UNIT 1 DIFFERENTIAL CALCULUS

Structure

1.1 INTRODUCTION

Calculus was related to meet the pressing mathematical needs of the 17th century science and is still the fundamental mathematics for solving problems in Science and Technology. In this unit, we will introduce the concept of functions and discuss different types of functions, algebra of functions and their limit and continuity. We will also introduce the concept of derivative as the instantaneous rate of change, and shall discuss methods of differentiation.

We propose to study the applications of derivative to geometry. We will define increasing and decreasing functions and shall discuss how derivatives can be used to determine the points where a differentiable function has maxima and minima. We shall also discuss some fundamental theorems of differential calculus.

Objectives

After studying this unit, you should be able to

- recall the basic properties of real numbers,
- define a function and examine whether a given function is one-one/onto,

- identify whether a given function has an inverse or not,
- determine whether a given function is even or odd,
- evaluate the limit of a function at a point,
- identify points of continuity and discontinuity of a function,
- obtain the derivative of a function at a point,
- find the tangent and normal to a given function at given points,
- determine whether a function is decreasing or increasing,
- solve some problems when it is required to minimize or maximize a function, and
- identify whether the derivative of a function can vanish once within an interval.

1.2 REAL NUMBER SYSTEM

The set of all real numbers is denoted by R. The real number system is the foundation on which a large part of mathematics including calculus rests. You are familiar with the operations of addition, subtraction, multiplication and division of real numbers and with inequalities. We shall recall some of their properties :

P1 - R is closed under addition.

P2 - Addition is associative and commutative.

P3 - R is closed under multiplication.

P4 - Multiplication is commutative and associative.

P5 - Multiplication is distributive over addition.

P6 - For any two real numbers a and b, either $a > b$ *or* $a < b$ *or* $a = b$.

You are also familiar with the following subsets of *R*.

(i) *N*, the set of natural numbers.

(ii) *Z*, the set of integers.

(iii) *Q*, the set of rational numbers.

Definition 1

If x is a real number, its absolute value, denoted by $|x|$ *is defined as*

$$|x| = x \text{ if } x \geq 0$$
$$= -x \text{ if } x < 0$$

For example

$$|5| = 5$$
$$|-5| = 5$$

The following theorems (without proof) gives some of the important properties of $|x|$.

Theorem 1

If x and y are real numbers, then

(i) $|x| = \max\{-x, x\}$

(ii) $|x| = |-x|$

(iii) $|x|^2 = x^2 = |-x|^2$

(iv) $|x + y| \le |x| + |y|$

(v) $|x - y| \ge ||x| - |y||$

Definition 2

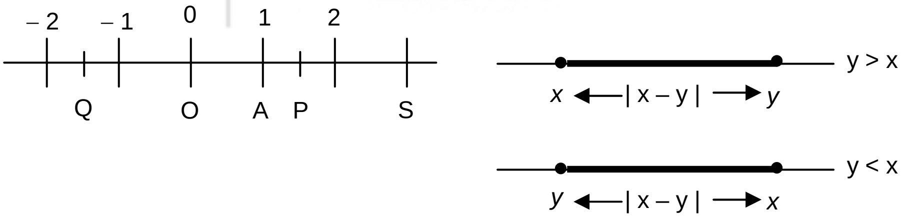
Let S be a non-empty subset of R. An element $u \in R$ is said to be upper bound of S if $u \ge x$ for all $x \in S$. If S has an upper bound, we say S is bounded above. On similar lines we can define a lower bound for a non-empty set S to be a real number v such that $v \le x$ for all $x \in S$ and we say that the set S is bounded below.

1.2.1 Intervals on the Real Line

Before we define an interval, let us see what is meant by a number line. The real numbers in set *R* can be put into one-to-one correspondence with the points on a straight line *L*. In other words, we shall associate a unique point on *L* to each real number and vice-versa.

Consider a straight line *L* (Figure 1.1(a)). Mark a point *O* on it. The point *O* divides the straight line into two parts. We shall use the part of the left of *O* for representing negative real numbers and the part of the right of *O* for representing positive real numbers. We choose a point *A* on *L* which is to the right of *O*. we shall represent the number 0 by *O* and 1 by *A*. *OA* can now serve as a unit. To each positive real number, *x* we can associate exactly one point *P* lying to the right of *O* on *L*, so that
$OP = x$ units (= x). A negative real number *y* will be represented by a point *Q* lying to the left of *O* on the straight line *L* so that $OQ = y$ units (= $-y$, since *y* is negative). We thus find that to each real number we can associate a point on the line. Also, each point *S* on the line represents a unique real number *z*, such that $z = OS$. Further, *z* is positive if *S* is to the right of *O*, and is negative if *S* is to the left of *O*.

This representation of real numbers by points on a straight line is often very useful. Because of this one-to-one correspondence between real numbers and the points of a straight line, we often call a real number "a point of *R*". Similarly, *L* is called a "number line". Note that the absolute value or the modulus of any number *x* is nothing but its distance from the point *O* on the number line. In the same way, $x - y$ denotes the distance between the two numbers *x* and *y* (Figure 1.1(b)).



(a) **(b) Distance between *x* and *y* is $|x - y|$**

Figure 1.1

Now let us consider the set of real numbers which lie between two given real numbers *a* and *b*. Actually, there will be four different sets satisfying this loose condition. These are :

(i) $(a, b) = \{x \mid a < x < b\}$

(ii) $[a, b] = \{x \mid a \le x \le b\}$

(iii) $(a, b] = \{x \mid a < x \le b\}$

(iv) $[a, b) = \{x \mid a \le x < b\}$

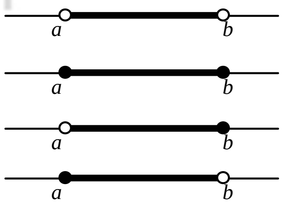


Figure 1.1 (c)

The representation of each of these sets is given alongside (Figure 1.1(c)). Each of these sets is called an interval, and *a* and *b* are called the end points of the interval. The interval (*a*, *b*), in which the end points are not included, is called an **open interval**. Note that in this case we have drawn a hollow circle around a and b to indicate that they are not included in the graph. The set $[a, b]$ contains both its end points and is called a **closed interval**. In the representation of this closed interval, we have put thick black dots at a and b to indicate that they are included in the set.

The sets $(a, b]$ and $[a, b)$ are called **half-open** (or **half-closed**) intervals or **semi-open** (or **semi-closed**) intervals, as they contain only one end point. This fact is also indicated in their geometrical representation.

Each of these intervals is bounded above by b and bounded below by a.

Can we represent the set

$$I = \{x : |x - a| < \delta\}$$

on the number line? Yes, we can. We know that $x - a$ can be thought of as the distance between x and a. This means I is set of all numbers x, whose distance from a is less than δ. Thus, I is the open interval $(a - \delta, a + \delta)$. Similarly,

$$I_1 = \{x : |x - a| \le \delta\}$$

is the closed interval $[a - \delta, a + \delta]$. Sometimes, we also come across sets like

$$I_2 = \{x : |0 < |x - a| < \delta\}$$

This means if $x \in I_2$, then the distance between x and a is less than δ, but is not zero. We can also say that the distance between x and a is less than, δ, but $x \ne a$. Thus,

$$I_2 = (a - \delta, a + \delta) - \{a\} = (a - \delta, a) \cup (a, a + \delta)$$

Apart from the four types of intervals listed above, there are a few more types. These are :

$(a, \infty) = \{x \mid a < x\}$		(open right ray)
$[a, \infty) = \{x \mid a \le x\}$		(closed right ray)
$(-\infty, b) = \{x \mid x < b\}$		(open left ray)
$(-\infty, b] = \{x \mid x \le b\}$		(closed left ray)
$(-\infty, \infty) = R$		(open interval)

As you can see easily, none of these sets is bounded. For instance, (a, ∞) is bounded below, but is not bounded above, $(-\infty, b)$ is bounded above, but is not bounded below. Note that ∞ does not denote a real number; it merely indicates that an interval extends without limit.

We note further that if S is any interval (bounded and unbounded) and if c and d are two elements of S then all numbers lying between c and d are also elements of S.

SAQ 1

(a) Prove the following using results of Theorem.

(i) $x = 0$, if $|x| = 0$

(ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \ne 0$

(iii) $|x - y| \le |x| + |y|$

(b) State whether the following are true or false.

(i) $0 \in [1, \infty]$ True/False

(ii) $-1 \in (-\infty, 2)$ True/False

(iii) $1 \in [1, 2]$ True/False

(iv) $5 \in (5, \infty)$ True/False

1.3 FUNCTIONS

Now let us move over to functions. Here we shall present some basic facts about functions which will help you refresh your knowledge. We shall look at various examples of functions and shall also define inverse functions. Let us start with the definition.

1.3.1 Definition and Examples

Definition 3

If X and Y are two non-empty sets, a function f from X to Y is a rule or a correspondence which connects every member of X to a unique member of Y. We write $f : X \to Y$ (reads as "f is a function from X to Y" or "f is a function of X into Y"). X is called the ***domain*** *and Y is called the* ***co-domain*** *of f. We shall denote by f (x) that unique element of Y, which corresponds to $x \in X$.*

The following examples will help you in understanding this definition better.

Example 1.1

Consider $f : N \to R$ defined by $f(x) = -x$. Is "f" a function?

Solution

"f" is a function since the rule $f(x) = -x$ associates a unique member $(-x)$ of R to every member x of N. The domain here is N and the co-domain is R.

Example 1.2

Consider $f : N \to Z$, defined by the rule $f(x) = \frac{x}{2}$. Is "f" a function?

Solution

"f" is not a function from N to Z as odd natural numbers like 1, 3, 5 . . . from N cannot be associated with any member of Z.

Example 1.3

Consider $f : N \to N$, under the rule $f(x)$ = a prime factor of x. Is "f" a function?

Solution

Here, since $6 = 2 \times 3$, $f(6)$ has two values : $f(6) = 2$ and $f(6) = 3$. This rule does not associate a unique member in the co-domain with a member in the domain and, hence, f, as defined, is not a function of N into N.

Thus, you see, to describe a function completely we have to specify the following three things :

(i) the domain,

(ii) the co-domain, and

(iii) the rule which assigns to each element x in the domain, a single fully determined element in the co-domain.

Given a function $f : X \to Y$, $f(x) \in Y$ is called the **image** of $x \in X$ under f or the f-**image** of x. The set of f-images of all number of X, i.e. $\{f(x) : x \in X\}$ is called the **range** of f and is denoted by $f(X)$. It is easy to see that $f(X) \subset Y$.

Remark

(i) Throughout this unit we shall consider functions whose domain and co-domain are both subsets of R. Such functions are often called real functions or real value functions.

(ii) The variable x used in describing a function is often called a dummy variable because it can be replaced by any other letter. Thus, for example, the rule $f(x) = -x$, $x \in N$ can as well be written in the form $f(t) = -t$, $t \in N$, or as $f(u) = -u$, $u \in N$. The variable x (or t or u) is also called an **independent variable** and $f(x)$, which is dependent on this independent variable, is called a **dependent variable**.

