UNIT 11 CASH-FLOW PROJECTION

Structure

11.0 OBJECTIVES

After going through this unit, you will be able to:

- get some idea about cash-flow techniques and projection;
- know the benefits of life insurance contract;
- estimate the payment of premium; and
- apply stochastic interest rate models in actuarial economics.

11.1 INTRODUCTION

Under cash-flow projections two areas are fundamental. One is theory of *compound interest* and the other is *probability theory*.

In life insurance contract, the benefit of insured consists of a single payment and the sum assured. The time and the amount of payment are functions of the random variable T (or, $T(x)$) where T is the future life time of persons of age x. The present value of the payment is denoted by Z. The expected value of the payment $E(Z)$ is the net single premium of the contract. The premium does not reflect the risk to be carried by the insurer. In order to assess this, one requires the different characteristics of Z.

We are introducing the models of stochastic interest rate and some simple uses in actuarial applications.

11.2 THE MATHEMATICS OF COMPOUND INTEREST

11.2.1 Effective Interest Rates

An interest rate is always related with a *basic time unit* like an annual interest rate of 5%. With the interest rate, the time interval at the end of which interest rate is credited or "Compounded" has to be stated, which is known as *conversion period.*

An interest rate is called *effective* if the basic time unit and the conversion period are identical. In this case interest is credited at the end of basic time unit.

Let i be the effective interest rate and let F_0 be the initial capital. Let r_k be the additional amount invested at the end of year k , k = 1,2,...,n.

We want to find out the balance at the end of n years.

Suppose F_k is the balance at the end of year k including r_k, interest credited to the previous year's balance is $i\ F_{k-1}$.

So,

$$F_k = F_{k-1} + iF_{k-1} + r_k, k = 1,2,...,n \tag{11.2.1}$$

which implies

$$F_k = (1+i)\ F_{k-1} + r_k \tag{11.2.2}$$

Multiplying (11.2.2) by $(1+i)^{n-k}$ and summing over all values of k we obtain,

$$F_n = (1+i)^n F_0 + \sum_{i=1}^{n} (1+i)^{n-k} r_k \tag{11.2.3}$$

The powers of (1 + i) are called *accumulation factors*. The *accumulated value of an initial capital C* after n years is $(1+i)^n$ C. Equation (11.2.3) gives the capital at the end of interval which is the sum of value of initial capital plus the sum of accumulated values of the intermediate deposits.

The *discount factor* is $v = \dfrac{1}{1+i}$

Equation (11.2.3) can be written as $v^n F_n = F_0 + \sum_{k=1}^{n} v^k r_k$ (11.2.4)

Hence the present value of a capital C due at time h is $v^h C$.

From equation (11.2.1), we have, $F_k - F_{k-1} = i\ F_{k-1} + r_k$ (11.2.5)

Now summing over k we get $F_n - F_0 = \sum_{k=1}^{n} \left(i F_{k-1} + r_k \right)$ (11.2.6)

i.e., *the increment of the fund is the sum of the total interest credited and the total deposits made.*

11.3 NOMINAL INTEREST RATES

If the conversion period does not coincide with the basic time unit, then the interest rate is called *nominal.*

An interest rate of 6% with conversion period 3 months means that interest of $\left(\frac{6\%}{4}\right) = 1\cdot5\%$ is credited at the end of each quarter. Thus an initial capital of 1 increases to $(1\cdot015)^4 = 1\cdot06136$ at the end of one year. Therefore an annual nominal interest rate of 6% is equivalent to an annual effective interest rate of $6\cdot136\%$.

11.3.1 Relation between Effective Interest Rate and Nominal Interest Rate

Let i denotes a given effective interest rate. We define $i^{(m)}$ as the nominal interest rate convertible m times per year which is equivalent to i.

Equality of accumulation factors for 1 year leads to the following relationship

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i \tag{11.3.1}$$

$$\Rightarrow \qquad i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right] \tag{11.3.2}$$

The limiting case correspond to continuous compounding.

Let

$$\delta = \lim_{m \to \infty} i^{(m)} \tag{11.3.3}$$

It is called the *force of interest* equivalent to i.

So from equation (11.3.2) we have

$$i^{(m)} = \frac{(1+i)^{\frac{1}{m}} - (1+i)^0}{\frac{1}{m}} \tag{11.3.4}$$

i.e., δ is a derivative of the function $(1+i)^x$ *at* $x = 0$.

So,

$$\delta = \ln(1+i) \tag{11.3.5}$$

or

$$e^{\delta} = 1 + i \tag{11.3.6}$$

From this we conclude that the accumulation factor for a period of h years is $(1+i)^h = e^{\delta h}$ and discount factor for the same period is $v^h = e^{-\delta h}$ where h, the length of the period is any real number.

Example: A deposit of 100,000 is made into a newly established fund. The fund pays nominal interest of 12% convertible quarterly. At the end of each six months, a withdrawal is made from the fund. The first withdrawal is X, the second is 2X, the third is 3X and so on. The last is sixth withdrawal which exactly exhausts the fund. Calculate X.

