbol{c}$.

Then, we have

$$\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$$

$$= \boldsymbol{a} \,.\, [\hat{\boldsymbol{n}}\, |\, \boldsymbol{b} \,|\; |\, \boldsymbol{c} \,| \sin \theta]$$

$$= (|\,\boldsymbol{a}\,| \cos \beta)\,(|\,\boldsymbol{b}\,|\;|\,\boldsymbol{c}\,| \sin \theta)$$

Now $(|\,\boldsymbol{b}\,|\;|\,\boldsymbol{c}\,| \sin \theta)$ is the area of the parallelogram formed by the vectors **b** and **c**, i.e., the base of the prallelopiped. Also $(|\,\boldsymbol{a}\,| \cos \beta)$ is the height *h* of the parallelopiped. Thus $\vec{a} \cdot (\vec{b} \times \vec{c})$ gives the volume of the parallelopiped.

Hence, *the absolute value of the scalar triple product* $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ *is equal to the volume of the parallelopiped with* ***a***, ***b***, ***c*** *as adjacent edges*.

From this geometrical consideration it follows that the value of the scalar triple product is a real number, which is independent of the choice of Cartesian coordinates in space.

In the parallelopiped in Figure 5.34 if we take the area of the face formed by **a** and **b** and multiply it by the height perpendicular (projection of **c** on $\boldsymbol{a} \times \boldsymbol{b}$), we get the volume of the same parallelopiped. Thus

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

Thus *the positions of dot and cross in a scalar triple product are interchangeable provided the cyclic order of the factors is maintained.*

$$\therefore \quad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$$

The change of cyclic order of factors brings about a change of sign in the value of the scalar triple product.

The scalar triple product of the orthonormal right handed vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ *is equal to unity* because

$$\hat{\boldsymbol{i}} \cdot (\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}}) = \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = 1$$

Next consider three coplanar vectors **a**, **b**, **c**. Now $\boldsymbol{b} \times \boldsymbol{c}$ is a vector perpendicular to the plane of **b** and **c** and is, therefore, perpendicular to **a** also.

$\therefore$ Scalar product of **a** and $\boldsymbol{b} \times \boldsymbol{c}$ must be zero

$$\therefore \quad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$$

Thus, *if three vectors are coplanar, their scalar triple product is zero.*

Further if we consider **a**, **a**, **b** and form a scalar triple product, then

$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \times \boldsymbol{a}) \cdot \boldsymbol{b}$$

$$= \boldsymbol{0} \cdot \boldsymbol{b}$$

$$= 0$$

If two vectors in a scalar triple product are equal or parallel, then their scalar triple product vanishes.

With respect to any right-handed Cartesian coordinate system, let

$$\boldsymbol{a} = a_1 \hat{\boldsymbol{i}} + a_2 \hat{\boldsymbol{j}} + a_3 \hat{\boldsymbol{k}},\; \boldsymbol{b} = b_1 \hat{\boldsymbol{i}} + b_2 \hat{\boldsymbol{j}} + b_3 \hat{\boldsymbol{k}},\; \boldsymbol{c} = c_1 \hat{\boldsymbol{i}} + c_2 \hat{\boldsymbol{j}} + c_3 \hat{\boldsymbol{k}}$$

Then
$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence $\quad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (a_1 \hat{\boldsymbol{i}} + a_2 \hat{\boldsymbol{j}} + a_3 \hat{\boldsymbol{k}}) \cdot \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\Rightarrow \quad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Since interchanging of two rows reverses the sign of the determinant, we have

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = -[\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}]$$

Let us consider some examples giving applications of scalar triple product.

Example 5.16

Show that the force acting on a charged particle q which moves with velocity $\boldsymbol{V}$ in the magnetic field $\boldsymbol{B}$ does not bring about any change in the energy of the charge.

Solution

The force acting on a charge particle q moving with velocity $\boldsymbol{V}$ in the magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F} = q\,(\vec{\boldsymbol{V}} \times \vec{\boldsymbol{B}})$$

We know that the change in energy is equal to the work done.

Now work done on the charge for an infinitesimal displacement $d\boldsymbol{r}$ is

$$\boldsymbol{W} = \boldsymbol{F} \cdot d\boldsymbol{r} = q\,(\vec{\boldsymbol{V}} \times \vec{\boldsymbol{B}}) \cdot d\vec{\boldsymbol{r}}$$

If the displacement $d\boldsymbol{r}$ takes place in time $d\boldsymbol{t}$, then

$$d\boldsymbol{r} = \boldsymbol{V}\,dt$$

$$\therefore \quad \boldsymbol{W} = q\,(\vec{\boldsymbol{V}} \times \overline{\boldsymbol{B}}) \cdot \vec{\boldsymbol{V}}\,dt$$

$$\therefore \quad = q\,d\boldsymbol{t}\,[\boldsymbol{V}, \boldsymbol{B}, \boldsymbol{V}]$$

$$\therefore \quad = q\,d\boldsymbol{t} \cdot 0$$

$$= 0$$

Since the work done is zero, hence the force $\boldsymbol{F}$ does not bring about any change in the energy of the charge.

Example 5.17

Find the volume of the tetrahedron whose three sides are given by

$$2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}},\ \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} \text{ and } 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

Solution

We know that volume of a tetrahedron is one-sixth the volume of the parallelopiped with three sides of tetrahedron as its three adjacent edges.

$$\therefore \quad \text{Reqd volume} = \frac{1}{6}\,\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \frac{1}{6}\,(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

where $\quad \boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

$$\boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$$

$$\boldsymbol{c} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

Now $$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{vmatrix} = -5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$$

$\therefore$ $$(\boldsymbol{a} \times \boldsymbol{b}) \,.\, \boldsymbol{c} = (-5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}) \,.\, (3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$$

$$= -15 - 6 + 14 = -7$$

Neglecting negative sign, the volume of tetrahedron $= \dfrac{7}{6}$.

Example 5.18

Find the constant λ so that the vectors

$$\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}},\ \boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}},\ \boldsymbol{c} = 3\hat{\boldsymbol{i}} + \lambda\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$$

are coplanar.

Solution

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar, then their scalar triple product is zero, i.e., $\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) = 0$.

$$\Rightarrow \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2\,(10 + 3\lambda) + (5 + 9) + (\lambda - 6) = 0$$

$$\Rightarrow \quad 20 + 6\lambda + 14 + \lambda - 6 = 0$$

$$\Rightarrow \quad 28 + 7\lambda = 0$$

$$\Rightarrow \quad \lambda = -4$$

You may now attempt a few exercise to test your knowledge.

SAQ 6

(a) Prove that $(\boldsymbol{a} + \boldsymbol{b}) \,.\, [(\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{c} + \boldsymbol{a})] = 2\,[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

(b) If the volume of the parallelopiped whose edges are $-12\hat{\boldsymbol{i}} + \alpha\hat{\boldsymbol{k}}$, $3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 15\hat{\boldsymbol{k}}$ is 546, determine α.