Graph of a Function

A convenient and useful method for studying a function is to study it through its graph. To draw the graph of function $f : X \to Y$, we choose a system of co-ordinate axes in the plane. For each $x \in X$, the ordered pair $(x, f(x))$ determines a point in the plane (Figure 1.2). The set of all the points obtained by considering all possible values of x, that is, $(x, f(x)) : x \in X)$ is the graph of f.

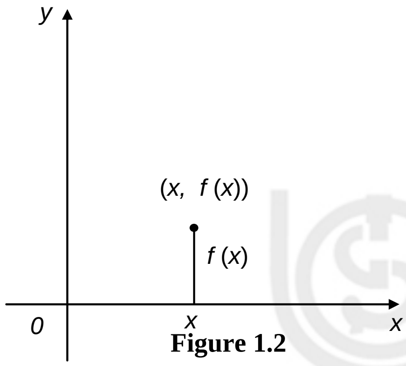


Figure 1.2

The role that the graph of a function plays in the study of the function will become clear as we proceed further. In the meantime let us consider some more examples of functions and their graphs.

A Constant Function

A simple example of a function is a constant function. A constant function sends all the elements of the domain to just one element of the co-domain.

For example, let $f : R \to R$ be defined by $f(x) = 1$.

Alternatively, we may write

$$f : x \to 1 \text{ for all } x \in R$$

The graph of f is as shown in Figure 1.3.

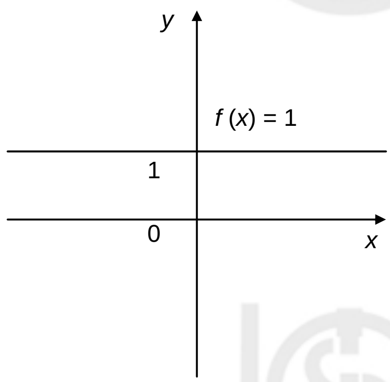


Figure 1.3

It is the line $y = 1$.

In general, the graph of a constant function $y : x \to c$ is a straight line which is parallel to the x-axis at a distance of c units from it.

The Identity Function

Another simple but important example of a function is a function which sends every element of the domain to itself.

Let X by any non-empty set, and f be the function of X defined by setting $f(x) = x$, for all $x \in X$.

This function is known as the *identity function* on X and is denoted by I_X.

The graph of I_R, the identity function of R, is shown in Figure 1.4. It is the line $y = x$.

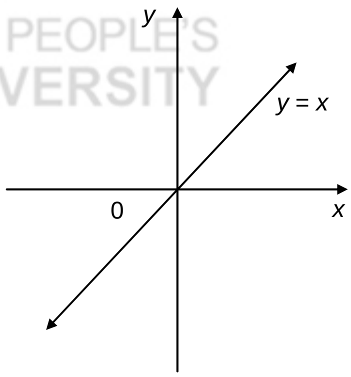


Figure 1.4

Absolute Value Function

Another interesting function is the absolute value function (or modulus function) which can be defined by using the concept of the absolute value of a real number as

$$f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

The graph of this function is shown in Figure 1.5. It consists of two rays, both starting at the origin and making angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively, with the positive direction of the x-axis.

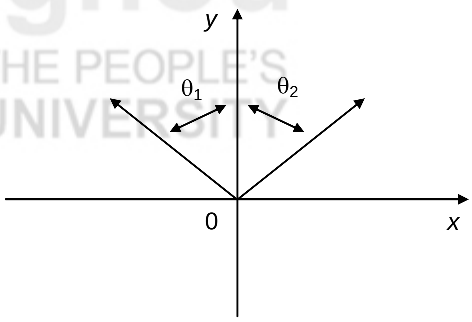


Figure 1.5

The Exponential Function

If a is a positive real number other than 1, we define a function $f : R \to R$ by $f(x) = a^x$ where $a > 0$, $a \neq 1$.

This function is known as the general exponential function. A special case of this function, where $a = e$, is often found useful. Figure 1.6 shows the graph of the function $f : R \to R$ such that $f(x) = e^x$. This function is called the natural exponential function. Its range is the set R^+ of positive numbers.

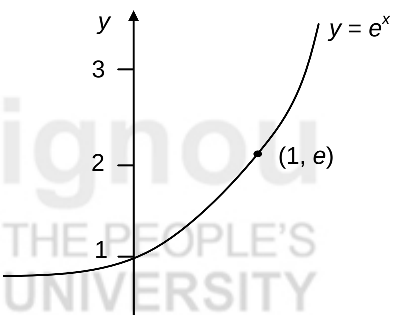


Figure 1.6

The Natural Logarithmic Function

This function $f : R^+ \rightarrow R$ is defined on the set R^+ of all real numbers such that $f(x) = \ln(x)$. The range of this function is R. Its graph is shown in Figure 1.7.

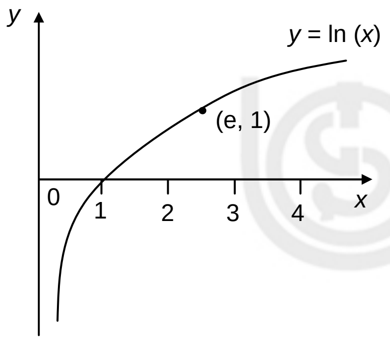


Figure 1.7

The Greatest Integer Function

Take a real number x. Either it is an integer, say n (so that $x = n$), or it is not integer. If it is not an integer, we can find an integer n, such that $n < x < n + 1$. Therefore, for each real number x we can find an integer n, such that $n < x < n + 1$. Further, for a given real number x, we can find only one such integer n. We say that n is the greatest integer not exceeding x and denote it by $[x]$. For example, $[3] = 3$ and $[3.5] = 3$, $[-3.5] = -4$.

Other Functions

The following are some important classes of functions.

Polynomial Functions	$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ where $a_0, a_1, \ldots, a_n$ are given real numbers (constants) and n is a positive integer. We say n is the degree of the polynomial $f(x)$ and assume $a_0 \neq 0$.
Rational Functions	$f(x) = \dfrac{g(x)}{k(x)}$, where $g(x)$ and $k(x)$ are polynomial functions of degree n and m respectively. This is defined for all real x for which $k(x) \neq 0$.
Trigonometrical or Circular Functions	$f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \tan x$, $f(x) = \cot x$, $f(x)$ sec x, and $f(x) = \text{cosec } x$.
Hyperbolic Functions	$f(x) = \cosh x = \dfrac{(e^x + e^{-x})}{2}$, $f(x) = \sinh x = \dfrac{(e^x - e^{-x})}{2}$.

We shall study these in detail in this unit.

1.3.2 Inverse Functions

In this sub-section we shall see what is meant by the inverse of a function. But before talking about the inverse, let us look at some special categories of functions. These special types of functions will then lead us to the definition of inverse of a function.

One-one and Onto Functions

Consider the function $h : x \to x^2$, defined on the set R. Here $h(2) = h(-2) = 4$; i.e. 2 and – 2 are distinct members of the domain R, but their h-images are the same (Can you find some more members whose h-images are equal?). In general, this may be expressed by saying : find x, y such that $x \neq y$ but $h(x) = h(y)$.

Now consider another function $g : x \to 2x + 3$. Here you will be able to see that if x_1 and x_2 are two distinct real numbers, then $g(x_1)$ and $g(x_2)$ are also distinct.

For,

$$x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow 2x_1 + 3 \neq 2x_2 + 3 \Rightarrow g(x_1) \neq g(x_2).$$

We have considered two functions here. While one of them, namely g, sends distinct members of the domain to distinct members of the co-domain, the other, namely h, does not always do so. We give a special name to functions like g above.

Definition 4

*A function $f : X \to Y$ is said to be a **one-one function** (a 1–1 function or an injective function) if images of distinct members of X are distinct members of Y.*

Thus, the function g above is one-one, whereas h is not one-one.

Remark

The condition that the images of distinct members of X are distinct members of Y in the above definition, can be replaced by either of the following equivalent conditions :

(i) For every pair of members x, y of X, $x \neq y \Rightarrow f(x) \neq f(y)$.

(ii) For every pair of members x, y of X, $f(x) = f(y) \Rightarrow x = y$.

We have observed earlier that for a function $f : X \to Y$, $f(X) \subseteq Y$.

This opens two possibilities :

(i) $f(X) = Y$, or

(ii) $f(X) \subset Y$, that is $f(X)$ is a proper subset of Y.

The function $h : x \to x^2$, for all $x \in R$ falls in the second category. Since the square of any real number is always non-negative, $h(R) = R^+ \cup (0)$ which is the set of non-negative real numbers. Thus $h(R) \subset R$.

On the other hand, the function $g : x \to 2x + 3$ belongs to the first category. Given any $y \in R$ (co-domain), if we take $x = \left(\frac{1}{2}\right) y - \frac{3}{2}$, we find that $g(x) = y$. This shows that every member of the co-domain is a g-image of some member of the domain and, thus, is in the range g. From this, we get that $g(R) = R$. The following definition characterises this property of the function.

Definition 5

*A function $f : X \to Y$ is said to be an **onto** function (or a **surjective** function) if every member of Y is the image of some members of X.*

Thus h is not an onto function, whereas g is an onto function. Functions which are both one-one and onto (or **bijective**) are of special importance in mathematics. Let us see what makes them special.

Consider a function $f : X \to Y$ which is both one-one and onto. Since f is an onto function, each $y \in Y$ is the image of some $x \in X$. Also, since f is one-one, y cannot be the image of two distinct members of X. Thus, we find that to each $y \in Y$ there corresponds a unique $x \in X$ such that $f(x) = y$. Consequently, f sets up one-to-one correspondence between the members of X and Y. It is the one-to-one correspondence between members of X and Y which makes a one-one and onto function so special, as we shall soon see.

Consider the function $f : N \to E$ defined by $f(x) = 2x$, where E is the set of even natural numbers. We can see that f is one-one as well as onto. In fact, to each $y \in E$ there exists $\frac{y}{2} \in N$ such that $f\left(\frac{y}{2}\right) = y$. The correspondence $y \to \frac{y}{2}$ defines a function, say g, from E to N such that $g(y) = \frac{y}{2}$.

The function g so defined is called as **inverse** of f. Since, to each $y \in E$ there corresponds a unique $x \in N$ such that $f(x) = y$ only one such function g can be defined corresponding to a given function f. For this reason, g is called the inverse of f.

As you will notice, the function g is also one-one and onto and therefore it will also have an inverse. You must have already guessed that the inverse of g is the function f.

From this discussion, we have the following :

If f is one-one and onto function from X to Y, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x \Leftrightarrow y = f(x)$. The function g so defined is called the inverse of f. Further, if g is the inverse of f, then f is the inverse of g, and the two functions f and g are said to be the inverse of each other. The inverse of a function f is usually denoted by f^{-1}.

To find the inverse of a given function f, we proceed as follows :

Solve the equation $f(x) = y$ for x. The resulting expression for x (in terms of y) defines the inverse function.

Thus, if $f(x) = \frac{x^5}{5} + 2$,

we solve $\frac{x^5}{5} + 2 = y$ for x

This gives us $x = (5(y-2))^{\frac{1}{5}}$.