Solution: Taking the notation of the text to half years we get $F_0 = 100{,}000$ and $F_1 = F_0(1\cdot 03)^2 - X$ and so on. This can be taken using (11.1.2). The 6[th] withdrawal exactly exhausts the fund. So $F_6 = 0$. Solving this we get X = 6128.

11.4 DISCOUNTED CASH-FLOW TECHNIQUES

One of the techniques is given by the equation (11.2.4) where the interest is credited at the end of each conversion period in arrears. The other technique is given below, where the interest is credited at the beginning of the conversion period. Interest credited in this way is also referred to as *discount* and the corresponding rate is called *discount rate* or *rate of interest in advance*. Let d be an annual effective discount rate. A person investing an amount of C will be credited interest equal to dC immediately and the invested capital C will be returned at the end of the period. Investing the interest dC at the same conditions, the investor will receive additional interest of $d(dC) = d^2C$, and the additional invested amount will be returned at the end of the year; reinvesting, the interest yields additional interest of $d(d^2C) = d^3\ C$ and so on.

Repeating this process infinitely we find the investor will receive the total sum of

$$C + dC + d^2C + \ldots + \ldots = \frac{1}{1-d}C \qquad (11.4.1)$$

at the end of the year in return for investing the initial capital C. The equivalent effective interest rate i is given by the equation.

$$\frac{1}{1-d} = 1 + i \qquad (11.4.2)$$

which implies

$$d = \frac{i}{1+i} \qquad (11.4.3)$$

This result has the following interpretation. If 1 unit of capital is invested, d (the interst payable at the beginning of the year) is the discounted value of the interst i to be paid at the end of year.

Further, $$i = \frac{d}{1-d}$$

Thus the interest payable at the end of the year is the accumulated value of interest payable at the beginning of the year.

In case of nominal interest rate, we have

$$\frac{1}{1-\frac{d^{(m)}}{m}}=1+\frac{i^{(m)}}{m}=(1+i)^{\frac{1}{m}} \tag{11.4.5}$$

which gives $$d^{(m)}=\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}} \tag{11.4.6}$$

or $$\frac{1}{d^{(m)}}=\frac{1}{m}+\frac{1}{i^{(m)}} \tag{11.4.7}$$

and $$\lim_{m\to\infty} d^{(m)}=\lim_{m\to\infty} i^{(m)}=\delta \tag{11.4.8}$$

when the interest is compounded continuously the difference between interest in advance and interest in arrears vanishes.

11.5 VALUATION OF CASH-FLOWS OF SIMPLE INSURANCE CONTRACT

In this section we will discuss elementary types of insurance.

11.5.1 Whole Life and Term Insurance

Consider a *whole life insurance.* This provides for payment of 1 unit at the end of year of death. In this case the amount of the payment is fixed, while the time of payment $(K+1)$ is random. Its present value is

$$Z=v^{K+1} \tag{11.5.1}$$

The random variable Z ranges over the values $v, v^2, v^3, \ldots$ and the distribution of Z is determined by (11.5.1) whereas the distribution of K is

$$\Pr\left(Z=v^{(K+1)}\right)=P(K=k)={}_kp_x\,q_{x+k}, k=0,1,2,\ldots \tag{11.5.2}$$

where ${}_kp_x$ is the probability that a person aged x survives upto age $x+k$ and q_{x+k} is the probability that a person of exact age $x+k$ will die within one year following the attainment of that age. The net single premium is denoted by A_x and given by

$$A_x=E(Z)=E\left(v^{K+1}\right)=\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x q_{x+k} \tag{11.5.3}$$

Var(Z) can be calculated by the identity

$$\begin{aligned}V(Z)&=E\left(Z^2\right)-\{E(Z)\}^2\\ &=E\left(Z^2\right)-A_x^2\end{aligned} \tag{11.5.4}$$

Replacing v by $e^{-\delta}$ we see that

$$E\left(Z^2\right)=E\left[e^{-2\delta(K+1)}\right] \tag{11.5.5}$$

which is the net single premium calculated at twice the original force of interest. Thus calculating the variance is no more difficult than calculating the net single premium.