(c) The position vectors of the points A, B, C, D are $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}}$ respectively. If the points A, B, C, D lie in a plane, find the value of λ.

Let us now discuss the second type of vector product, that is, the vector triple product.

5.6.2 Vector Triple Product

We first give the definition of vector triple product.

Definition

The vector product of two vectors, one of which is itself the vector product of two vectors, is a vector quantity called the ***vector triple product***.

Thus if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three vectors, then $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ are vector triple products.

We next give geometrical interpretation for vector triple product.

Geometrical Interpretation of Vector Triple Product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

You know that $\boldsymbol{b} \times \boldsymbol{c}$, by definition, is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$, that is, $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to the plane containing $\boldsymbol{b}$ and $\boldsymbol{c}$. Also $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, being the vector product of vector $\boldsymbol{a}$ and vector $(\boldsymbol{b} \times \boldsymbol{c})$ is perpendicular to both $\boldsymbol{a}$ and $(\boldsymbol{b} \times \boldsymbol{c})$.

Hence $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ must be parallel to the plane determined by $\boldsymbol{b}$ and $\boldsymbol{c}$ and is perpendicular to $\boldsymbol{a}$.

From the above geometrical interpretation of vector triple product, you may note that where as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is a vector parallel to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$ and is perpendicular to the vector $\boldsymbol{a}$, the vector triple product $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is a vector parallel to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$ is perpendicular to the vector $\boldsymbol{c}$.

Hence, in general,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$$

Thus *cross (or vector) product is not associative*. As a word of caution we may mention that always keep track of the brackets for correct interpretation and expansion of vector triple product.

The geometrical interpretation of vector triple product can also be used to find its expansion formula.

Expansion Formula for Vector triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

Since $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ lie in the plane containing $\boldsymbol{b}$ and $\boldsymbol{c}$, we can, therefore, resolve $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ into components parallel to $\boldsymbol{b}$ and $\boldsymbol{c}$, i.e., it is possible to find scalars m and n such that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = m\boldsymbol{b} + n\boldsymbol{c} \quad \ldots (5.11)$$

Multiplying both sides by $\boldsymbol{a}$, we get

$$\vec{a} \,.\, [\vec{a} \times (\vec{b} \times \vec{c})] = m\,(\boldsymbol{a} \,.\, \boldsymbol{b}) + n\,(\boldsymbol{a} \,.\, \boldsymbol{c}) \quad \ldots (5.12)$$

The left-hand side of Eq. (5.12) is a scalar triple product of $\boldsymbol{a}$, $\boldsymbol{a}$ and $(\boldsymbol{b} \times \boldsymbol{c})$. Since in this scalar triple product, two of the vectors are equal, therefore it is zero.

Hence $\quad m\,(\boldsymbol{a} \,.\, \boldsymbol{b}) + n\,(\boldsymbol{a} \,.\, \boldsymbol{c}) = 0$

or $\quad \dfrac{m}{(\boldsymbol{a} \,.\, \vec{c})} = \dfrac{n}{-(\boldsymbol{a} \,.\, \boldsymbol{b})} = p$, say

Putting the values of m and n in Eq. (5.11), we get

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = p\,(\boldsymbol{a} \,.\, \boldsymbol{c})\,\boldsymbol{b} - p\,(\boldsymbol{a} \,.\, \boldsymbol{b})\,\boldsymbol{c} \quad \ldots (5.13)$$

We note that both the sides of Eq. (5.13) are equally balanced in $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Hence p must be some numerical constant independent of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Also Eq. (5.13) is true for any three vectors **a**, **b**, **c**. Thus to determine p, we take the special case where

$$\boldsymbol{a} = \boldsymbol{b} = \hat{\boldsymbol{i}} \text{ and } \boldsymbol{c} = \hat{\boldsymbol{j}}$$

Then $\hat{\boldsymbol{i}} \times (\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}) = p\,(\hat{\boldsymbol{i}} \,.\, \hat{\boldsymbol{j}})\,\hat{\boldsymbol{i}} - p\,(\hat{\boldsymbol{i}} \,.\, \hat{\boldsymbol{i}})\,\hat{\boldsymbol{j}}$

$\Rightarrow \quad \hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}} = p\,(0)\,\hat{\boldsymbol{i}} - p\,(1)\,\hat{\boldsymbol{j}}$

$\Rightarrow \quad -\hat{\boldsymbol{j}} = -p\,\hat{\boldsymbol{j}}$

$\Rightarrow \quad p = 1$

Substituting for $p = 1$ in Eq. (5.13), we get the required expansion formula as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \,.\, \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\,\boldsymbol{c}$$

Let us consider a physical example of the vector triple product from electromagnetism.

Example 5.19

Show that parallel wires carrying current in the same direction attract each other.

Solution

Consider two wires w_1 and w_2 carrying current i_1 and i_2 respectively. Let $d\boldsymbol{I}_1$ and $d\boldsymbol{I}_2$ be the infinitesimal elements of the wire in the direction of current flow. Then the force experienced by the infinitesimal element $d\boldsymbol{I}_2$ due to $d\boldsymbol{I}_1$ is given by

$$\boldsymbol{F} = \frac{\mu}{4\pi}\, i_1\, i_2\, \frac{d\vec{\boldsymbol{I}}_2 \times (d\vec{\boldsymbol{I}}_1 \times \vec{\boldsymbol{r}})}{r^3},$$

where μ is the permeability of the free space and $\boldsymbol{r}$ is the position vector of $d\boldsymbol{I}_2$ w.r.t. to $d\boldsymbol{I}_1$.

If we take $d\boldsymbol{I}_1$ and $\boldsymbol{r}$ in the plane of this paper, then $d\boldsymbol{I}_1 \times \boldsymbol{r}$ is a vector perpendicular to the plane of this paper and points to it. Thus $d\boldsymbol{I}_2 \times (d\boldsymbol{I}_1 \times \boldsymbol{r})$ is a vector in the plane of this paper and perpendicular to $d\boldsymbol{I}_2$. This means that parallel wires carrying current in the same direction attract each other.

In case the direction of any one of the currents is reversed, the wires will repulse each other.

Let us take another example to illustrate the use of vector triple products.

Example 5.20

Prove that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0$$

Solution

From the expansion of vector triple products, we have

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \,.\, \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\,\boldsymbol{c} \quad \ldots (5.14)$$

$$\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = (\boldsymbol{b} \,.\, \boldsymbol{a})\,\boldsymbol{c} - (\boldsymbol{b} \,.\, \boldsymbol{c})\,\boldsymbol{a} \quad \ldots (5.15)$$

$$\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{c} \,.\, \boldsymbol{b})\,\boldsymbol{a} - (\boldsymbol{c} \,.\, \boldsymbol{a})\,\boldsymbol{b} \quad \ldots (5.16)$$

Since scalar product is commutative, hence adding Eqs. (5.14), (5.15) and (5.16) we see that the terms in the right cancel and we get

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0$$

You may now attempt the following exercises.