Hence f^{-1} is the function defined by $f^{-1}(y) = (5(y-2))^{\frac{1}{5}}$.

1.3.3 Graphs of Inverse Functions

There is an interesting relation between the graphs of a pair of inverse functions because of which, if the graph of one of them is known, the graph of the other can be obtained easily.

Let $f : X \to Y$ be a one-one and onto function, and let $g : Y \to X$ be the inverse of f. A point (p, q) lies on the graph of $f \Leftrightarrow q = f(p) \Leftrightarrow p = g(q)$ (q, p) lies on the graph of g. Now, the point (p, q) and (q, p) are reflections of each other with respect to (w. r. t.) the line $y = x$. Therefore, we can say that the graphs of f and g are reflections of each other w. r. t. the line $y = x$.

Therefore, it follows that if the graph of one of the functions f and g is given, that of the other can be obtained by reflecting it w. r. t. the line $y = x$. As an illustration, the graphs of the functions x^3 and $y = x^{1/3}$ are given in Figure 1.8.

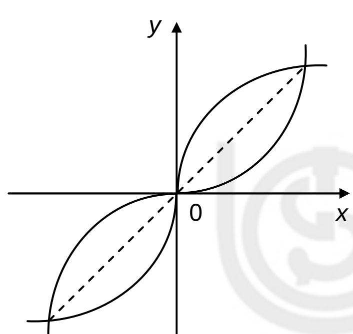


Figure 1.8

Do you agree that these two functions are inverses of each other? If the sheet of paper on which the graph have been drawn is folded along the line $y = x$, the two graphs will exactly coincide.

If a given function is not one-one on its domain, we can choose a subset of the domain on which it is one-one and then define its inverse function. For example, consider the function $f : x \to \sin x$.

Since we know that $\sin (x + 2\pi) = \sin x$, obviously this function is not one-one on R. But if we restrict it to the intervals $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we find that it is one-one.

Thus, if

$$f(x) = \sin (x), \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then we can define

$$f^{-1}(x) = \sin^{-1}(x) = y, \text{ if } \sin y = x.$$

Similarly, we can define $\cos^{-1}$ and $\tan^{-1}$ functions as inverse of cosine and tangent functions if we restrict the domains to $[0, \pi]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ respectively.

SAQ 2

(a) Compare the graphs of $\ln x$ and e^x given in Figures 1.6 and 1.7 and verify that they are inverses of each other.

(b) Which of the following functions are one-one?

  (i) $f : R \to R$ defined by $f(x) = |x|$.

  (ii) $f : R \to R$ defined by $f(x) = 3x - 1$.

  (iii) $f : R \to R$ defined by $f(x) = x$.

  (iv) $f : R \to R$ defined by $f(x) = 1$.

(c) Which of the following functions are onto?

  (i) $f : R \to R$ defined by $f(x) = 3x + 7$.

  (ii) $f : R^+ \to R$ defined by $f(x) = \sqrt{x}$.

  (iii) $f : R \to R$ defined by $f(x) = x^2 + 1$.

  (iv) $f : X \to R$ defined by $f(x) = \frac{1}{x}$ where X stands for the set of non-zero real numbers.

(d) Show that the function $f : X \to X$ such that $f(x) = \frac{x+1}{x-1}$ where X is the set of all real number except 1, is one-one and onto. Find its inverse.

(e) Give one example of each of the following :

(i) a one-one function which is not onto?

(ii) onto function which is not one-one?

(iii) a function which is neither one-one nor onto?

1.3.4 New Functions from Given Functions

In this sub-section, we shall see how we can construct new functions from some given functions. This can be done by operating upon the given functions in a variety of ways. We give a few such ways here.

1.3.5 Operations on Functions

Scalar Multiple of a Function

Consider the function $f : x \to 3x^2 + 1$, for all $x \in R$. The function $g : x \to 2\,(3x^2 + 1)$ for all $x \in R$ is such that $g\,(x) = 2f\,(x)$, for all $x \in R$. We say that $g = 2f$ and that g is a scalar multiple of f by 2. In the above example, there is nothing special about the number 2. We could have taken any real number to construct a new function from f. Also, there is nothing special about the particular function that we have considered. We could as well as have taken any other function. This suggests the following definition : Let f be a function with domain D and let k any real number. The scalar multiple of f by k is a function with domain D. It is denoted by kf and is defined by setting $(kf)\,(x) = kf\,(x)$.

Two special cases of the above definition are important.

(i) Given any function f, if $k = 0$, the function kf turns out to be the zero function. That is, $kf = 0$.

(ii) If $k = -1$, the function kf is called the negative of f and is denoted simply by $-f$.

Absolute Value Function (or Modulus Function) of a Given Function

Let f be a function with domain D. The absolute value of function f denoted by $|\,f\,|$ and read as mod f is defined by setting $(|\,f\,|)\,(x) = |\,f\,(x)\,|$ for all $x \in D$.

Since $|\,f\,(x)\,| = f\,(x)$, if $f\,(x) \geq 0$, thus $|\,f\,|$ have the same graph for those values of x for which $f\,(x) \geq 0$.

Now let us consider those values of x for which $f\,(x) < 0$. Here $|\,f\,(x)\,| = -f\,(x)$. Therefore, the graphs of f and $|\,f\,|$ are reflections of each other w. r. t. the x-axis for those values of x for which $f\,(x) < 0$.

As an example, consider the graph in Figure 1.9(a). The portion of the graph below the x-axis, that is, the portion for which $f\,(x) < 0$ has been shown as dotted.

To draw the graph of $|\,f\,|$ we retain the undotted portion in Figure 1.9(a) as it is, and replace the dotted portion by its reflection w. r. t. the x-axis (Figure 1.9(b)).

(a) **(b)**

Figure 1.9

Sum, Difference, Product and Quotient of Two Functions

If we are given two functions with a common domain, we can form several new functions by applying the four fundamental operations of addition, subtraction, multiplication and division on them.

(i) Define a function δ on D by setting $\delta(x) = f(x) + g(x)$. The functions is called the sum of the functions f and g, and is denoted by $f + g$. Thus, $(f + g)(x) = f(x) + g(x)$.

(ii) Define a function d on D by setting $d(x) = f(x) - g(x)$. The function d is the function obtained by subtracting g from f, and is denoted by $f - g$. Thus, for all $x \in D$, $(f - g)(x) = f(x) - g(x)$.

(iii) Define a function p on D by setting $p(x) = f(x)\, g(x)$. The function p is called the product of the functions f and g, and is denoted by fg. Thus, for all $x \in D$, $(fg)(x) = f(x)\, g(x)$.

(iv) Define a function q on D by setting $q(x) = \dfrac{f(x)}{g(x)}$. The function $q(x)$ exists provided $g(x) \neq 0$ for any $x \in D$. The function q is called the quotient of f by g and is denoted by $\dfrac{f}{g}$. Thus, $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$, $(g(x) \neq 0$ for any $x \in D)$.

Remark

In case $g(x) = 0$ for some $x \in D$, we can consider the set, say D' of all those values of x for which $g(x) \neq 0$ and define $\dfrac{f}{g}$ on D' by setting $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ for all $x \in D'$.

Example 1.4

Consider the functions $f : x \to x^2$ and $g : x \to x^3$. Find the functions which are defined as

(i) $f + g$,

(ii) $f - g$,

(iii) fg,

(iv) $(2f + 3g)$, and

(v) $\dfrac{f}{g}$.

Solution

(i) $(f + g)(x) = x^2 + x^3$

(ii) $(f - g)(x) = x^2 - x^3$

(iii) $(fg)(x) = x^5$

(iv) $(2f + 3g)(x) = 2f(x) + 3g(x)$

$= 2f(x) + 3g(x)$

$= 2x^2 + 3x^3$.

(v) Now, $g(x) = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x = 0$. Therefore, in order to define the function $\frac{f}{g}$, we shall consider only non-zero values of x. If $x \neq 0$, $\frac{f(x)}{g(x)} = \frac{x^2}{x^3} = \frac{1}{x}$.

Therefore, $\frac{f}{g}$ is the function $\frac{f}{g} : x \to \frac{1}{x}$, whenever $x \neq 0$.

1.3.6 Composition of Functions

We shall now describe a method of combining two functions which is somewhat different from the ones studied so far. Uptil now, we have considered functions with the same domain. We shall now consider a pair of functions such that the co-domain of one is the domain of the other.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We define a function $h : X \to Z$ by setting $h(x) = g(f(x))$.

To obtain $h(x)$, we first take the f-image $f(x)$ of an element x of X. Thus $f(x) \in Y$, which is the domain of g. We then take the g image of $f(x)$, that is, $g(f(x))$, which is an element of Z. This scheme has been shown in Figure 1.10.

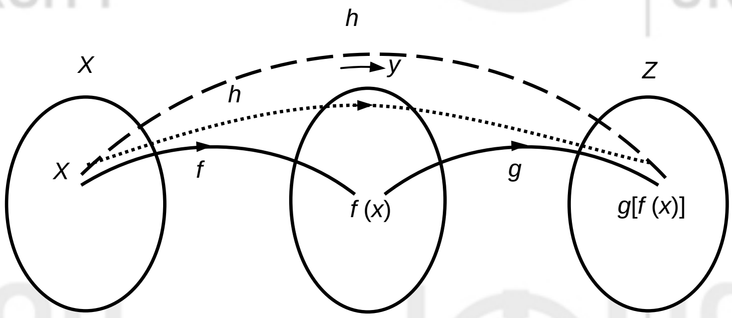


Figure 1.10

The function h, defined above, is called the **composition** of f and g and is written as "*gof*". Note the order. We first find the f-image and then its g-image. Try to distinguish the composite function *gof* from the composite function *fog* which will be defined only when Z is a subset of X.

Example 1.5

For functions $f : x \to x^2, \forall x \in R$ and $g : x \to 8x + 1$, for all $x \in R$, obtain functions "*gof* " and "*fog*".

Solution

"*gof* " is a function from R to itself, defined by

$$(gof)(x) = g(f(x)) = g(x^2) = 8x^2 + 1 \text{ for all } x \in R$$

"*fog* " is a function from R to itself, defined by

$$(fog)(x) = f(g(x)) = f(8x + 1) = (8x + 1)^2.$$

Thus *gof* and *fog* are both defined but are different from each other.

The concept of composite functions is used not only to combine functions, but also to look upon a given function as made up of two simpler functions. For example, consider the function

$$h : x \to \sin(3x + 7).$$

We can think of it as the composition (*gof*) of the functions $f : x \to 3x + 7$, for all $x \in R$ and $g : u \to \sin u$, for all $u \in R$.