An insurance which provides for payment only if death occurs within n years is known as *term insurance of duration n*. For example: 1 unit is payable only if death occurs during the first n years, the actual time of payment still being the end of the year of death. So

$$Z = \begin{cases} v^{K+1} \text{ for } & K = 0, 1, 2, \ldots, n-1 \\ 0 \quad \text{for } & K = n, \; n+1, \; n+2, \ldots \end{cases} \tag{11.5.6}$$

The net single premium is denoted by

$$A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k}p_x q_{x+k} \tag{11.5.7}$$

and

$$E\left(Z^2\right) = E\left[e^{-2\delta(K+1)}\right] \tag{11.5.8}$$

11.5.2 Pure Endowments

A pure endowment of duration n provides for payment of the sum insured only if the insured is alive at the end of n years,

$$Z = \begin{cases} 0 \quad \text{for} & K = 0, 1, \ldots, n-1 \\ v^n \quad \text{for} & K = n, \; n+1, n+2, \ldots \end{cases} \tag{11.5.9}$$

The net premium is denoted by $A_{x,\overline{n|}}^{\;\;1}$ and is given by

$$A_{x,\overline{n|}}^{\;\;1} = v^n \, {}_{n}p_x \tag{11.5.10}$$

The formula for the variance of a Bernoulli random variable gives

$$Var(Z) = v^{2n} \cdot {}_{n}p_x \cdot {}_{n}q_x \tag{11.5.11}$$

11.5.3 Endowments

Assume that the sum insured is payable at the end of the year of death, if this occurs within the first n years, otherwise at the end of the nth year

$$Z = \begin{cases} v^{K+1} \text{ for } K = 0, 1, 2, \ldots, n-1 \\ v^n \text{ for } K = n, \; n+1, \; n+2, \ldots \end{cases} \tag{11.5.12}$$

The net single premium is denoted by $A_{x:\overline{n|}}$. Denoting the present value of (11.5.6) by Z_1 and (11.5.9) by Z_2, we have

$$Z = Z_1 + Z_2 \tag{11.5.13}$$

As a consequence

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x,\overline{n}|^{1}} \quad (11.5.14)$$

and

$$V(Z) = V(Z_1) + V(Z_2) + 2Cov(Z_1, Z_2) \quad (11.5.15)$$

The product Z_1, Z_2 is always zero, hence

$$Cov(Z_1,\ Z_2) = E(Z_1, Z_2) - E(Z_1),\ E(Z_2)$$

$$= -A^{1}_{x:\overline{n}|}\ A_{x,\overline{n}|^{1}} \quad (11.5.16)$$

Hence

$$V(Z) = V(Z_1) + V(Z_2) - 2A^{1}_{n:\overline{n}|}\ A_{x,\overline{n}|^{1}} \quad (11.5.17)$$

As a consequence of the last identity the risk in selling an endowment policy, measured by the variance is less than that in selling a term insurance to one person and a pure endowment to another.

Let us finally consider an m year *deferred whole life insurance*. Its present value is

$$Z = \begin{cases} 0 & for \quad K = 0,1,\ldots,m-1 \\ v^{K+1} & for \quad K = m, m+1, m+2, \ldots \end{cases} \quad (11.5.18)$$

The net single premium is denoted by ${}_{m|}A_x$. Alternative formula for its net single premium are

$${}_{m|}A_x = {}_{m}p_x v^m A_{x+m} \quad (11.5.19)$$

and

$${}_{m|}A_x = A_x - A^{1}_{x:\overline{m}|} \quad (11.5.20)$$

The second moment $E(Z^2)$ again equals the net single premium at twice the original force of interest.

11.5.4 Insurance Payable at the Moment of Death

In previous sections we have discussed that the sum insured was payable at the end of the year of death. This assumption does not reflect insurance practice in a realistic way, but has the advantage that the formulae may be evaluated directly from a life table.

Let us now assume that the sum insured is payable at the instant of death. i.e. at time T. The present value of a payment of payable at the instant of death is at time T. The present value of a payment of 1 payable immediately on death is

$$Z = v^T \quad (11.5.21)$$

The net single premium is denoted by A_x. Using the formula from force of mortality

$$\bar{A}_x = \int_0^\infty v^t \; {}_tp_x \; \mu_{x+t} \, dt \qquad (11.5.22)$$

The force of mortality at age x is defined as the ratio of instantaneous rate of decrease in l_x to the value of $l_x \cdot is$. $\mu_x = -\frac{1}{l_x}\frac{d}{dx} l_x = -\frac{d}{dx}(\log l_x)$. l_x, is the number of person's living at age x.

One practical approximation of T is given by

$$T = K + S = (K+S) = (K+1) - (1-S) \qquad (11.5.23)$$

where K is the number of complete future years lived by (x) and S is the fraction of a year during which (x) is alive in the year of death.

Making use of assumed independence of K and S, as well as the uniform distribution of S, so that

$$E\left[(1+i)^{1-S}\right] = \int_0^1 (1+i)^u \, du = \bar{S}_{\overline{1}|} = \frac{i}{\delta} \qquad (11.5.24)$$

we find

$$\bar{A}_x = E\left[v^{K+1}\right] E\left[(1+i)^{1-S}\right] = \frac{i}{\delta} A_x \qquad (11.5.25)$$

Thus calculation of $\bar{A}_x$ is a simple extension of A_x.

A similar formula may be derived for term insurance. For endowments the factors $\frac{i}{\delta}$ is only used in the term insurance part.