SAQ 7

(a) If $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}},\ \boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}},\ \boldsymbol{c} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, find

(i) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and

(ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

(b) Prove that

$$\hat{\boldsymbol{i}} \times (\boldsymbol{a} \times \hat{\boldsymbol{i}}) + \hat{\boldsymbol{j}} \times (\boldsymbol{a} \times \hat{\boldsymbol{j}}) + \hat{\boldsymbol{k}} \times (\boldsymbol{a} \times \hat{\boldsymbol{k}}) = 2\boldsymbol{a}$$

for any arbitrary vector $\boldsymbol{a}$.

(c) If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are three unit vectors, such that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \frac{1}{2}\boldsymbol{b}$, find the angle which $\boldsymbol{a}$ makes with $\boldsymbol{b}$ and $\boldsymbol{c}$, given that $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-parallel.

(d) Show that $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ if and only if $\boldsymbol{a}$ and $\boldsymbol{c}$ are collinear vectors.

Using the results of the Sub-sections 5.6.1 and 5.6.2 the product of four (or more) vectors can also be evaluated. It will be seen that other repeated products which occur in applications may be expressed in terms of dot product or vector product or scalar triple products. We now discuss briefly the product of four vectors.

5.6.3 Quadruple Product of Vectors

Quadruple product means the product of four vectors. Some of the relevant quadruple products are :

(i) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d})$

(ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d})$

(iii) $\boldsymbol{a} \times [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})]$

Let us now find their simplified expressions.

(i) Let $\boldsymbol{p} = \boldsymbol{c} \times \boldsymbol{d}$

$\therefore \quad (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{p}$

$= \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{p})$,

since dot and cross product are interchangeable in scalar triple product.

$\Rightarrow \quad (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{a} \cdot [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})]$

$= \boldsymbol{a} \cdot [(\boldsymbol{b} \cdot \boldsymbol{d})\, \boldsymbol{c} - (\boldsymbol{b} \cdot \boldsymbol{c})\, \boldsymbol{d}]$

$= (\boldsymbol{a} \cdot \boldsymbol{c})\, (\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{b} \cdot \boldsymbol{c})\, (\boldsymbol{a} \cdot \boldsymbol{d})$

Hence, $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, (\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{b} \cdot \boldsymbol{c})\, (\boldsymbol{a} \cdot \boldsymbol{d})$ . . . (5.17) which is known as *Identity of Lagrange*.

Next consider

(ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{p}$, where $\boldsymbol{p} = \boldsymbol{c} \times \boldsymbol{d}$

$$= -\, \boldsymbol{p} \times (\boldsymbol{a} \times \boldsymbol{b})$$

$$= -\, [(\boldsymbol{p} \,.\, \boldsymbol{b})\, \boldsymbol{a} - (\boldsymbol{p} \,.\, \boldsymbol{a})\, \boldsymbol{b}]$$

$$= (\boldsymbol{p} \,.\, \boldsymbol{a})\, \boldsymbol{b} - (\boldsymbol{p} \,.\, \boldsymbol{b})\, \boldsymbol{a}$$

$$= \{\boldsymbol{a} \,.\, (\boldsymbol{c} \times \boldsymbol{d})\}\, \boldsymbol{b} - \{\boldsymbol{b} \,.\, (\boldsymbol{c} \times \boldsymbol{d})\}\, \boldsymbol{a}$$

Hence, $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]\, \boldsymbol{b} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\, \boldsymbol{a}$ . . . (5.18)

In case we start with $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{q}$ and expand vector triple product $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d})$ as $\boldsymbol{q} \times (\boldsymbol{c} \times \boldsymbol{d})$, we get another expression for this quadruple product as

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\, \boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, \boldsymbol{d} \quad . . . (5.19)$$

Lastly, we consider

(iii) $\boldsymbol{a} \times [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})] = \vec{a} \times [(\vec{b} \,.\, \vec{d})\, \vec{c} - (\vec{b} \,.\, \vec{c})\, \vec{d}]$

$$= (\boldsymbol{b} \,.\, \boldsymbol{d})\, (\boldsymbol{a} \times \boldsymbol{c}) - (\boldsymbol{b} \,.\, \boldsymbol{c})\, (\boldsymbol{a} \times \boldsymbol{d})$$

Hence $\boldsymbol{a} \times [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})] = (\boldsymbol{b} \,.\, \boldsymbol{d})\, (\boldsymbol{a} \times \boldsymbol{c}) - (\boldsymbol{b} \,.\, \boldsymbol{c})\, (\boldsymbol{a} \times \boldsymbol{d})$ . . . (5.20)

Let us consider some example illustrating the use of quadruple products.

Example 5.21

Prove that

$$(\boldsymbol{a} \times \boldsymbol{b}) \,.\, \{(\boldsymbol{b} \times \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{a})\} = [\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})]^2$$

Solution

Using result (5.18), we have

$$(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}]\, \boldsymbol{c} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{c}]\, \boldsymbol{a}$$

$$= [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}]\, \boldsymbol{c} \qquad (\because\ [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{c}] = 0)$$

Hence $(\boldsymbol{a} \times \boldsymbol{b}) \,.\, [(\boldsymbol{b} \times \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{a})]$

$$= (\boldsymbol{a} \times \boldsymbol{b}) \,.\, \{[\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}]\, \boldsymbol{c}\}$$

$$= [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}]\, \{(\boldsymbol{a} \times \boldsymbol{b}) \,.\, \boldsymbol{c}\}$$

$$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$$

Cyclic change in scalar triple product

$$= [\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})]^2$$

Hence from the above result, it immediately follows that *if* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are three non-coplanar vectors, then* $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{b} \times \boldsymbol{c}$ *and* $\boldsymbol{c} \times \boldsymbol{a}$ *are also non-coplanar.*

You may now try this exercise.

SAQ 8

Prove that

$$[(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})] \,.\, \boldsymbol{d} = (\boldsymbol{a} \,.\, \boldsymbol{d})\ [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$$

On the basis of the above knowledge, we shall like to discuss in brief in the next section the two categories of vectors into which all vector quantities of mechanics/physics can be grouped. These are termed as proper (or polar) vectors and pseudo vectors (or axial vectors).

5.7 POLAR AND AXIAL VECTORS

The direction of some of the vector quantities is clearly indicated by the direction of motion of a system. Examples of such vectors are displacements, velocity,

acceleration, etc. *Such vectors are called polar vectors.*

Vectors, namely, angular velocity, angular acceleration, angular momentum, etc. *associated with the rotation motion are such that their direction does not indicate the direction of rotation of the body*. Their direction is taken to be along the axis of rotation. *Such vectors are called axial vectors*.