Now let us try to find the composite functions "*fog*" and "*gof*" of the functions :

$$f : x \to 2x + 3, \text{ for all } x \in R, \text{ and } g : x \to \left(\frac{x}{2}\right) - \left(\frac{3}{2}\right), \text{ for all } x \in R.$$

Note that *f* and *g* are inverses of each other. Now

$$gof\,(x) = g\,(f\,(x)) = g\,(2x + 3) = \frac{1}{2}\,(2x + 3) - \frac{3}{2} = x.$$

Similarly, $$fog\,(x) = f\,(g\,(x)) = f\left(\frac{x}{2} - \frac{3}{2}\right) = 2\left(\frac{x}{2} - \frac{3}{2}\right) + 3 = x.$$

Thus, we see that $gof\,(x) = x$ and $fog\,(x) = x$ are the identity functions on *R*. What we have observed here is true for any two functions *f* and *g* which are inverses of each other. Thus, if $f : X \to Y$ and $g : Y \to X$ are inverses of each other, then *gof* and *fog* are identity functions. Since the domain of *gof* is *X* and that of *fog* is *Y*, we can write this as :

$$gof = I_x,\ fog = I_y$$

This fact is often used to test whether two given functions are inverses of each other or not.

1.3.7 Types of Functions

In this section, we shall talk about various types of functions, namely, even, odd, increasing and decreasing. In each case, we shall also try to explain the concept through graphs.

Even and Odd Functions

We shall first introduce two important classes of functions : even functions and odd functions.

Consider the function *f* defined on *R* by setting

$$f\,(x) = x^2, \text{ for all } x \in R.$$

You will notice that

$$f\,(-x) = (-x)^2 = x^2 = f\,(x), \text{ for all } x \in R.$$

This is an example of an even function. Let's take a look at the graph (Figure 1.11) of this function. We find that the graph (a parabola) is symmetrical about the *y*-axis. If we fold the paper along the *y*-axis, we shall see that the parts of the graph on both sides of the *y*-axis completely coincide with each other. Such functions are called even functions. Thus, a function *f*, defined on *R* is **even** if, for each $x \in R$, $f\,(-x) = f\,(x)$.

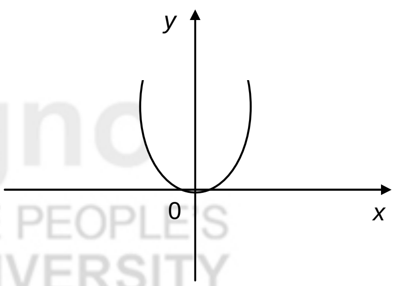


Figure 1.11

The graph of an even function is symmetric with respect to the *y*-axis. We also note that if the graph of a function is symmetric with respect to the *y*-axis, the function must be an even function. Thus, if we are required to draw the graph of an even function, we can use this property to our advantage. We only need to draw that part of the graph which lies to the right of the *y*-axis and then just take its reflection w. r. t. the *y*-axis to obtain the part of the graph which lies on the left of the *y*-axis.

Now let us consider the function *f* defined by setting $f\,(x) = x^3$, for all $x \in R$. We observe that $f\,(-x) = (-x)^3 = -x^3 = -f\,(x)$, for all $x \in R$. If we consider another

function g given by $g(x) = \sin x$, we shall be able to note again that
$g(-x) = \sin(-x) = -\sin x = -g(x)$.

The functions f and g above are similar in one respect : the image of $-x$ is the negative of the image of x. Such functions are called **odd** functions. Thus, a function f defined on R is said to be an **odd** function if $f(-x) = -f(x)$, for all $x \in R$.

If $(x, f(x))$ is a point on the graph of an odd function f, then $(-x, -f(x))$ is also a point on it. This can be expressed by saying that the graph of odd function is symmetric with respect to the origin. In other words, if you turn the graph of an odd function through 180^o about the origin you will find that you get the original graph again. Conversely, if the graph of a function is symmetric with respect to the origin, the function must be an odd function. The above facts are often useful while handling odd functions.

While many of the functions that you will come across in this course will turn out to be either even or odd, there will be many more which will be neither even nor odd. Consider, for example, the function $f : x \rightarrow (x+1)^2$.

Here $\quad f(-x) = (-x+1)^2 = x^2 - 2x + 1$.

Is $\quad f(x) = f(-x)$, for all $x \in R$?

The answer is 'no'. Therefore, f is not an even function.

Further is $f(x) = -f(-x)$, for all $x \in R$?

Again, the answer is 'no'. Therefore f is not an odd function. The same conclusion could have been drawn by considering the graph of f which is given in Figure 1.12.

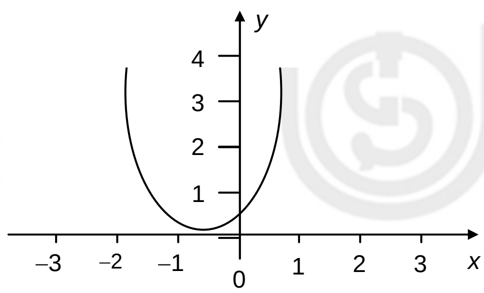


Figure 1.12

You will observe that the graph is symmetric neither with respect to the y-axis, nor with respect to the origin.

Now, there should be no difficulty in solving the exercise below.

SAQ 3

(a) Given below are two examples of even functions, alongwith their graphs. Try to convince yourself, by calculations as well as by looking at the graphs, that both the functions are, indeed, even functions.

(i) The absolute value function on R

$$f : x \rightarrow |x|$$

The graph of f is shown in Figure 1.13.

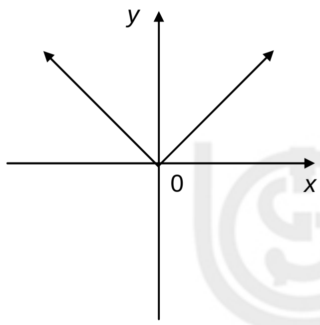


Figure 1.13

(ii) The function g defined on the set of non-zero real number by setting

$g(x) = \frac{1}{x^2}, \ x \neq 0.$ The graph of g is shown in Figure 1.14.

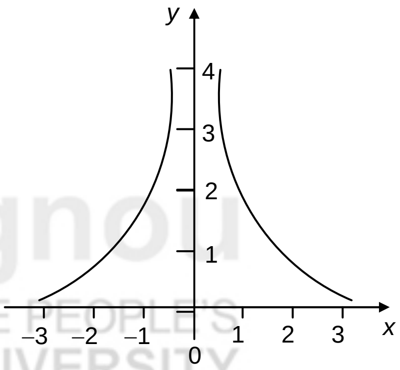


Figure 1.14

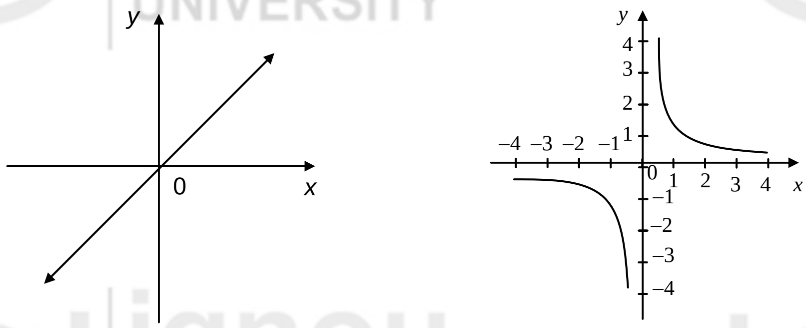
(b) We are giving two functions alongwith their graphs (Figures 1.15(a) and (b)). By calculations as well as by looking at the graphs, prove that each is an odd function.

(i) The identity function on R :

$$f : x \to x$$

(ii) The function g defined on the set of non-zero real numbers by setting

$$g(x) = \frac{1}{x}, \ x \neq 0.$$



(a) **(b)**

Figure 1.15

(c) Which of the following functions are even, which are odd, and which are neither even nor odd?

(i) $x \to x^2 + 1$, for all $x \in R$.

(ii) $x \to x^2 - 1$, for all $x \in R$.

(iii) $x \to \cos x$, for all $x \in R$.

(iv) $x \to |x|$, for all $x \in R$.

(v) $f(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational} \end{cases}.$

1.4 LIMITS

In Section 1.3, we introduced you to the concept of a function and discussed several useful types of functions. In this section, we consider the value "to which $f(x)$ approaches as x gets closer and closer to some number a". The phrase in inverted commas is to be understood intuitively and through practice. We do not give a formal definition here. We call such a value the limit of $f(x)$ and denote it by $\lim_{x \to a} f(x)$.

Sometimes, this value may not exist, as you will see in an example later.

Example 1.6

Consider the function $f : \boldsymbol{R} \to \boldsymbol{R}$ given by $f(x) = x + 4$. We want to find $\lim_{x \to 2} f(x)$.

Solution

Look at Tables 1.1 and 1.2. These give values of $f(x)$ as x gets closer and closer at 2 through values less than 2 and through values greater than 2, respectively.

Table 1.1

x	1	1.5	1.9	1.99	1.999
$f(x)$	5	5.5	5.90	5.99	5.999

Table 1.2

x	3	2.5	2.1	2.01	2.001
$f(x)$	7	6.5	6.1	6.01	6.001

From the above tables, it is clear that as x approaches 2, $f(x)$ approaches 6. In fact, the nearer x is chosen to 2, the closer $f(x)$ will be to 6. Thus, 6 is limit of $x + 4$ as x approaches 2, that is, $\lim_{x \to 2} (x + 4) = 6$.

In the above example, the value of $\lim_{x \to 2} (x + 4)$ coincides with the value $x + 4$ when $x = 2$, that is, $\lim_{x \to 2} f(x) = f(2)$.

Numbers x near 2 fall into two natural categories; those which are < 2, that is, those that lie to the left of 2, and those which are > 2, that is, those which lie to the right of 2.

We write

$$\lim_{x \to 2^-} f(x) = 6$$

to indicate that as x approaches 2 from the left, $f(x)$ approaches 6.

We shall describe this limit as the *left-hand limit of* $f(x)$ *as* x *approaches* (or tends to) 2.

Similarly,

$$\lim_{x \to 2^+} f(x) = 6$$

indicates that as x tends to 2 from the right, $f(x)$ approaches 6.

We shall call this limit as the *right-handed limit of f (x) as x approaches* 2.

The left and right-hand limits are called *one-sided* limits.

It is clear now that

$$\lim_{x \to 2} f(x) = 6$$

if and only if, both

$$\lim_{x \to 2^-} f(x) = 6 \text{ and } \lim_{x \to 2^+} f(x) = 6.$$

In the above example, the value of $\lim_{x \to 2} (x + 4)$ coincides with the value of $x + 4$ when $x = 2$, that is

$$\lim_{x \to 2} f(x) = f(2).$$

Likewise $\lim_{x \to 2} (x^2 - 3x + 1) = -1.$

as also $$\lim_{x \to 2^-} (x^2 - 3x + 1) = -1 = \lim_{x \to 2^+} (x^2 - 3x + 1).$$

Now consider another function $f : \boldsymbol{R} - \{2\} \to \boldsymbol{R}$ given by $f(x) = \dfrac{x^2 - 4}{x - 2}$. This function is not defined at the point $x = 2$, since division by zero is undefined. But $f(x)$ is defined for values of x which approach 2. So it makes sense to evaluate $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$. Again, we consider the following Tables 1.3 and 1.4 which give the values of $f(x)$ as x approaches 2 through values less than 2 and through values greater than 2, respectively.