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;\;1} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;\;1}$$

$$= A_{x:\overline{n}|} + \left(\frac{i}{\delta} - 1\right) A^{1}_{x:\overline{n}|} \qquad (11.5.26)$$

Let us finally assume that the sum insured is payable at the end of the *mth* part of the year in which death occurs i.e. time $K + S^{(m)}$. The present value of a whole life insurance of 1 unit then becomes

$$Z = v^{K+S^{(m)}} \qquad (11.5.27)$$

and $$K + S^{(m)} = (K+1) - \left(1 - S^{(m)}\right) \qquad (11.5.28)$$

So,

$$E\left[(1+i)^{1-S^{(m)}}\right] = S^{(m)}_{\overline{1}|} = \frac{i}{i^{(m)}} \qquad (11.5.29)$$

and we obtain

$$A_x^{(m)} = E\left[v^{K+1}\right] E\left[(1+i)^{1-S^{(m)}}\right] = \frac{i}{i^{(m)}} A_x. \qquad (11.5.30)$$

Example: Z_1 is the present value random variable for an n-year continuous endowment insurance of 1 issued to (x). Z_2 is the present value random variable for an n year continuous term insurance of 1 issued to (x). Given $Var(Z_2) = 0.01,\ v^n = 0\cdot 3,\ {}_np_x = 0.8, E(Z_2) = 0\cdot 04$, calculate $Var(Z_1)$.

Solution: Let Z_3 be the present value random variable for the pure endowment

$$Z_1 = Z_2 + Z_3$$

we want to find out $Var(Z_1)$.

But $\quad Var(Z_1) = Var(Z_2 + Z_3)$

$$= Var(Z_2) + 2\operatorname{cov}(Z_2, Z_3) + Var(Z_3).$$

$$Cov(Z_2, Z_3) = -E(Z_2)\cdot E(Z_3) \quad (\because Z_2 Z_3 = 0)$$

$$Z_3 = \begin{cases} v^n \cdot 1 \ \textit{with probability} \ 1 \\ \\ 0 \ \textit{otherwise} \end{cases}$$

Now

$$E(Z_3) = v^n\, {}_np_x = (0\cdot 3)(0\cdot 8) = (0\cdot 24)$$

$$Var(Z_3) = v^{2n}\ {}_np_x(1 - {}_np_x)$$

$$= (0\cdot 3)^2\,(0\cdot 8)\,(0\cdot 2)$$

$$= (0\cdot 09)(0\cdot 16) = 0\cdot 0144$$

$$Var(Z_1) = 0\cdot 01 - 2(0\cdot 04)(0\cdot 24) + 0\cdot 01484$$

$$= 0\cdot 0244 - 0\cdot 192 = 0\cdot 0052.$$

Check Your Progress 1

1) What is the difference between effective interest rate and nominal interest rate? Find the relation between them.

....

....

....

....

2) For m = 1,2,3,4,6,12,and ∞ calculate $i^{(m)}$ for i = 6%.

....

....

....

....

3) Z_1 is the present value random variable for an n-year continuous endowment insurance of 1 issued to (x). Z_2 is the present value random variable for an n-year continuous term insurance of 1 issued to (x). Given values $Var(Z_2)$, ${}_n p_x$, $E(Z_2)$ obtain a formula to find out $Var(Z_1)$.

....

....

....

11.6 ANALYSIS AND VALUATION OF ANNUITIES, BONDS, LOANS AND OTHER SECURITIES

To have a clear idea on annuity and its analysis we introduce certain types of perpetual streams known as perpetuities. The resulting formulae are very simple and will later be useful for calculating the present value of annuities with a finite term.

11.6.1 Perpetuities

First we consider perpetuities consisting of annual payments of 1 unit. If the first payment occurs at time zero, the perpetuity is called *perpetuity-due* and its present value is denoted by $\ddot{a}_{\overline{\infty}|}$. Thus

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \ldots = \frac{1}{1-v} = \frac{1}{d}. \tag{11.6.1}$$

If the first payment is made at the end of year 1, we call the perpetuity an *immediate perpetuity*. Its present value is denoted by $a_{\overline{\infty}|}$ and is given by

$$a_{\overline{\infty}|} = v + v^2 + \ldots = \frac{v}{1-v} = \frac{1}{i} \tag{11.6.2}$$

Let us now consider perpetuity where payments of $\frac{1}{m}$ are made m times each year. If the payments are made in advance (first payment of $\frac{1}{m}$ at time 0), the present value is denoted by $\ddot{a}^{(m)}_{\overline{\infty}|}$ and is

$$\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{m} + \frac{1}{m} v^{\frac{1}{m}} + \frac{1}{m} v^{\frac{2}{m}} + \ldots = \frac{1}{m} \frac{1}{\left(1 - v^{\frac{1}{m}}\right)} = \frac{1}{d^{(m)}} \tag{11.6.3}$$

If the payments are made in arrears (first payment of $\frac{1}{m}$ at time 0), the present value is denoted by $a^{(m)}_{\overline{\infty}|}$ and given by

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{m} v^{\frac{1}{m}} + \frac{1}{m} v^{\frac{2}{m}} + \frac{1}{m} v^{\frac{3}{m}} + \ldots$$

$$= \frac{1}{m}\frac{v^{\frac{1}{m}}}{\left(1-v^{\frac{1}{m}}\right)} = \frac{1}{m\left[(1+i)^{\frac{1}{m}}-1\right]} = \frac{1}{i^{(m)}} \tag{11.6.4}$$

The results in (11.6.3) and (11.6.4) lead to an interpretation of the identity (11.4.7). Since a perpetuity-due and an immediate perpetuity differ only by a payment of $\frac{1}{m}$ at time 0, their present value differ by $\frac{1}{m}$.