The transformation of coordinates

$$x' = -\,x,\ \ y' = -\,y,\ \ z' = -\,z$$

is called the **parity transformation.**

The difference between polar and axial vectors can be expressed in terms of parity transformation as follows:

If a vector changes sign under the parity transformation, it is called a proper or a polar vector. The vectors which do not change sign under a parity transformation are called axial vectors.

It may be observed that

The cross product of two polar vectors is an axial vector.

Before we move on to the next units, wherein we shall discuss vector differential calculus and vector integral calculus, it will be worthwhile to look briefly to the three coordinate systems in space and give their relations. We take up these coordinate systems briefly in the next section.

5.8 COORDINATE SYSTEMS FOR SPACE

Let us look briefly at three coordinate systems for space. The first, Cartesian coordinates, is the system we have been using the most in our discussion so far. Cylindrical and spherical polar coordinates will come handy when we study integration of vectors in Units 6 and 7 because surfaces that have complicated representation in Cartesian coordinates sometimes have simpler equations in one of these other systems.

5.8.1 Cartesian Coordinates

Consider a system of mutually orthogonal coordinate axes *Ox*, *Oy* and *Oz* (see Figure 5.35). The Cartesian coordinates of a point $P(x, y, z)$ in space may be read from the coordinate axes by passing planes through *P* perpendicular to each axis.

The three coordinate planes $x = 0$, $y = 0$, $z = 0$ divide the space into eight cells, called octants.

Figure 5.35 : Cartesian Coordinates

The octant in which all the three coordinates are positive is called the first octant. We next take up cylindrical coordinates.

5.8.2 Cylindrical Coordinates

It is frequently convenient to use cylindrical coordinates (r, θ, z) to locate a point in space. There are first the polar coordinates (r, θ) used instead of (x, y) in the plane $z = 0$ coupled with the *z*-coordinate (refer Figure 5.36).

Equations relating Cartesian and Cylindrical Coordinates are

$$x = r \cos \theta, \qquad r^2 = x^2 + y^2$$

$$y = r \sin \theta, \qquad \tan \theta = \frac{y}{x}$$

$$z = z$$

Here $r = 0$ is the equation for *z*-axis and r = constant describes a circular cylinder of radius *r* whose axis is the *z*-axis.

Figure 5.36 : Cylindrical Coordinates

The equation θ = constant describes a plane containing the z-axis and making an angle θ with the positive x-axis.

Cylindrical coordinates are convenient when there is an axis of symmetry in a physical problem.

5.8.3 Spherical Coordinates

Spherical coordinates are useful when there is a center of symmetry. The spherical coordinates (R, θ, ϕ) of a point in space are shown in Figure 5.37 below.

Figure 5.37 : Spherical Coordinates

The first coordinate $R = |OP|$ is the distance from the origin to the point P. It is never negative.

The second coordinate θ is the same as in cylindrical coordinates, namely, the angle from xz-plane to the plane through P and the z-axis.

The third spherical coordinate ϕ is the angle measured down from the z-axis to the line OP.

Every point in space can be represented in terms of spherical coordinates restricted to the ranges.

$$R \geq 0,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi$$

Equations relating Cartesian and Cylindrical Coordinates to Spherical Polar Coordinates are

$$r = R \sin \phi, \quad x = r \cos \theta, \quad x = R \sin \phi \cos \theta$$

$$z = R\cos\phi, \quad y = r\sin\theta, \quad y = R\sin\phi\sin\theta$$

$$\theta = \theta, \qquad z = z, \qquad z = R\cos\phi$$

The equation R = constant describes the surface of a sphere of radius R with center at 0.

The equation θ = constant in spherical polar coordinates defines a half-plane

$\because$ $(R \geq 0, \text{ and } 0 \leq \phi \leq \pi)$

The equation ϕ = constant describes a cone with vertex at 0, axis OZ, and generating angle ϕ provided we broaden our interpretation of the word 'cone' to include the xy-plane for which $\phi = \frac{\pi}{2}$ and comes with generating angles greater than $\frac{\pi}{2}$.

5.9 SUMMARY

In this unit, you have learnt

- A physical quantity completely specified a single number (with a suitable choic e of units of measure) is called a *scalar*.
- Quantities specified by a magnitude and a direction are called *vector* quantities.
- Length, support and sense characterize a *directed line segment*. Length of a directed line segment is called *magnitude* or *modulus* or norm of the vector it represents. *Direction* of a vector is from its initial to terminal point.
- A vector with a certain fixed initial point is called a *bounded vector* and when there is no restriction to choose the initial point, it is called a *free vector*.
- A vector whose length is zero is called a *null vector*.
- A vector whose length is unity is called *unit vector*.
- All vectors having the same initial point are called *coinitial vectors*.
- Vectors having the same or parallel line of action are called *like* or *parallel* or *collinear vectors* and vectors are called *unlike* if they have opposite directions.
- Vectors parallel to the same plane or lying in the same plane are called *coplanar vectors*. Three vectors are coplanar if their scalar triple product is zero.
- Two vectors are said to be equal if they have the same length, same or parallel supports and the same sense.
- Projections of a vector on the axes of an orthogonal Cartesian coordinate system are called its components. If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ in component form, then

  $$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

  and direction cosines of the vector are $\frac{a_1}{|\boldsymbol{a}|}, \frac{a_2}{|\boldsymbol{a}|}, \frac{a_3}{|\boldsymbol{a}|}$.
- If initial point of a vector is chosen to be the origin of a Cartesian coordinate system, then components of the vector are the coordinates of its terminal point and the vector is called *position vector* of its terminal point.

- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be added graphically using *triangle law or parallelogram law of vector addition.*
- *Vector addition is commutative and associative.* There exist *additive identity* (0) and *additive inverse* (negative of the vector).
- The *difference* $\boldsymbol{a} - \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is obtained by adding $(-\boldsymbol{b})$ to $\boldsymbol{a}$.
- Given $\boldsymbol{a} = a_1 \hat{\boldsymbol{i}} + a_2 \hat{\boldsymbol{j}} + a_3 \hat{\boldsymbol{k}},\ \boldsymbol{b} = b_1 \hat{\boldsymbol{i}} + b_2 \hat{\boldsymbol{j}} + b_3 \hat{\boldsymbol{k}}$

  $$\boldsymbol{a} \pm \boldsymbol{b} = (a_1 \pm b_1)\, \hat{\boldsymbol{i}} + (a_2 \pm b_2)\, \hat{\boldsymbol{j}} + (a_3 \pm b_3)\, \hat{\boldsymbol{k}}.$$