Table 1.3

x	1	1.5	1.9	1.99	1.999
***f* (*x*)**	3	3.5	3.9	3.99	3.999

Table 1.4

x	3	2.5	2.1	2.01	2.001
***f* (*x*)**	5	4.5	4.1	4.01	4.001

As you can see

$$\lim_{x \to 2^-} f(x) = 4 \text{ and } \lim_{x \to 2^+} f(x) = 4.$$

Hence, we shall say

$$\lim_{x \to 2} f(x) = 4.$$

Now we are in a position to define the limit of a function.

Let f be a function and let a be a real number. We do not require that f be defined at a, but we do require that f be defined on a set of the form $(a - p, a) \cup (a, a + p)$.

(This guarantees that we can form $f(x)$ for all $x \neq a$ that are "sufficiently close" to a.)

Definition 6

$f(x)$ is said to tend to the limit l as x approach a, written as

$$\lim_{x \to a} f(x) = l,$$

if, and only if,

$$\lim_{x \to a^-} f(x) = l, \quad and \quad \lim_{x \to a^+} f(x) = l.$$

There is another definition of the limit of a function, equivalent to the above definition.

Definition 7

Let f be a function defined on some set $(a - p, a) \cup (a, a + p)$.

Then $$\lim_{x \to a} f(x) = l$$

if, and only if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $| f(x) - l | < \varepsilon$ when $0 < | x - a | < \delta$. Note that we do not deny that f may be possibly defined at the point a. All we are saying is that the definition does not require it.

We illustrate the above definitions with the help of examples.

Example 1.7

Show that

$$\lim_{x \to a} (mx + n) = ma + n; \ m \neq 0.$$

Solution

Let $\varepsilon > 0$. We seek a number $\delta > 0$ such that, if $0 < | x - a | < \delta$, then $| (mx + n) - (ma + n) | < \varepsilon$.

What we have to do first is establish a connection between

$$| (mx + n) = (ma + n) | \text{ and } | x - a |.$$

The connection is simple :

$$| (mx + n) - (ma + n) | = | m | \ | x - a |.$$

To make $| (mx + n) - (ma + n) | < \varepsilon$, we need only make $|x - a| < \dfrac{\varepsilon}{|m|}$.

This suggests that we choose $\delta = \dfrac{\varepsilon}{|m|}$.

Thus, if $0 < | x - a | < \delta$, then $| (mx + n) - (ma + n) | < \varepsilon$.

Hence $$\lim_{x \to a} (mx + n) = ma + n.$$

Example 1.8

Given $f(x) = \begin{cases} 3, & x \text{ an integer} \\ 1, & \text{otherwise} \end{cases}$, evaluate $\lim_{x \to 2} f(x)$, if it exists.

Solution

$$\lim_{x \to 2^-} f(x) = 1 = \lim_{x \to 2^+} f(x).$$

Hence $$\lim_{x \to 2} f(x) = 1.$$

Example 1.9

Consider the function $f : R - \{0\} \to R$ given by $f(x) = \dfrac{1}{x}$. As you see, $f(0)$ is not defined. In contrast to Examples 1.6, 1.7 and 1.8 where f is defined at the point a where we are evaluating the limit. We try to evaluate $\lim\limits_{x \to 0} f(x)$.

Look at Tables 1.5 and 1.6.

Table 1.5

x	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
$f(x)$	2	10	100	1000

Table 1.6

x	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	$-\frac{1}{1000}$
$f(x)$	– 2	– 10	– 100	– 1000

We see that $f(x)$ does not approach any fixed number as x approaches 0. In this case we say that $\lim\limits_{x \to 0} f(x)$ does not exist.

Example 1.10

Evaluate the following limits :

(i) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

(ii) $\lim\limits_{x \to -3} \dfrac{x^3 - 27}{x + 3}$

Solution

(i) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x+3)(x-3)}{(x-3)}$

$= \lim\limits_{x \to 3} (x+3) = 6$

(ii) $\lim\limits_{x \to -3} \dfrac{x^3 - 27}{x + 3}$

$= \lim\limits_{x \to -3} \dfrac{(x+3)(x^2 - 3x + 9)}{(x+3)}$

$= \lim\limits_{x \to -3} (x^2 - 3x + 9) = (-3)^2 - 3(-3) + 9 = 27$

We state below a theorem giving six properties of limits.

Theorem 2 : Properties of Limits

Let f and g be two functions of x. Then

(i) For any constant c, $\lim\limits_{x \to a} c = c$.

(ii) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iv) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$, k **is any real number**

(v) $\lim_{x \to a} [f(x)\, g(x)] = [\lim_{x \to a} f(x)] \, . \, [\lim_{x \to a} g(x)]$

(vi) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{[\lim_{x \to a} f(x)]}{[\lim_{x \to a} g(x)]}$ **provided** $\lim_{x \to a} g(x) \neq 0.$

Using the above properties, we try a few examples.

Example 1.11

Evaluate $\lim_{x \to 2} (x^2 - 4)(x^2 + x + 1)$

Solution

$$\lim_{x \to 2} (x^2 - 4)(x^2 + x + 1) = [\lim_{x \to 2} (x^2 - 4)][\lim_{x \to 2} (x^2 + x + 1)]$$

$$= (4 - 4)(4 + 2 + 1) = 0.$$

Example 1.12

Evaluate $\lim_{x \to 0} \frac{x^3 + 2x^2 + x}{x^2 + 2x}$.

Solution

First we reduce $\frac{x^3 + 2x^2 + x}{x^2 + 2x}$ by cancelling the common factor :

$$\frac{x^3 + 2x^2 + x}{x^2 + 2x} = \frac{x(x+1)^2}{x(x+2)} = \frac{(x+1)^2}{x+2}$$

Hence, $\lim_{x \to 0} \frac{x^3 + 2x^2 + x}{x^2 + 2x} = \lim_{x \to 0} \frac{(x+1)^2}{x+2}$

$$= \frac{\lim_{x \to 0} (x+1)^2}{\lim_{x \to 0} (x+2)} = \frac{1}{2}.$$

We now state another theorem, the importance of which will become clear by the examples taken later in this unit.

Theorem 3

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

(ii) $\lim_{x \to 0} \cos x = 1.$

We are now in a position to evaluate a variety of trigonometric limits.

Example 1.13

Find $\lim_{x \to 0} \frac{\sin 4x}{x}$

Solution

You have seen that $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

From this, it follows that

$$\lim_{x \to 0} \frac{\sin 4x}{4x} = 1$$

and $$\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} (4) \left(\frac{\sin 4x}{4x} \right) = 4.$$

Example 1.14

Find $\lim_{x \to 0} x \cot 4x$.

Solution

We write $x \cot 4x = x \dfrac{\cos 4x}{\sin 4x}$

Since $$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

it follows that

$$\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1$$

Thus $$\lim_{x \to 0} x \cot 4x = \lim_{x \to 0} \left(\frac{1}{4} \right) \cos 4x \left(\frac{4x}{\sin 4x} \right) = \left(\frac{1}{4} \right) (1)\ (1) = \frac{1}{4}.$$

Example 1.15

Find $\lim_{x \to 0} \dfrac{x^2}{\sec x - 1}$

Solution

We cannot deal with the expression $\dfrac{x^2}{\sec x - 1}$ as it stands since both the numerator and denominator tend to zero with x. As such we rewrite the expression in a certain form which will make things easy.

$$\lim_{x \to 0} \frac{x^2}{\sec x - 1} = \lim_{x \to 0} \frac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \frac{x^2 \cos x}{2 \sin^2 \frac{x}{2}}$$

$$= \lim_{x \to 0} (2)\ (\cos x) \left(\frac{\frac{x}{2}}{\sin \left(\frac{x}{2} \right)} \right)^2$$

$$= (2)\ (1)\ (1) = 2.$$

SAQ 4

Evaluate the limits that exist :

(i) $\lim_{x \to -2} \dfrac{x^2 - x - 6}{(x + 2)^2}$ (ii) $\lim_{x \to -4} \left(\dfrac{3x}{x + 4} + \dfrac{8}{x + 4} \right)$

(iii) $\lim_{x \to 3} \dfrac{(x^2 + x - 12)^2}{x - 3}$ (iv) $\lim_{x \to 0} \dfrac{\tan x}{x}$

(v) $\lim_{x \to 0} \frac{x^2 - 2x}{\sin 3x}$ (vi) $\lim_{x \to 0} \frac{\sin^2 x}{x(1 - \cos x)}$

(vii) $\lim_{x \to a} \frac{\sin (x - a)}{(x - a)^2}$ (viii) $\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x}$

(ix) $\lim_{x \to 0} \frac{\sin \alpha x}{\sin \beta x}$.

1.5 CONTINUITY

In ordinary language, to say that a certain process is "continuous" means that it goes on without interruption and without abrupt changes. In mathematics, continuity of a function can also be interpreted in a similar way.

Like limits, the idea of continuity is basic to calculus. First we introduce the idea of continuity at a point (or number) *a*, and then define continuity on an interval.

Continuity at a Point

Definition 8

Let f be a function defined at least on an open interval $(a - p, a + p)$.

We say f is continuous at a if, and only if,

$$\lim_{x \to a} f(x) = f(a).$$

or, equivalently,

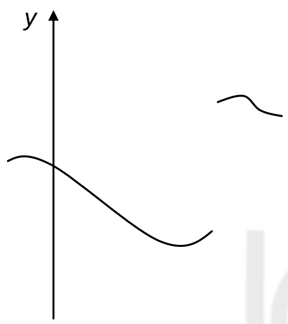
$$\lim_{h \to 0} f(a + h) = f(a).$$

If the domain of f contains an interval $(a - p, a + p)$, *then f can fail to be continuous at a for one of the following two reasons :*

(i) *Either* $\lim_{x \to a} f(x)$ *does not exist.*

(ii) $\lim_{x \to a} f(x)$ *exists but is not equal to* $f(a)$.

The function graphed in Figure 1.16 is discontinuous at *a* because it does not have a limit at *a*.



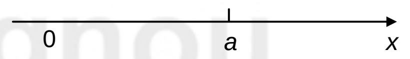


Figure 1.16

The function depicted in Figure 1.17 does have a limit at a. It is discontinuous at a only because $\lim_{x \to a} f(x)$ is different from $f(a)$, the value of f at a. The discontinuity at a is removable; it can be removed by lowering the dot into place (or by redefining f at a).

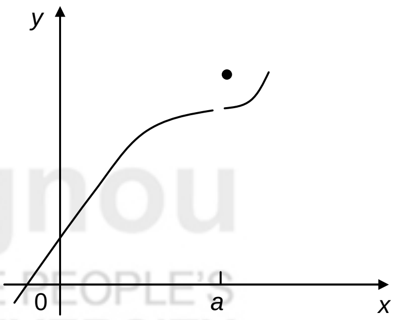


Figure 1.17

An ε, δ characterization of continuity at a reads as follows :

Definition 9

A function f is continuous at a if, and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Clearly, δ will depend both on ε and a; if any of these are changed the same δ may not work.