Let us now consider a continuous perpetuity with constant rate of payment r = 1 and starting at time 0. Its present value is denoted by $\ddot{a}_{\overline{\infty}|}$ and given by

$$\ddot{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t}dt = \frac{1}{\delta} \tag{11.6.5}$$

Letting $m \to \infty$ in (11.6.4) or (11.6.3) we will get the same result.

The systematic pattern in formulae (11.6.1) to (11.6.5) is evident.

A certain type of perpetuities with increasing payments is defined by two parameters m and q. i.e. the number of payments per year and the number of increase per year. We assume that q is a factor of m. if $m = 12$, and $q = 4$, it means that payments are made monthly and increase quarterly. In general, the payments of such as increasing perpetuity due are defined as follow.

Time				Payment
0	$\frac{1}{m}$	...	$\frac{1}{q}-\frac{1}{m}$	$\frac{1}{mq}$
$\frac{1}{q}$	$\frac{1}{q}+\frac{1}{m}$	...	$\frac{2}{q}-\frac{1}{m}$	$\frac{2}{mq}$
$\frac{2}{q}$	$\frac{2}{q}+\frac{1}{m}$	...	$\frac{3}{q}-\frac{1}{m}$	$\frac{3}{mq}$

and so on

In particular, the last $\frac{m}{q}$ payments of year k are $\frac{k}{m}$ each.

Let us denote the present value of such a perpetuity by $\left(I^{(q)}\ \ddot{a}\right)^{(m)}_{\overline{\infty}|}$. We can calculate it by representing the sequence of increasing payments as sum of perpetuities with constants payments of $\frac{1}{(mq)}$ payable m times per year and beginning at times $0, \frac{1}{q}, \frac{2}{q}, \ldots$. Thus we obtain the following formula:

$$\left(I^{(q)}\ddot{a}\right)_{\overline{\infty}|}^{(m)} = \frac{1}{q}\ddot{a}_{\overline{\infty}|}^{(m)}\left[1 + v^{\frac{1}{q}} + v^{\frac{2}{q}} + \ldots\right]$$

$$= \ddot{a}_{\overline{\infty}|}^{(m)} \cdot \ddot{a}_{\overline{\infty}|}^{(q)} = \frac{1}{d^{(m)}} \cdot \frac{1}{d^{(q)}} \qquad (11.6.6)$$

The corresponding immediate annuity differs only in that each payment is made one *mth* year later, thus giving

$$\left(I^{(q)} a\right)_{\overline{\infty}|}^{(m)} = v^{\frac{1}{m}} \left(I^{(q)} \ddot{a}\right)_{\overline{\infty}|}^{(m)} = \frac{v^{\frac{1}{m}}}{d^{(m)}} \cdot \frac{1}{d^{(q)}} = \frac{1}{i^{(m)}} \cdot \frac{1}{d^{(q)}} \qquad (11.6.7)$$

A superscript 1 is always omitted. For instance, the present value of a perpetuity due with annual payments of 1,2,... is

$$(I\ddot{a})_{\overline{\infty}|} = \left(I_{\ddot{a}}^{(1)}\right)_{\overline{\infty}|}^{(1)} = \frac{1}{d^2} \qquad (11.6.8)$$

For continuous payment streams

$$(I\ddot{a})_{\overline{\infty}|} = \int_0^\infty t e^{-\delta t} dt = \frac{1}{\delta^2} \qquad (11.6.9)$$

$$(I\ddot{a})_{\overline{\infty}|} = \int_0^\infty [t+1] e^{-\delta t} dt = \frac{1}{\delta d} \qquad (11.6.10)$$

without actually calculating the integrals.

But when the payments are arbitrary with annual payments r_0, r_1, r_2,... (at times 0,1,2,...), its present value is denoted by $\ddot{a}$.

$$\ddot{a} = r_0 + v r_1 + v^2 r_2 \qquad (11.6.11)$$

Such a variable perpetuity may be represented as a sum of constant perpetuities in the following way.