- If m is a scalar and $\boldsymbol{a}$ a vector, then $m\,\boldsymbol{a}$ is a vector whose magnitude $= |m|\ |\boldsymbol{a}|$, support is same or parallel to $\boldsymbol{a}$ and direction of vector $m\,\boldsymbol{a}$ is same as $\boldsymbol{a}$ if $m > 0$ and opposite to $\boldsymbol{a}$ if $m < 0$.
- Two vectors are *linearly dependent* if and only if they are parallel, otherwise they are linearly independent. The condition of linear dependence is that their vector product is zero.
- The *scalar product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} \,.\, \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos \gamma$, where $\gamma\ (0 \le \gamma \le \pi)$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. In component form, $\vec{\boldsymbol{a}} \,.\, \vec{\boldsymbol{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3$.
- Two non zero vectors are perpendicular to each other if and only if their dot product is zero.
- $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \,.\, \boldsymbol{a}}$ and $\cos \gamma = \dfrac{\boldsymbol{a} \,.\, \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$ for angle γ between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Geometrically, scalar product of two vectors is the product of the modulus of either vector and projection of the other in its direction.
- The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

  $$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin \gamma\ \hat{\boldsymbol{n}}, \qquad (0 \le \gamma \le \pi),$$

  where γ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\hat{\boldsymbol{n}}$ is a unit vector perpendicualr to both $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ in the direction such that $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ form a right-handed trial.
- $\boldsymbol{a} \times \boldsymbol{b} = 0$ is the condition for vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to be parallel.
- Geometrically, $\boldsymbol{a} \times \boldsymbol{b}$ represents the vector area of the parallelogram having adjacent sides represented by $\boldsymbol{a}$ and $\boldsymbol{b}$.
- $|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(\boldsymbol{a} \,.\, \boldsymbol{a})\ (\boldsymbol{b} \,.\, \boldsymbol{b}) - (\boldsymbol{a} \,.\, \boldsymbol{b})^2}$.
- In the component form

  $$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- The scalar triple product $\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$ is given by

  $$\boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) = |\boldsymbol{a}|\ |\boldsymbol{b} \times \boldsymbol{c}| \cos \beta,$$

  where β is the angle between $\boldsymbol{a}$ and $(\boldsymbol{b} \times \boldsymbol{c})$.

  In the component form,

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Geometrically, $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is the volume of the parallelopiped with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as adjacent sides.

- The positions of dot and cross products in a scalar triple product are interchangeable provided the cyclic order of the factors is maintained.

- The vector triple product of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is given by

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{c}$$

- Geometrically, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is parallel to the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$ and is perpendicular to $\boldsymbol{a}$.

- Some quadruple products of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\, \boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, \boldsymbol{d}$$

$$\boldsymbol{a} \times [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})] = (\boldsymbol{b} \cdot \boldsymbol{d})(\boldsymbol{a} \times \boldsymbol{c}) - (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \times \boldsymbol{d})$$

- The component of *polar* vector change their sign under a parity transformation, while those of an *axial* vector do not.

- Equations relating Cartesian and Cylindrical Coordinates to Spherical Polar Coordinates are

$$r = R \sin\phi, \quad x = r\cos\theta, \quad x = R\sin\phi\cos\theta$$

$$z = R\cos\phi, \quad y = r\sin\theta, \quad y = R\sin\phi\sin\theta$$

$$\theta = \theta, \qquad z = z, \qquad z = R\cos\phi$$

5.10 ANSWERS TO SAQs

SAQ 1

(a) If a_1, a_2, a_3 are the components of a vector $\boldsymbol{a}$ with initial point $P(3, -2, 1)$ and terminal point $Q(1, 2, -4)$ then

$$a_1 = 1 - 3 = -2, \; a_2 = 2 - (-2) = 4, \; a_3 = -4 - 1 = -5$$

and

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$= \sqrt{(-2)^2 + (4)^2 + (-5)^2}$$

$$= \sqrt{4 + 16 + 25}$$

$$= \sqrt{45}$$

$$= 3\sqrt{5}$$

(b) Since $\frac{1}{2}, 1, \frac{3}{2}$ are components of a vector $\boldsymbol{a}$ and

$P\left(-\frac{1}{2}, 1, \frac{3}{2}\right)$ is the initial point of ***a*** and if $Q\ (x, y, z)$ is the terminal point of ***a*** then, $x = -\frac{1}{2} + \frac{1}{2} = 0,\ y = 1 + 1 = 2,\ z = \frac{1}{2} + \frac{3}{2} = 2$ i.e., terminal point of (0, 2, 2)

Also length of $\boldsymbol{a} = |\boldsymbol{a}| = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + \left(\frac{3}{2}\right)^2}$

$$= \sqrt{\frac{1}{4} + 1 + \frac{9}{4}}$$

$$= \sqrt{\frac{14}{4}}$$

$$= \sqrt{\frac{7}{2}}$$

SAQ 2

Figure

In $\Delta\ ABC$, by triangle law of addition, $\boldsymbol{AB} = \boldsymbol{AC} + \boldsymbol{CB}$

In $\Delta\ BCD$ by triangle law of addition, $\boldsymbol{DC} = \boldsymbol{DB} + \boldsymbol{BC}$

Adding, we get $\boldsymbol{AB} + \boldsymbol{DC} = \boldsymbol{AC} + \boldsymbol{CB} + \boldsymbol{DB} + \boldsymbol{BC}$

$$= \boldsymbol{AC} + \boldsymbol{DB} + \boldsymbol{CB} - \boldsymbol{CB} \qquad (\because\ \vec{a} + (-\vec{a}) = 0)$$

$$= \boldsymbol{AC} + \boldsymbol{DB}$$

Hence the result.

SAQ 3

(a) (i) Let A, B, C be three non-linear point. Draw $a\ \Delta\ ABC$ such that

$$\boldsymbol{AB} = \boldsymbol{a} \quad \text{and} \quad \boldsymbol{BC} = \boldsymbol{b}$$

Figure

As sum of two sides of a triangle is greater than the third side,

$\therefore \quad |\boldsymbol{AC}| < |\boldsymbol{AB}| + |\boldsymbol{BC}|$

When A, B, C are collinear, then

$$\boldsymbol{a} = \boldsymbol{AB}, \ \boldsymbol{b} = \boldsymbol{BC}$$

$$\therefore \quad \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{AC}$$

$$\because \quad \boldsymbol{AC} = \boldsymbol{AB} + \boldsymbol{BC}$$

$$\therefore \quad |\boldsymbol{AC}| = |\boldsymbol{AB}| + |\boldsymbol{BC}|$$

$$\Rightarrow \quad |\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}| + |\boldsymbol{b}|$$

Combining the results for collinear and non-collinear points $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, we get

$$|\boldsymbol{a} + \boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$$

(ii) Here $\quad |\boldsymbol{a}| = |\boldsymbol{a} + (\boldsymbol{b} - \boldsymbol{b})| = |(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b}|$

$$\le |\boldsymbol{a} - \boldsymbol{b}| + |\boldsymbol{b}|$$

$$\Rightarrow \quad |\boldsymbol{a}| - |\boldsymbol{b}| \le |\boldsymbol{a} - \boldsymbol{b}|$$

(b) Here $\quad \boldsymbol{a} = 3\sqrt{3}\,\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}, \ \boldsymbol{b} = 6\hat{\boldsymbol{j}}, \ \boldsymbol{c} = 3\sqrt{3}\,\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

Now $\quad \boldsymbol{a} + \boldsymbol{b} = 3\sqrt{3}\,\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{j}}$

$$= 3\sqrt{3}\,\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$$

$$= \boldsymbol{c}$$

Also $\quad |\boldsymbol{a}| = \sqrt{27 + 9} = \sqrt{36} = 6$

$$|\boldsymbol{b}| = \sqrt{36} = 6$$

$$|\boldsymbol{c}| = \sqrt{27 + 9} = \sqrt{36} = 6$$

Hence $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form an equilateral triangle.