Continuity of the function $f(x)$ at an end point of an interval $[a, b]$ of its domain is defined below :

(i) $f(x)$ is continuous at the left end point 'a' if $\lim_{x \to a^+} f(x) = f(a)$.

(ii) $f(x)$ is continuous at the right end point 'b' if $\lim_{x \to b^-} f(x) = f(b)$.

If a function is continuous at each point of an interval, it is continuous over the interval.

Remark

If the interval is closed, the limit in the continuity test over the interval is two-sided at an interior point and the appropriate one-sided at the end points.

Example 1.16

Prove that $f(x) = \sin x$ is continuous at $x = 0$.

Solution

(i) $f(0) = \sin 0 = 0$

(ii) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin x = 0$ and

(iii) $\lim_{x \to 0} f(x) = f(0) = 0$

Therefore, $f(x) = \sin x$ is continuous at $x = 0$. (In fact $\sin x$ and $\cos x$ are continuous at each real x.)

Example 1.17

Examine the continuity of the function $f(x) = \dfrac{1}{x}$.

Solution

The function is defined for all non-zero real value of x. It is not defined at $x = 0$.

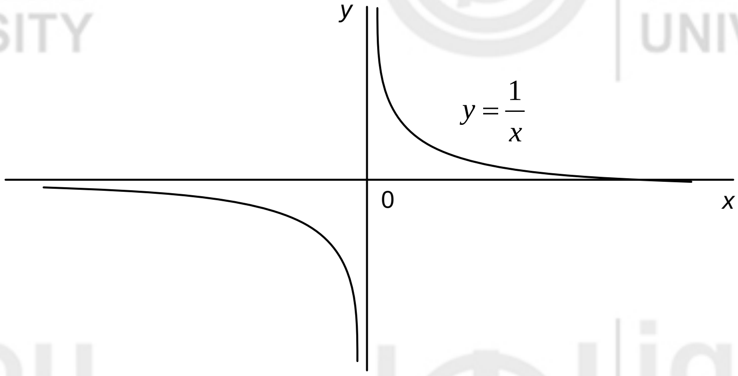


Figure 1.18

Also $\lim_{x \to x_0} f(x) = \frac{1}{x_0} = f(x_0)$, if $x_0 \neq 0$

Therefore, the function is continuous for all real $x \neq 0$. It fails to be continuous at $x = 0$ (why)? (Figure 1.18). This function is continuous on any interval which does not include $x = 0$ as an element of it.

Theorem 4 : Properties of Continuous Functions

(a) If the functions $f(x)$ and $g(x)$ are continuous at $x = a$, then

(i) $f(x) \pm g(x)$,

(ii) $f(x)\, g(x)$, and

(iii) $\frac{f(x)}{g(x)}$, $g(a) \neq 0$,

are continuous at $x = a$.

(b) If $f(x)$ is continuous at $x = x_0$ and if $g(y)$ is continuous at $y = y_0 = g(x_0)$, then the composite function $F(x) = g(f(x))$ is continuous at $x = x_0$.

These results are in fact the immediate corollaries of the corresponding limit theorems discussed in Section 1.4.

If a function is not continuous at a point x_0, it is said to be discontinuous at x_0.

Example 1.18

Let $f(x) = \frac{\sin x}{x}$ $(x \neq 0)$. Now f is not defined at $x = 0$. If we define $f(0) = 1$ which is same as $\lim_{x \to 0} \frac{\sin x}{x}$, then f is continuous at $x = 0$.

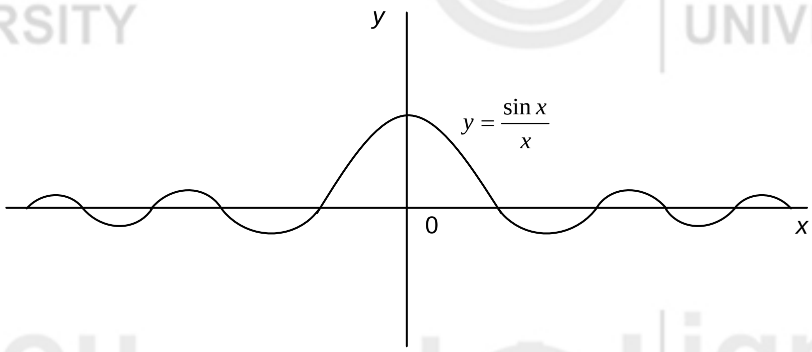


Figure 1.19

Example 1.19

The function f defined by

$$f(x) = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x \le 0 \end{cases}$$

is discontinuous at $x = 0$.

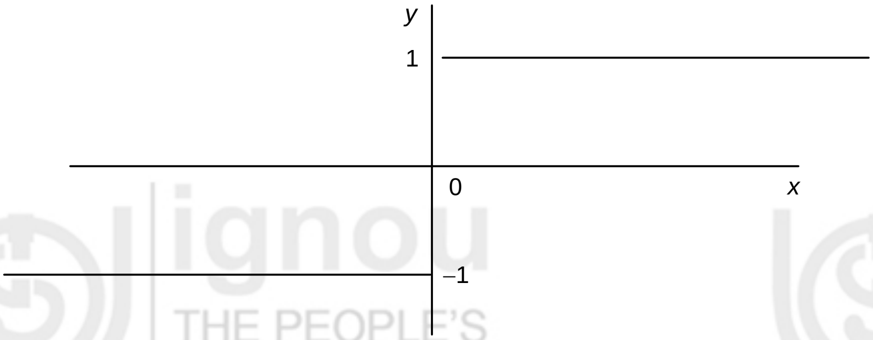


Figure 1.20

Example 1.20

Draw the graph of the function

$$f(x) = \begin{cases} -1 + x^2, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 2, & x = 1 \\ -x + 2, & 1 < x < 2 \\ 0, & 2 < x \le 3 \end{cases}$$

and examine its continuity on [– 1, 3].

Solution

The possible points of discontinuity are $x = 0$, $x = 1$ and $x = 2$. We examine the continuity of the function $f(x)$ on each of these points.

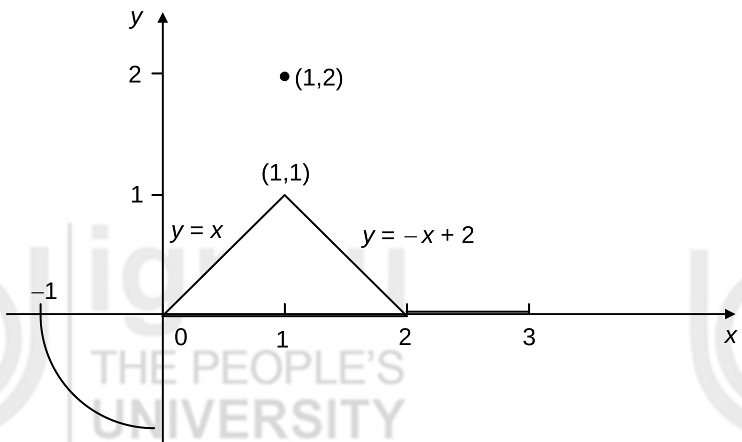
(i) $\lim\limits_{x \to 0^-} f(x) = -1 \ne 0 = f(0) = \lim\limits_{x \to 0^+} f(x)$

Therefore, $x = 0$ is a discontinuity.

(ii) $\lim\limits_{x \to 1^-} f(x) = (\lim\limits_{x \to 1^+} f(x) = 1)$

But $\lim\limits_{x \to 1} f(x) = 1 \ne f(1) = 2$. Therefore, $x = 1$ is a discontinuity.

(iii) The function is not defined at $x = 2$, but $\lim\limits_{x \to 2} f(x) = 0$ exists. Therefore, $x = 2$ is also a removable discontinuity.



$y = -1+x^2$

-1

Figure 1.21

SAQ 5

(a) Which of the following functions are continuous

(i) $f(x) = \begin{cases} 2x - 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$

(ii) $f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(b) Find the value of *b* for which the function

$$f(x) = \begin{cases} x^3 + 1 & \text{when } x < 2 \\ bx + \dfrac{2}{x} & \text{when } x \geq 2 \end{cases}$$

is continuous at $x = 2$.

(c) Prove that the greatest integer function $[x]$ is not continuous at $x = 0$ and $x = 1$.

1.6 DERIVATIVE

You are familiar with notions like velocity, acceleration, slope of a tangent line, etc. You also know that differential calculus is the right mathematical tool to obtain formulae to calculate velocity and acceleration of moving bodies, the slope of the tangent line to a curve, etc. In fact differential calculus deals with the problem of calculating rates of change and also helps to express the physical laws in precise mathematical terms for studying their consequences. The problems associated with a moving body are mainly responsible for development of the concept of derivative in calculus. To illustrate the fact we consider the rectilinear motion of a body. We know that the distance '*s*' traversed by the body and measured from a fixed point depends on the time '*t*'. Let the distance $s = f(t)$ be a function of time '*t*'. The average of velocity during a time interval $(t, t + \Delta t)$ is

$$V_{av} = \frac{\Delta s}{\Delta t}, \text{ where } \Delta s = f(t + \Delta t) - f(t).$$

To obtain the velocity at time *t*, we need to calculate

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

The above limit, if exists, is called the derivative of $f(t)$ with respect to *t* and we write

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = f'(t).$$

The derivative $\frac{ds}{dt}$ is the velocity of the moving body at time t.

1.6.1 Derivative of a Function

We now define the derivative of a given function by using the notion of limit.

Definition 10

The function $y = f(x)$ is said to have a derivative (or be differentiable) at a point x if, and only if, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite. This limit, written as $\frac{dy}{dx}$ or $f'(x)$, is called the derivative of $f(x)$ with respect to x.

If the derivative of a function exists at every point of an interval, then we say that the function is *differentiable in that interval*. However, while considering the derivatives at the end points of an interval, we evaluate suitable one-sided derivatives. For example, if the interval is $[a, b]$, then the derivative at $x = a$ is calculated as

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ and at } b \text{ as } \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

If $f'(x)$ is continuous at a point, then we say that $f(x)$ is continuously differentiable at that point.

Example 1.21

Let $y = x^n$ (n is a positive integer). Prove that $\frac{dy}{dx} = nx^{n-1}$.

Solution

$$y = f(x) = x^n$$

Then $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$= \frac{(x+h-x)}{h}\left[(x+h)^{n-1} + (x+h)^{n-2}\, x + \ldots + (x+h)\, x^{n-2} + x^{n-1}\right]$$

The above step is derived by using the result

$$a^n - b^n = (a-b)\left[a^{n-1} + a^{n-2}\, b + \ldots + ab^{n-2} + b^{n-1}\right]$$

We now have

$$\frac{f(x+h) - f(x)}{h} = (x+h)^{n-1} + (x+h)^{n-2}\, x + \ldots + (x+h)\, x^{n-2} + x^{n-1}$$

Taking the limit $h \to 0$, we get

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = x^{n-1} + x^{n-2}\, x + \ldots + x^{n-1} \text{ (n number of terms)}$$

Therefore, $$\frac{dy}{dx} = nx^{n-1}.$$

In the following theorem, we make an important observation (a necessary condition) about any differentiable function.