Annual Payments	Starts at time
r_0	0
$r_1 - r_0$	1
$r_2 - r_1$	2
$r_3 - r_2$	3
and so an	

The present value of this perpetuity may therefore be expressed as

$$\ddot{a} = \frac{1}{d}\left\{r_0 + v\left(r_1 - r_0\right) + v^2\left(r_2 - r_1\right)\ldots\right\}. \qquad (11.6.12)$$

We can use (11.6.11) to calculate the present value of exponentially growing payments with $r_k = e^{\tau k}$ for k = 0,1,2,... and obtain (11.6.13)

$$\ddot{a} = \frac{1}{1 - e^{-(\delta - \tau)}} \qquad (11.6.14)$$

provided $\tau < \delta$.

Annuities

In practice annuities are more frequently used than perpetuities. An annuity is defined as a sequence of payments of a limited duration, which we denote by n. Here we consider some standard type of annuities or annuities-certain.

The present value of an annuity due with n-annual payments of 1 starting at time 0, is denoted by $\ddot{a}_{\overline{\infty}|}$. It is given by

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1} \qquad (11.6.15)$$

Representing, the annuity as the difference of two perpetuities (one starting at time 0, the other at time n), we find that

$$\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v^n \ddot{a}_{\overline{\infty}|} = \frac{1}{d} - v^n \frac{1}{d} = \frac{1 - v^n}{d} \qquad (11.6.16)$$

This result can be verified by evaluating the geometric sum (11.6.15). Similarly from (11.6.2), (11.6.3) and (11.6.4) the following formulae can be obtained

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \qquad (11.6.17)$$

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}} \qquad (11.6.18)$$

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}} \qquad (11.6.19)$$

Note that only the denominator varies, depending on the payment made (immediate / due) and frequency. Note that n must be an integer in (11.6.16) and (11.6.17), and a multiple of $\frac{1}{m}$ in (11.6.18) and (11.6.19).

The final or accumulated value of annuities is also of interest. This is defined as the accumulated value of the payment stream at time n, and the usual symbol used is S. The final value is obtained by multiplying the initial value with the accumulator factor

$$\ddot{S}_{\overline{n}|} = \frac{(1 + i)^n - 1}{d} \qquad (11.6.20)$$

$$S_{\overline{n|}} = \frac{(1+i)^n - 1}{i} \qquad (11.6.21)$$

$$\ddot{S}^{(m)}_{\overline{n|}} = \frac{(1+i)^n - 1}{d^{(m)}} \qquad (11.6.22)$$

$$S^{(m)}_{\overline{n|}} = \frac{(1+i)^n - 1}{i^{(m)}} \qquad (11.6.23)$$

Another simple relation between the initial value and final value of a constant annuity may be verified.

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{S_{\overline{n|}}} + i \qquad (11.6.24)$$

Let us now consider an increasing annuity due with parameter q and m.

Time				**Payment**
0	$\frac{1}{m}$	$\dots$	$\frac{1}{q} - \frac{1}{m}$	$\frac{1}{mq}$
$\frac{1}{q}$	$\frac{1}{q} + \frac{1}{m}$	$\dots$	$\frac{2}{q} - \frac{1}{m}$	$\frac{2}{mq}$
$\frac{2}{q}$	$\frac{2}{q} + \frac{1}{m}$	$\dots$	$\frac{3}{q} - \frac{1}{m}$	$\frac{3}{mq}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$n - \frac{1}{q}$	$n - \frac{1}{q} + \frac{1}{m}$	$\dots$	$n - \frac{1}{m}$	$\frac{n}{m}$

Such an increasing annuity can be represented as an increasing perpetuity starting at time 0, minus an identical increasing perpetuity starting of time n, minus a constant annuity starting at n.

$$\left(I^{(q)}\ddot{a}\right)^{(m)}_{\overline{n|}} = \left(1^{(q)}\ddot{a}\right)^{(m)}_{\overline{\infty|}} v^n \left(I^{(q)}\ddot{a}\right)^{m}_{\overline{\infty|}} - v^n n \ddot{a}^{(m)}_{\overline{\infty|}} \qquad (11.6.25)$$

Substituting (11.6.6) and (11.6.3) and using (11.6.19) we obtain the equation

$$\left(I^{(q)}\ddot{a}\right)^{(m)}_{\overline{n|}} = \frac{\ddot{a}^{(q)}_{\overline{n|}} - nv^n}{d^{(m)}} \qquad (11.6.26)$$

Similarly the present value of the corresponding immediate annuity is calculated as

$$\left(I^{(q)}a\right)^{(m)}_{\overline{n|}} = \frac{\ddot{a}^{(q)}_{\overline{n|}} - nv^n}{i^{(m)}} \qquad (11.6.27)$$

In these equations n must be a multiple of $1/q$. Special cases are the combinations of $m = 1$ and $q = 1$, $m = 12$ and $q = 12$, $m = \infty$ and $q = 1$, and $m = \infty$ and $q = \infty$.