(c) Let us denote the three points by A, B, C. With reference to some origin of regerence O, we have

$$\boldsymbol{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}, \ \boldsymbol{OB} = 3\hat{\boldsymbol{i}} + \frac{9}{4}\hat{\boldsymbol{j}}, \ \boldsymbol{OC} = 5\hat{\boldsymbol{i}} + \frac{3}{4}\hat{\boldsymbol{j}}$$

Now $\quad \boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \left(3\hat{\boldsymbol{i}} + \frac{9}{4}\hat{\boldsymbol{j}}\right) - (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

$$= \hat{\boldsymbol{i}} - \frac{3}{4}\hat{\boldsymbol{j}}$$

$$\boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA} = \left(5\hat{\boldsymbol{i}} + \frac{3}{4}\hat{\boldsymbol{j}}\right) - (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$$

$$= 3\hat{\boldsymbol{i}} - \frac{9}{4}\hat{\boldsymbol{j}}$$

$$= 3\left(\hat{\boldsymbol{i}} - \frac{3}{4}\hat{\boldsymbol{j}}\right)$$

$$= 3\,\boldsymbol{AB}$$

$\therefore$ $\boldsymbol{AC}$ and $\boldsymbol{AB}$ are parallel, but $\boldsymbol{AC}$ and $\boldsymbol{AB}$ have one point A common.

Hence A, B, and C are collinear.

(d) If the three vectors are coplanar, they should be linearly dependent. Thus one of them can be expressed as linear combination of the remaining two, i.e., there should exist two scalars x and y such that

$$5\boldsymbol{a} + 6\boldsymbol{b} + 7\boldsymbol{c} = x\,(7\boldsymbol{a} - 8\boldsymbol{b} + 9\boldsymbol{c}) + y\,(3\boldsymbol{a} + 20\boldsymbol{b} + 5\boldsymbol{c})$$

Equating coefficients of like vectors, we get

$$5 = 7x + 3y$$

$$6 = -8x + 20y$$

$$7 = 9x + 5y$$

Solving the first two of these equations, we get

$$x = \frac{1}{2},\; y = \frac{1}{2}$$

These values of x and y also satisfy the third of the above equations.

Here the three vectors are coplanar.

SAQ 4

(a) Let O be the centre of the semicircle with $\boldsymbol{AB}$ as diameter.

Let P be the any point on the circumference of the semi-circle.

Let O be the origin of reference

and $\boldsymbol{OA} = \boldsymbol{a}$ so that $\boldsymbol{OB} = -\boldsymbol{a}$

let $\boldsymbol{OP} = \boldsymbol{r}$

Now for the semi-circle,

$$|\boldsymbol{OP}| = |\boldsymbol{OA}|$$

$$\therefore \quad \boldsymbol{OP}^2 = \boldsymbol{OA}^2$$

Figure

$$\Rightarrow \quad \boldsymbol{r}^2 = \boldsymbol{a}^2$$

$$\Rightarrow \quad (\boldsymbol{r} - \boldsymbol{a}) \,.\, (\boldsymbol{r} + \boldsymbol{a}) = 0$$

$$\Rightarrow \quad (\boldsymbol{OP} - \boldsymbol{OA}) \,.\, (\boldsymbol{OP} - \boldsymbol{OB}) = 0$$

$$\Rightarrow \quad \boldsymbol{AP} \,.\, \boldsymbol{BP} = 0$$

$\Rightarrow$ $\boldsymbol{AP}$ is perpendicular to $\boldsymbol{BP}$

Hence $\angle\, \boldsymbol{APB} = 90^{\circ}$

(b) $\boldsymbol{F}$ = Total force = Sum of forces

$$= (-3\hat{i} + 2\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 3\hat{k})$$

$$= -\hat{i} + 3\hat{j} + 2\hat{k}$$

$\boldsymbol{d}$ = Total displacement $= (4\hat{i} - 3\hat{j} + 7\hat{k}) - (2\hat{i} - \hat{j} - 3\hat{k})$

$$= 2\hat{i} - 2\hat{j} + 10\hat{k}$$

Work done $\boldsymbol{F} \cdot \boldsymbol{d} = (-\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (2\hat{i} - 2\hat{j} + 10\hat{k})$

$$= -2 - 6 + 20 = 12$$

(c) Here $\boldsymbol{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\boldsymbol{b} = -2\hat{i} + 2\hat{j} + 4\hat{k}$

(i) $\therefore$ $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{9 + 4 + 1} = \sqrt{14}$

$|\boldsymbol{b}| = \sqrt{\boldsymbol{b} \cdot \boldsymbol{b}} = \sqrt{4 + 4 + 16} = \sqrt{24}$

If α is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\cos \alpha = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{-6 - 4 + 4}{\sqrt{14}\,\sqrt{24}} = -\frac{3}{\sqrt{84}} \Rightarrow \alpha = \cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)$$

(ii) Here $\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} = (3\hat{i} - 2\hat{j} + \hat{k}) + \frac{1}{2}(-2\hat{i} + 2\hat{j} + 4\hat{k})$

$$= 2\hat{i} - \hat{j} + 3\hat{k}$$

Now projection of $\left(\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ on $\boldsymbol{a}$

$$= \frac{\left(\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) \cdot \boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= \frac{(2\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{14}}$$

$$= \frac{(6 + 2 + 3)}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

(iii) Vector $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Here $\boldsymbol{a} \cdot \boldsymbol{c} = (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (-\hat{i} - 4\hat{j} + 2\hat{k}) = -3 + 8 + 2 = 7 \neq 0$

$\therefore$ $\boldsymbol{a}$ is not perpendicular to $\boldsymbol{c}$.

Also $\boldsymbol{a} \cdot \boldsymbol{d} = (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (-3\hat{i} + \hat{k}) = -9 + 0 + 1 = -8 \neq 0$

$\therefore$ $\boldsymbol{a}$ is not perpendicular to $\boldsymbol{d}$.

Now, $\boldsymbol{a} \cdot \boldsymbol{c} = (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} + 2\hat{j} - 2\hat{k}) = 6 - 4 - 2 = 0$

$\therefore$ $\boldsymbol{a}$ in not perpendicular to $\boldsymbol{c}$.