Theorem 5

If a function $y = f(x)$ is differentiable at some point $x = x_0$, it is continuous at that point.

Proof

By definition, the derivative of $f(x)$ at $x = x_0$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Now $\lim_{h \to 0} [f(x_0 + h) - f(x_0)]$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot h$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot \lim_{h \to 0} h$$

$$= f'(x_0) \cdot 0$$

$$= 0$$

$\therefore$ $$\lim_{h \to 0} f(x_0 + h) - \lim_{h \to 0} f(x_0) = 0$$

i.e. $$\lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} f(x_0)$$

i.e. $f(x)$ is continuous at x_0.

The converse of the above result is not true. Indeed, there exist functions which are continuous but not differentiable.

Example 1.22

Examine the differentiability of the function

$$y = \begin{cases} x & \text{whenever } x > 0 \\ -x & \text{whenever } x \le 0 \end{cases}$$

Solution

The function is continuous for $-\infty < x < \infty$. The graph of the function is given below.

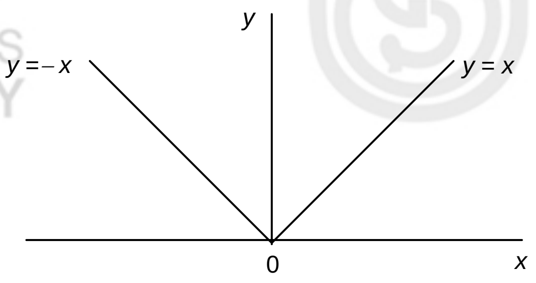


Figure 1.22

Further $$\frac{f(0 + h) - f(0)}{h} = \frac{h - 0}{h} = 1$$ when h is positive.

$$= \frac{-h-0}{h} = -1 \text{ when } h \text{ is negative.}$$

$$\therefore \quad \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

i.e. $f'(0)$ does not exist.

Remark

The above example will help you to conclude that a function can never have derivative at a point of discontinuity. In the light of this remark, look at the following example.

Example 1.23

Examine the differentiability of the function

$$y = \begin{cases} x\,, & -\infty < x < 0 \\ 1\,, & 0 \le x < 2 \\ 3 - x\,, & 2 \le x \end{cases}$$

Solution

From Figure 1.23, we observe as follows :

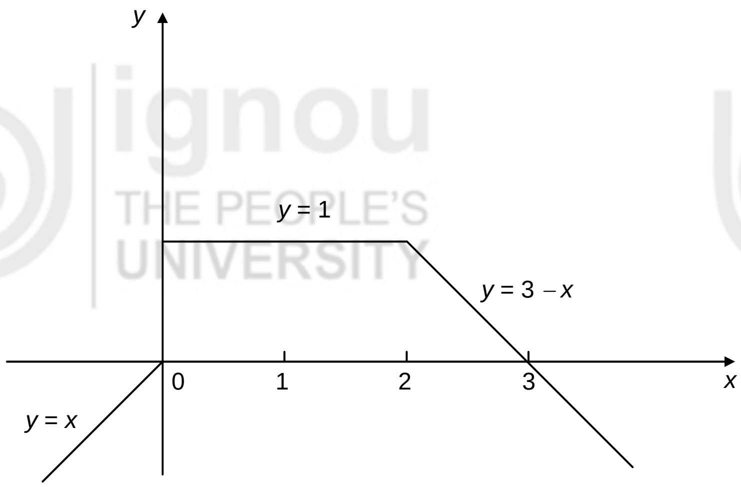


Figure 1.23

(i) The function is not continuous at $x = 0$. Hence it is not differentiable there.

(ii) Though the function is continuous at $x = 2$, the one-sided limits

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0^+} \frac{[(3-(2+h)] - (3-2)}{h}$$

$$= \lim_{h \to 0^+} \frac{3-2-h-1}{h}$$

$$= \lim_{h \to 0^+} \frac{-h}{h} = -1$$

and $\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0^-} \frac{1 - (3-2)}{h}$$

$$= \lim_{h \to 0^-} = 0 = 0 \text{ are not equal.}$$

Therefore, the derivative does not exist at $x = 2$.

At all points ($x \neq 0$ and $x \neq 2$) the derivatives exist.

SAQ 6

(a) Show that the derivative of a constant function is zero.

(b) Show that $y = |x|$ is differentiable at all points except at $x = 0$.

(c) Find the derivative of the function

(i) $y = x^2$

(ii) $y = 3x^2 + 5x - 1$.

1.6.2 Algebra of Derivatives

We now state some important rules regarding the derivatives of the sum, difference, product and quotient of functions in terms of their derivatives.

Theorem 6

If $f(x)$ and $g(x)$ are differentiable functions at a point x, then

(i) $$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

(ii) $$\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$$

(iii) $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$$

provided $g(x) \neq 0$.

We shall give the proof of the result (ii) below. Other results we leave as an exercise for you.

Proof

Let $y = f(x) \cdot g(x)$

Then $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

Now $f(x+h)\,g(x+h) - f(x)\,g(x)$

$$= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$$

$$\therefore \quad \frac{dy}{dx} = \lim_{h \to 0} \left\{ f(x+h) \frac{[g(x+h) - g(x)]}{h} + g(x) \frac{[f(x+h) - f(x)]}{h} \right\}$$

$$\frac{dy}{dx} = \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \, . \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

That is,

$$\frac{dy}{dx} = f(x) \frac{dg(x)}{dx} + g(x) \frac{df(x)}{dx},$$

Use of the product rule (ii) and the result which states that 'derivative of a constant is zero' gives

$$\frac{d}{dx} = [cf(x)] = c \frac{d}{dx} f(x),$$

where c is a constant.

Example 1.24

Find the derivative of $y = 5x^2 \sin x$.

Solution

$$\frac{dy}{dx} = 5 [x^2 \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (x^2)]$$

We derive $\frac{d}{dx} (\sin x)$ as under :

$$\frac{d}{dx} (\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{\frac{h}{2} \to 0} \cos \left(x + \frac{h}{2} \right) \left(\frac{\sin \left(\frac{h}{2} \right)}{\frac{h}{2}} \right)$$

$$= \lim_{\frac{h}{2} \to 0} \cos \left(x + \frac{h}{2} \right) . \lim_{\frac{h}{2} \to 0} \frac{\sin \left(\frac{h}{2} \right)}{\frac{h}{2}}$$

$$= \cos x \, . \, 1 = \cos x$$

and $$\frac{d}{dx} (x^2) = 2x$$

Hence, $$\frac{dy}{dx} = 5 [x^2 \cos x + 2x \sin x] .$$

Example 1.25

Find $\frac{dy}{dx}$, if $y = \frac{\cos x}{x^2}$.

Solution

By formula (iii)

$$\frac{dy}{dx} = \frac{x^2 \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} (x^2)}{x^4}$$

To obtain $\frac{d}{dx}(\cos x)$, we have

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{\frac{h}{2} \to 0} \left(-\sin\left(x + \frac{h}{2}\right) \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right)$$

$$= -\lim_{\frac{h}{2} \to 0} \left[\sin\left(x + \frac{h}{2}\right)\right] \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= -\sin x \,.\, 1 = -\sin x$$

and $\frac{d}{dx}(x^2) = 2x$

Therefore, $\frac{dy}{dx} = -\frac{x \sin x + 2\cos x}{x^3}$

1.6.3 Derivative of a Composite Function (Chain Rule)

Theorem 7

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ as a function of x is differentiable and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof

Let $u = g(x)$ be differentiable at x_0 and $y = f(u)$ be differentiable at u_0 where $u_0 = g(x_0)$.

Let $h(x) = f(g(x))$. The function $g(x)$, being differentiable, is continuous at x_0; so also the function $f(u)$ at u_0 and, therefore, at x_0.

$$\Delta u = g(x_0 + \Delta x) - g(x_0)$$

where $\Delta u \to 0$ as $\Delta x \to 0$

and at u_0

$$\Delta y = f(u_0 + \Delta u) - f(u_0)$$
$$= f(g(x_0 + \Delta x)) - f(g(x_0))$$
$$= h(x_0 + \Delta x) - h(x_0)$$

Assuming $\Delta u \neq 0$ (i.e. $g(x)$ is not a constant function in neighbourhood of x_0), we write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

In the limit, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

or $$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

You can also show that in the case of $\Delta u = 0$ the above formula holds true. In fact it becomes an identity 0 = 0. Note that it cannot be divided by Δu in this case.

Example 1.26

If $y = \sin(x^2)$, find $\frac{dy}{dx}$.

Solution

Let $y = \sin(u)$ and $u = x^2$. By the chain-rule

$$\frac{dy}{dx} = \frac{d}{du}(\sin u) \cdot \frac{du}{dx}$$
$$= (\cos u)(2x)$$
$$= 2x \cos x^2.$$

1.6.4 Second Order Derivatives

You know that the derivative of a function $y = f(x)$, i.e. $f'(x)$ is also a function. Hence we can calculate the derivative of $f'(x)$ at a point of its domain, if it exists, by using the same limiting procedure. That is, if $\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ exists, it is called the *second order derivative* of $f(x)$ with respect to x. We write it as $\frac{d^2y}{dx^2}$ or $f''(x)$. You can also calculate higher order derivatives like 3rd order, 4th order etc., of a function $f(x)$.

Example 1.27

Find the (i) 2nd order and (ii) 5th order derivatives of $y = x^4 - 3x^2 + 5x - 1$.

Solution

(i) $\frac{dy}{dx} = y' = 4x^3 - 6x + 5$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3 - 6x + 5) = 12x^2 - 6$$

(ii) $y^{(3)} = 24x,\ y^{(4)} = 24,\ y^{(5)} = 0$

1.6.5 Implicit Differentiation

If we are given an equation like $x^6 + 4xy^4 - 3y^3 + 1 = 0$, it is not easy to solve it for y in terms of x. However, it is often possible to calculate $\frac{dy}{dx}$ straight from the equation by the method of implicit differentiation. For example, if we differentiate both the sides of the given equation, we get

$$6x^5 + 4y^4 + 4x\left(4y^3 \frac{dy}{dx}\right) - 3\left(3y^2 \frac{dy}{dx}\right) + 0 = \frac{d}{dx}(0)$$

$$\Rightarrow 6x^5 + 4y^4 + (16xy^3 - 9y^2)\frac{dy}{dx} = 0$$

Therefore, $$\frac{dy}{dx} = \frac{-(6x^5 + 4y^4)}{16xy^3 - 9y^2}$$

This holds for all points where $16xy^3 - 9y^2 \neq 0$.