The annuities just considered are known as *standard increasing annuities* (I). *Standard decreasing annuities* (D) are similar, but the payments are made in the reversed order. The sum of these two annuities is a constant annuity. This relation carries over to the present values and we obtain

$$\left(I^{(q)}\ddot{a}\right)_{\overline{n}|}^{(m)} + \left(D_{\ddot{a}}^{(q)}\right)_{\overline{n}|}^{(m)} = \left(n + \frac{1}{q}\right)\ddot{a}_{\overline{n}|}^{(m)} \tag{11.6.28}$$

Using (11.6.26) and (11.6.27)

$$\ddot{a}_{\overline{n}|}^{(q)} = a_{\overline{n}|}^{(q)} + \left(1 - v^{n}\right)\frac{1}{q} \tag{11.6.29}$$

we get

$$\left(D^{(q)}\ \ddot{a}\right)_{\overline{n}|}^{(m)} = \frac{n - a_{\overline{n}|}^{(q)}}{d^{(m)}} \tag{11.6.30}$$

The standard decreasing annuity due may be interpreted as a constant perpetuity with m^{th} ly payments of n/m, minus a series of deferred perpetuities-due, each with constant m^{th} ly payments of $\left(\frac{1}{mq}\right)$ and starting at times $\frac{1}{q}, \frac{2}{q}, \ldots, n$.

Loans, Bonds and Other Securities

Let S be the value at 0 of a loan that is to be repaid by payments $r_1, r_2, \ldots, r_n$, made at the end of years $k = 1, 2, \ldots, n$. Then S must be the present value of the payments.

$$S = vr_1 + v^2 r_2 + \ldots + v^n r_n \tag{11.6.31}$$

Let S_k be the principal outstanding i.e., the remaining debt immediately after r_k has been paid. It consists of the previous years debt, accumulated for one year minus r_k.

$$S_k = (1 + i)\ S_{k-1} - r_k,\ k = 1, 2, \ldots, n \tag{11.6.32}$$

or,

$$r_k = i S_{k-1} + \left(S_{k-1} - S_k\right) \tag{11.6.33}$$

It consists of two components, one is interest on the *running debt* and *reduction of principal*. Substituting $-S_k$ for F_k (11.6.32) is equivalent to (11.2.2). From (11.2.3)

$$S_k = (1 + i)^k\ S - \sum_{h=1}^{k} (1 + i)^{k-h}\ r_k \tag{11.6.34}$$

From this one can get $S_n = 0$ and $S_k = v\, r_{k+1} + v^2 r_{k+2} + \ldots + v^{n-k} r_n$ (11.6.35)

Formula (11.6.34) is the *retrospective formula* and (11.6.35) is *the prospective formula* for the outstanding principal.

The payments r_1, r_2, ..., r_k may be chosen arbitrarily, subject to the constraint (11.6.31). By proper choice of payment steam the following formula of annuities can be obtained. For example, $S = 1$ can be repaid by the payments, The payments $r_1, r_2, \ldots, r_k$ may be chosen arbitrarily, subject to the constraint (11.6.31). By proper choice of payment steam the following formula of annuities can be obtained. For example, $S = 1$ can be repaid by the payments,

$$r_1 = r_2 = \ldots = r_{n-1} = i, \; r_n = 1 + i \tag{11.6.36}$$

In this case only interest is paid for the first (n – 1) years and the entire debt together with the last year's interest, is paid of the end of the n^{th} year. From (11.6.31) $\quad 1 = i\, a_{\overline{n}|} + v^n$ (11.6.37)

The debt $S = 1$ may also be repaid by constant payments of

$$r_1 = r_2 = \ldots = r_n = \frac{1}{a_{\overline{n}|}} \tag{11.6.38}$$

As an alternative to repaying the creditor at times $1,2,\ldots,n$-1, one could pay only the interest on S as in (11.6.36). In order to cover the final repayment one can make equal deposits to a fund that is to accumulate to 1 of the end of n years, from this it is obvious that the annual deposit must be $\frac{1}{S_n}$. Since the total annual outgo must be the same in both the cases and we arrive at

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{S_{\overline{n}|}} + i.$$

Suppose now that we repay a debt of $S = n$, so that the principal outstanding decreases linearly to D, $S_k = n - k$ for $k = 0, 1,2,\ldots,n$. From (11.6.33) it is evident that $r_k = i(n-k+1) + 1$. Using (11.6.31)

$$n = i(Da)_{\overline{n}|} + a_{\overline{n}|} \tag{11.6.39}$$

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \tag{11.6.40}$$

This result is a special case ($m = q = 1$) of (11.6.30)

The loan itself may consist of a series of payments. Assume that equal payments of 1 are received by the debtor at times $0,1,2,\ldots,n$-1. At the end of each year interest on the received amounts is paid and the total amount received is repaid at time n.

$$r_k = ik \;\; for \;\; k = 1,2,\ldots,n-1, \;\; r_n = in + n \tag{11.6.41}$$

From the equality of present values

$$\ddot{a}_{\overline{n}|} = i(Ia)_{\overline{n}|} + nv^n \tag{11.6.42}$$

For security, suppose an investor pays a price p which entitles him to n future payments. The payments are denoted by $r_1, r_2, \ldots, r_n$ and payment r_k is due at time T_k, for $k = 1,2,\ldots,n$. What is the resulting rate of return?