SAQ 5

(a) Torque $= \boldsymbol{r} \times \boldsymbol{F}$

$$= (7\hat{i} + 3\hat{j} + \hat{k}) \times (-3\hat{i} + \hat{j} + 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\,(15-1) + \hat{j}\,(-3-35) + \hat{k}\,(7+9)$$

$$= 14\hat{i} - 38\hat{j} + 16\hat{k}$$

(b) **L. H. S.** $= (\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

$$= \boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a} - \boldsymbol{b} \times \boldsymbol{b}$$

$$= \boldsymbol{0} + \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{0}$$

$$= 2\,(\boldsymbol{a} \times \boldsymbol{b})$$

Interpretation

Let the diagonals AC and BC of the parallelogram $ABCD$ intersect at O.

Figure

Let $\boldsymbol{AO} = \boldsymbol{a}$ and $\boldsymbol{OD} = \boldsymbol{b}$

$\therefore$ $\boldsymbol{OB} = -\,\boldsymbol{b}$

Now $\boldsymbol{AB} = \boldsymbol{AO} + \boldsymbol{OB} = \boldsymbol{a} - \boldsymbol{b}$

and $\boldsymbol{AD} = \boldsymbol{AO} + \boldsymbol{OD} = \boldsymbol{a} + \boldsymbol{b}$

Now $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{AB} \times \boldsymbol{AD}$ = area of parallelogram $ABCD$.

Again $\boldsymbol{a} \times \boldsymbol{b}$ = area of parallelogram whose sides are $\boldsymbol{a}$ and $\boldsymbol{b}$.

= area of parallelogram with sides as semi-diagonals of parallelogram $ABCD$.

Hence the area of the parallelogram $ABCD$ is equal to twice the area of the parallelogram whose adjacent sides are semi-diagonals of the first parallelogram.

(c) (i) If the given vectors are perpendicular, then

$$(3\hat{i} + 2\hat{j} + 9\hat{k}) \cdot (\hat{i} + a\,\hat{j} + 3\hat{k}) = 0$$

$\Rightarrow$ $3 + 2a + 27 = 0$

$\Rightarrow$ $2a = -30$

$\Rightarrow$ $a = -15$

(ii) If the given vectors are parallel, then

$$(3\hat{i} + 2\hat{j} + 9\hat{k}) \times (\hat{i} + a\,\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow \quad = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & a & 3 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \quad = \hat{i}\,(6-9a) - \hat{j}\,(9-9) + \hat{k}\,(3a-2) = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

$$\Rightarrow \quad = (6-9a) = 0 \text{ and } 3a - 2 = 0$$

$$\Rightarrow \quad a = \frac{2}{3}$$

(d) Let A be the point $(9\hat{i} - \hat{j} + 2\hat{k})$ and B the point $(3\hat{i} + 2\hat{j} + \hat{k})$

Then $\boldsymbol{BA} = \boldsymbol{r} = (9\hat{i} - \hat{j} + 2\hat{k}) - (3\hat{i} + 2\hat{j} + \hat{k}) = 6\hat{i} - 3\hat{j} + \hat{k}$

Also the force $= \boldsymbol{F} = 5\hat{i} + \hat{k}$

$\therefore$ The required moment $= \boldsymbol{r} \times \boldsymbol{F}$

$$= (6\hat{i} - 3\hat{j} + \hat{k}) \times (5\hat{i} + \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & -3 & 1 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= -3\hat{i} - \hat{j} + 15\hat{k}$$

SAQ 6

(a) **L. H. S.** $= (\boldsymbol{a} + \boldsymbol{b}) \,.\, [(\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{a})]$

$= (\boldsymbol{a} + \boldsymbol{b}) \,.\, [\boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}]$

$= (\boldsymbol{a} + \boldsymbol{b}) \,.\, [\boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a}]$ $\quad (\because \boldsymbol{c} \times \boldsymbol{c} = 0)$

$= \boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{a}) + \boldsymbol{a} \,.\, (\boldsymbol{c} \times \boldsymbol{a})$

$+ \boldsymbol{b} \,.\, (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \,.\, (\boldsymbol{b} \times \boldsymbol{a}) + \boldsymbol{b} \,.\, (\boldsymbol{c} \times \boldsymbol{a})$

$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] + [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}]$

($\because$ a scalar triple products is zero when the vectors are equal).

$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] + [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ $\quad (\because [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}] = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}])$

$= 2\,[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

= **R. H. S.**

(b) Let $\boldsymbol{a} = -12\hat{i} + \alpha\hat{k}$

$\boldsymbol{b} = 3\hat{i} - \hat{k}$

$\boldsymbol{c} = 2\hat{i} + \hat{j} - 15\hat{k}$

$\therefore$ Volume $= \boldsymbol{a} \,.\, (\boldsymbol{b} \times \boldsymbol{c})$

$$= \begin{vmatrix} -12 & 0 & \alpha \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}$$

$$= -12\,(-45+1) + \alpha\,(0-6)$$

$$= 528 - 6\alpha$$

But given volume = 546

$\therefore \qquad 528 - 6\alpha = 546$

$\Rightarrow \qquad -6\alpha = 546 - 528 = +18$

$\Rightarrow \qquad \alpha = -3$

(c) Here $\quad \boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}) - (3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

$$= -\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$$

$$\boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA} = (-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) - (3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

$$= -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$$

$$\boldsymbol{AD} = \boldsymbol{OD} - \boldsymbol{OA} = (4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}}) - (3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

$$= \hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} + (\lambda+1)\hat{\boldsymbol{k}}$$

Since *A*, *B*, *C*, *D* are coplanar, therefore $\boldsymbol{AB}, \boldsymbol{AC}$ and $\boldsymbol{AD}$ lie in the same plane.

Thus $\qquad \boldsymbol{AB} \,.\, (\boldsymbol{AC} \times \boldsymbol{AD}) = 0$

$$\Rightarrow \qquad \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

$\Rightarrow \qquad = -1\,(3\lambda + 3 - 21) - 5\,(-4\lambda - 4 - 3) - 3\,(-28 - 3) = 0$

$\Rightarrow \qquad = -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Rightarrow \qquad = 17\lambda + 146 = 0$

$\Rightarrow \qquad \lambda = -\dfrac{146}{17}$

SAQ 7

(a) Here $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}},\ \boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}},\ \boldsymbol{c} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