SAQ 7

(a) Differentiate

(i) $\dfrac{3x+1}{2x-5}$

(ii) $\dfrac{1}{x^3+3x-8}$

(iii) $(x^2 + x + 6) \sin x$

(iv) $(x + 2)(x^2 + 1) + \cos x$

(b) Find the derivatives of the functions

(i) $\dfrac{2x+3}{x^2-5}$

(ii) $\tan x$

(iii) $\dfrac{(1-\sin x)}{1+\cos^2 x}$

(c) Find the 2nd order derivatives of the functions

(i) $\sin^2 x$

(ii) $ax^3 + bx^2 + cx + d$

1.6.6 Derivatives of Some Elementary Functions

Power Function $y = x^\alpha$ (α is real)

We have already proved that the power function $y = x^n$, where n is a positive integer, has the derivative

$$\frac{dy}{dx} = n\,x^{n-1}$$

We shall show now that the above rule also holds when $y = x^{\frac{p}{q}}$ (q, p are integers with $q > 0$).

Let us assume $p, q > 0$.

Then $y^q = x^p$. By the method of implicit differentiation, we get

$$q\,y^{q-1}\,\frac{dy}{dx} = p\,x^{p-1}$$

or $$\frac{dy}{dx} = \frac{p\,x^{p-1}}{q\,y^{q-1}} = \frac{p\,x^{p-1}}{q\,x^{p-\frac{p}{q}}} = \frac{p}{q}\,x^{\frac{p}{q}-1} = nx^{n-1},\ \text{if}\ \frac{p}{q} = n$$

In fact the formula $\frac{d}{dx}(x^{\alpha}) = \alpha\,x^{\alpha-1}$ holds true for any real α. We shall accept this result without proof.

Trigonometric Functions

We have already seen that

$$\frac{d}{dx}(\sin x) = \cos x,\quad \frac{d}{dx}(\cos x) = -\sin x$$

By using the quotient rule you can show that

$$\frac{d}{dx}(\tan x) = \sec^2 x,\quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x\,\tan x,\quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\,\cot x$$

Logarithmic Function

We shall prove that $\frac{d}{dx}(\log x) = \frac{1}{x}$

Let $f(x) = \log x$

Then $$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}[\log(x+h) - \log(x)] = \frac{1}{h}\log\left(1+\frac{h}{x}\right)$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\log(x+h) - \log x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\cdot\frac{1}{x}$$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\cdot\lim_{h\to 0}\frac{1}{x} = 1\times\frac{1}{x} = \frac{1}{x}$$

$$\left(\because \lim_{y\to 0}\frac{\log(1+y)}{y} = 1\right)$$

Exponential Function (with respect to the base *a*)

Let $y = a^x$ with $a > 0$, but $a \neq 1$

Then $\log_e y = x\log_e a$

By the method of implicit differentiation, we get

$$\frac{1}{y}\frac{dy}{dx} = \log_e a$$

Therefore, $\frac{dy}{dx} = y \log_e a = a^x \log_e a$

In particular if $y = e^x$, then $\frac{dy}{dx} = e^x$ $\qquad (\because \log_e e = 1)$

Inverse Function

Let $\qquad y = \sin^{-1} x$

Then $\qquad \sin y = x$

Differentiating both sides w. r. t. x, we have

$$\cos y \frac{dy}{dx} = 1$$

i.e. $$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{\cos^2 y}}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

As we know that the range of $\sin^{-1} x$ is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, i.e. y lies between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, so cos y is positive.

Similarly, $$y = \cos^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

$$y = \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$y = \cot^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{1}{1 + x^2}$$

Example 1.28

If $y = \cos^{-1}(\tan x), -\frac{\pi}{4} < x < \frac{\pi}{4}$, find $\frac{dy}{dx}$.

Solution

Here $$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - \tan^2 x)}} \frac{d}{dx} (\tan x) = -\frac{1}{\sqrt{(1 - \tan^2 x)}} \sec^2 x$$

$$= -\frac{1}{\cos x \sqrt{\cos 2x}}$$

1.6.7 Derivatives of a Function Represented Parametrically

Sometimes we represent a function say $y = f(x)$, $a \le x \le b$ by its parametric equations,

$$\left.\begin{array}{l} x = \phi(t) \\ y = \psi(t) \end{array}\right], \quad \alpha \le t \le \beta$$

Let us assume $\phi(t)$, $\psi(t)$ are differentiable and $x = \phi(t)$ has an inverse $t = h(x)$. Then we can consider the equation of the function $y = f(x)$ as a composite function.

$$y = \psi(t),\ t = h(x)$$

Using the differentiation rule for a composite function (chain-rule), we get

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d\psi}{dt}\,\frac{dh}{dx}$$

Since $t = h(x)$ is inverse of $x = \phi(t)$, we arrive at $\dfrac{dy}{dx} = \dfrac{\frac{d\psi}{dt}}{\frac{d\phi}{dt}}$.

Example 1.29

The parametric equation of a semi-circular arc of radius 'a' with its centre at (0, 0) is

$$\left.\begin{array}{l} x = a\cos t \\ y = a\sin t \end{array}\right], \quad 0 \le t \le \pi$$

Find $\dfrac{dy}{dt}$ at $t = \dfrac{\pi}{4}$.

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a(-\sin t)} = -\cot t$$

At $\quad t = \dfrac{\pi}{4}, \dfrac{dy}{dx} = -\cot\dfrac{\pi}{4} = -1$

1.6.8 Logarithmic Differentiation

In some problems, it is easier to find $\dfrac{dy}{dx}$ by first taking logarithmic and then differentiating. Such process is called logarithmic differentiation. This is usually done in two kinds of problems. First when the function is a product of many simpler functions. In this case logarithm converts the product into a sum and facilitates differentiation. Secondly, when the variable x occurs in the exponent. In this case logarithm brings it to a simpler form.

Example 1.30

Differentiate $\sin x \sin 2x \sin 3x$.

Solution

Let $y = \sin x \sin 2x \sin 3x$

Then $\log y = \log \sin x + \log \sin 2x + \log \sin 3x$.

Differentiating both sides, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x + \frac{1}{\sin 2x} \cdot \cos 2x \cdot 2 + \frac{1}{\sin 3x} \cdot \cos 3x \cdot 3$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{\cos x}{\sin x} + \frac{2\cos 2x}{\sin 2x} + 3\,\frac{\cos 3x}{\sin 3x}\right]$$

$$= \sin x \sin 2x \sin 3x\,[\cot x + 2\cot 2x + 3\cot 3x]$$

Example 1.31

Differentiate $(\sin x)^x$

Solution

Let $\quad y = (\sin x)^x$

Then $\quad \log y = x \log \sin x$

Differentiating both the sides, we have

$$\frac{1}{y}\frac{dy}{dx} = \log \sin x + x \cdot \frac{1}{\sin x} \cdot \cos x$$

i.e. $$\frac{dy}{dx} = y\,[\log \sin x + x \cot x]$$

$$= (\sin x)^x\,[\log \sin x + x \cot x]$$

1.6.9 Differentiation by Substitution

Sometimes it is easier to differentiate by making substitution. Usually these examples involve inverse trigonometric functions.

Example 1.32

Differentiate $\tan^{-1}\left(\sqrt{1+x^2} - x\right)$

Solution

Put $\quad x = \tan\theta$

Then $\quad 1 + x^2 = \sec^2\theta$

$$\therefore \quad \sqrt{1+x^2} - x = \sec\theta - \tan\theta$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}} = \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$\therefore$ $$\tan y = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

i.e. $$y = \frac{\pi}{4} - \frac{\theta}{2}$$

$\therefore$ $$\frac{dy}{dx} = -\frac{1}{2}\frac{d\theta}{dx} = -\frac{1}{2}\frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{-1}{2(1 + x^2)}$$

SAQ 8

(a) A particle moves along a straight line. At any time t the distance s travelled by the particle is given by the formula : $s = 32t^2 + 9$. Find the velocity and acceleration of the particle at time, $t = 2$ units.

(b) Use the method of implicit differentiation to calculate $\frac{dy}{dx}$, if

$$2x^2 - 3xy + 4y^2 = 8$$

(c) Differentiate the following

(i) $x^2 e^x \sin x$

(ii) $(x + 1)^2 (x + 2)^3 (x + 3)^4$

(iii) $\tan^{-1} \frac{2x}{1 - x^2}$

(iv) $\cos^{-1} \frac{1 - x^2}{1 + x^2}$

ENGINEERING MATHEMATICS-I

This course on Engineering Mathematics-I (BME - 001) is in your hand. The whole course is comprises four blocks : Calculus, Vector Calculus, Matrices and Determinants and Probability and Statistics.

In first block, you will be introduced to basic concepts of differential and integral calculus. This includes limit, continuity, mean value theorem and application of calculus in engineering field. The matter of differential and integral calculus is presented in single block because one has to continuously refer to the results of differential calculus while studying integral calculus.

Block 2 covers vector algebra, vector differential calculus, line, surface and volume integral. The usefulness of vector calculus in engineering mathematics results from the fact that many physical quantities – for example, force, velocity, momentum etc. – may be represented by vectors.

Block 3 includes linear equations and Euclidean spaces, linear transformation and linear equations, matrices and determinants. Euclidean spaces are introduced as generalization of concepts of two or three dimensional spaces. Linear transformation from one Euclidean space to another Euclidean space are described in this block. The computational aspects of linear transformation by using matrices and determinants have been explained.

Probability and Statistics have relevance in our lives which have been introduced in Block 4. But more than that, it is itself inherently interesting. Our need to cope with uncertainty leads us to study and use of probability theory. Under the topic of hypothesis testing, we are trying to determine when it is reasonable to conclude, from analysis of a sample, that the entire population possesses a certain property, and when it is not reasonable to reach such a conclusion.

We believe that the course on engineering mathematics will provide insights which help you in understanding other subjects in Diploma in Computer Integrated Manufacturing.

CALCULUS : BASIC CONCEPTS

The knowledge and concepts of mathematics help in handling any engineering problem. In block of Mathematics, which is in your hands, you will be studying some concepts of mathematics, which are a must for you. This block will be dealing with the concepts of Differentiation and Integration, and consists of four units.

Unit 1 is devoted to Differential Calculus. In fact, calculus was created to meet the pressing mathematical needs of 17th century for solving problems in Science and Technology.

In Unit 1, the concept of function has been introduced and different types of functions and inverse functions have been discussed. This unit also lays the foundations of calculus in terms of definitions and theorems centering around limit and continuity of functions and algebra of derivatives. We will also be dealing with some of the applications of the derivatives to geometry and maxima and minima of functions. Some important theorems namely Rolle's Theorem and Lagranges Mean-value Theorem have also been discussed.

Unit 2 deals in function of several variables. We have extended the notions of limit, continuity, partial derivatives, total differentiation, maxima and minima, Jacobians and then applications for the functions of two or more variables in this unit.

Unit 3 deals with the concept of integrals and its applications. Indefinite integral of a function as an antiderivative has been defined and the different techniques for finding integrals of functions have been discussed. After introducing the concept of definite integral, some applications have been dealt such as finding the area of a curved surface.

The firm grounding in integrals provided by Unit 3 shall be helpful for easy understanding of Unit 4, dealing with ordinary differential equations.

For the want of clarity in concepts, number of solved examples has been introduced in each unit. To help you check your understanding and to assess yourself, each unit contains SAQs. The answers to these SAQs are given at the end of each unit. We suggest that you look at them only after attempting the exercises.

At the end, we wish you all the best for your all educational endeavours.