(i) Now $\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = \hat{\boldsymbol{i}}\,(-1+3) - \hat{\boldsymbol{j}}\,(-2+1) + \hat{\boldsymbol{k}}\,(6-1)$

$$= 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$$

$\therefore \qquad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -2 & -3 \\ 2 & 1 & 5 \end{vmatrix} = \hat{\boldsymbol{i}}\,(-10+3) - \hat{\boldsymbol{j}}\,(5+6) + \hat{\boldsymbol{k}}\,(1+4)$

$$= -7\hat{i} - 11\hat{j} + 5\hat{k}$$

(ii) Also $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \hat{i}\ (2+3) - \hat{j}\ (-1+6) + \hat{k}\ (1+4)$

$$= 5\hat{i} - 5\hat{j} + 5\hat{k}$$

$$\therefore \quad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -5 & 5 \\ 1 & 3 & -1 \end{vmatrix} = \hat{i}\ (5-15) - \hat{j}\ (-5-5) + \hat{k}\ (15+5)$$

$$= -10\hat{i} + 10\hat{j} + 20\hat{k}$$

(b) **L. H. S.** $= \hat{i} \times (\boldsymbol{a} \times \hat{i}) + \hat{j} \times (\boldsymbol{a} \times \hat{j}) + \hat{k}\ (\boldsymbol{a} \times \hat{k})$

$$= [(\hat{i}\ .\ \hat{i})\ \boldsymbol{a} - (\hat{i}\ .\ \boldsymbol{a})\ \hat{i}] + [(\hat{j}\ .\ \hat{j})\ \boldsymbol{a} - (\hat{j}\ .\ \boldsymbol{a})\ \hat{j}] + [(\hat{k}\ .\ \hat{k})\ \boldsymbol{a} - (\hat{k}\ .\ \boldsymbol{a})\ \hat{k}]$$

$$= \boldsymbol{a} - (\hat{i}\ .\ \boldsymbol{a})\ \hat{i} + \boldsymbol{a} - (\hat{j}\ .\ \boldsymbol{a})\ \hat{j} + \boldsymbol{a} - (\hat{k}\ .\ \boldsymbol{a})\ \hat{k}$$

$$= 3\boldsymbol{a} - [(\hat{i}\ .\ \boldsymbol{a})\ \hat{i} + (\hat{j}\ .\ \boldsymbol{a})\ \hat{j} + (\hat{k}\ .\ \boldsymbol{a})\ \hat{k}]$$

Let $\boldsymbol{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\therefore\ \hat{i}\ .\ \boldsymbol{a} = a_1,\ \hat{j}\ .\ \boldsymbol{a} = a_2,\ \hat{k}\ .\ \boldsymbol{a} = a_3$

$\therefore$ **L. H. S**

$$= 3\boldsymbol{a} - (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$= 3\boldsymbol{a} - \boldsymbol{a}$$

$$= 2\boldsymbol{a}$$

= **R. H. S.**

(c) We have

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \frac{1}{2}\boldsymbol{b}$$

$$\Rightarrow \quad (\boldsymbol{a}\ .\ \boldsymbol{c})\ \boldsymbol{b} - (\boldsymbol{a}\ .\ \boldsymbol{b})\ \boldsymbol{c} = \frac{1}{2}\boldsymbol{b}$$

$$\Rightarrow \quad \left(\boldsymbol{a}\ .\ \boldsymbol{c} - \frac{1}{2}\right)\boldsymbol{b} - (\boldsymbol{a}\ .\ \boldsymbol{b})\ \boldsymbol{c} = 0$$

$$\Rightarrow \quad \boldsymbol{a}\ .\ \boldsymbol{c} - \frac{1}{2} = 0 \quad \text{and} \quad \boldsymbol{a}\ .\ \boldsymbol{b} = 0$$

$$\Rightarrow \quad \boldsymbol{a}\ .\ \boldsymbol{c} = \frac{1}{2} \quad \text{and} \quad \boldsymbol{a}\ .\ \boldsymbol{b} = 0$$

(Since ***b*** and ***c*** are non-parallel, hence coefficients of ***b*** and ***c*** must vanish separately.)

If θ_1 and θ_2 are the angles between ***a*** and ***c*** and ***a*** and ***b*** respectively, then

$$\cos\theta_1 = \frac{1}{2} \Rightarrow \theta_1 = \frac{\pi}{3} \quad (\because\ \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \text{ are unit vectors})$$

$$\cos\theta_2 = 0 \Rightarrow \theta_2 = \frac{\pi}{2}$$

(d) Given $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$,

we have to prove that $\boldsymbol{a}$ and $\boldsymbol{c}$ are collinear

Now $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

$\Rightarrow$ $(\boldsymbol{a} \,.\, \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{b} \,.\, \boldsymbol{c})\, \boldsymbol{a} = (\boldsymbol{a} \,.\, \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\, \boldsymbol{c}$

$\Rightarrow$ $(\boldsymbol{a} \,.\, \boldsymbol{b})\, \boldsymbol{c} - (\boldsymbol{b} \,.\, \boldsymbol{c})\, \boldsymbol{a} = 0$

$\Rightarrow$ $(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b} = 0$

$\Rightarrow$ Either $\boldsymbol{a} \times \boldsymbol{c} = 0$ or $\boldsymbol{b} = 0$

But $\boldsymbol{b} \neq 0$

$\therefore$ $\boldsymbol{a} \times \boldsymbol{c} = 0$

$\Rightarrow$ $\boldsymbol{a}$ and $\boldsymbol{c}$ are collinear

($\therefore$ Cross product of collinear vectors is zero.)

Conversely, given $\boldsymbol{a}$ and $\boldsymbol{c}$ are collinear, we have to prove that

$$(\boldsymbol{a} \times \boldsymbol{b})\, \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

Since $\boldsymbol{a}$ and $\boldsymbol{c}$ are collinear,

let $\boldsymbol{c} = t\, \boldsymbol{a}$, where t is some scalar

Now $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{t}\, \boldsymbol{a}$

$= \boldsymbol{t}\, [(\boldsymbol{a} \,.\, \boldsymbol{a})\, \boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\, \boldsymbol{a}]$ . . .

(1)

and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{t}\, \boldsymbol{a})$

$= \boldsymbol{t}\, [(\boldsymbol{a} \,.\, \boldsymbol{a})\, \boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\, \boldsymbol{a}]$ . . .

(2)

From Eqs. (1) and (2)

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$$

SAQ 8

Let $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{r}$

$\therefore$ $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{r} \times (\boldsymbol{a} \times \boldsymbol{c})$

$= (\boldsymbol{r} \,.\, \boldsymbol{c})\, \boldsymbol{a} - (\boldsymbol{r} \,.\, \boldsymbol{a})\, \boldsymbol{c}$

$= [(\boldsymbol{a} \times \boldsymbol{b}) \,.\, \boldsymbol{c}]\, \boldsymbol{a} - [(\boldsymbol{a} \times \boldsymbol{b}) \,.\, \boldsymbol{a}]\, \boldsymbol{c}$

$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, \boldsymbol{a} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}]\, \boldsymbol{c}$

$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, \boldsymbol{a} - (\because [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}] = 0)$

$\therefore$ $[(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})] \,.\, \boldsymbol{d} = \{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, \boldsymbol{a}\} \,.\, \boldsymbol{d}$

$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\, (\boldsymbol{a} \,.\, \boldsymbol{d})$

$= (\boldsymbol{a} \,.\, \boldsymbol{d})\, [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

Hence the result.