The present value of the payment stream to be received by the investor is a function of the force of interest δ. Define

$$a(\delta) = \sum_{k=1}^{n} \exp(-\delta T_k)\, r_k \tag{11.6.43}$$

Let t be the solution of the equation

$$a(t) = p \tag{11.6.44}$$

The *internal rate of return* or *investment yield* is defined as $i = e^t - 1$.

Equation (11.6.44) can be solved by standard numerical methods such as interval bisection or the Newton-Raphson method. We shall present a more efficient and simpler method than other methods.

Consider

$$f(\delta) = \ln\left(\frac{a(\delta)}{r}\right), \tag{11.6.45}$$

here $r = r_1 + \ldots + r_n$ denotes the undiscounted sum of payments. It is not difficult to verify that

$$f(0) = 0, \quad f'(\delta) = \frac{a'(\delta)}{a(\delta)} < 0$$

$$f''(\delta) = \frac{a''(\delta)}{a(\delta)} - \left(\frac{a'(\delta)}{a(\delta)}\right)^2 > 0 \tag{11.6.46}$$

The last inequality may be verified by interpreting $f''(\delta)$ as a variance. Interpreting $\frac{f(\delta)}{\delta}$ as the slope of the secant and noting that f is a convex function by (11.6.46) we see that $\frac{f(\delta)}{\delta}$ is an increasing function of δ. Hence for $0 < s < t < u$,

$$\frac{f(s)}{s} < \frac{f(t)}{t} < \frac{f(u)}{u} \tag{11.6.47}$$

giving

$$\frac{f(t)}{f(s)} \cdot s < t < \frac{f(t)}{f(u)} \cdot u \tag{11.6.48}$$

Thus we have proved that

$$\frac{\ln\left(p/r\right)}{\ln\left(a(s)/r\right)} s < t < \frac{\ln\left(p/r\right)}{\ln\left(a(u)/r\right)} u \tag{11.6.49}$$

The process may be iterated yielding the following algorithm. Starting with δ_0, calculate

$$\delta_{k+1} = \frac{\ln\left(p/r\right)}{\ln\left(a(\delta_k)/r\right)} \delta_k, \; k = 0,1,\ldots \tag{11.6.50}$$

recursively.

For $\delta_0 < t$, the resulting sequence will be monotonically increasing and will converge to t. For $\delta_0 > t$, the sequence will decrease monotonically to t. Thus any arbitrary positive value may be chosen for δ_0.

Example: Security = Rs.5000/- (Face amount) yearly coupons = Rs.300/-. Assume that the security has been bought for Rs.5250 for a remaking running time of 9 years. Thus we have

$$P = 5250$$

$$T_k = k(k = 1,2,\ldots,9)$$

$$r_k = 300(k = 1,2,\ldots,8)$$

$$r_9 = 5300$$

$$r = 7700.$$

Assuming we know that the current yield of similar securities lies between 5% and 5.5% we have $s = \ln 1\cdot 05$, $u = \ln\ 1\cdot 055$. The bounds for the investment yield is

$$0\cdot 051461 < t < 0\cdot 051572$$

and

$$5\cdot 2808\% < i < 5\cdot 2925\% \quad \left(i = i' - 1\right) \qquad (11.6.51)$$

The algorithm defined by (11.6.50) may be used to obtain greater precision.

11.7 YIELD CURVES

There are many classes of bonds in financial markets. For example, we can restrict ourselves to various risk classes, such as Treasury bonds, municipal bonds, corporate bonds or junk bonds. Given the class of Treasury bonds, we see Treasury bills that mature within a year, Treasury notes with maturities between one and five years and Treasury bonds with maturities extending to 30years.

If risk-fee instantaneous interest rates were constant at r, the time t price of a pure discount bond with maturity u would be given by

$$B(u,t) = 100e^{-r(u-t)} \qquad (11.7.1)$$

where $B(\cdot)$ denotes the present value of 100 discounted at a rate r. A pure discount bond does not make any coupon payment, does not contain any implicit option and has a par value of 100. The function $e^{-r(u-t)}$ at time t plays the role of a discount factor. At time

$$t = u \qquad (11.7.2)$$

the bond matures and the exponential function equals 1. At all t < u, it is less than 1.

The bond price formula depends on the whole spectrum of future short rates. Hence the yield curve at time t contains all the available information concerning future short rates and the bond prices depend on the whole yield curve or on the term structure of interest rates, which we define below.

Definition: Assume that at time t there exists zero coupon bonds with a spectrum of maturities $u \in [t,T]$. Let their price be $B(u,t)$ and their yield be given by R_t^u. Then the spectrum of yields $\{R_t^u,\ u \in [t,\ T]\}$ is called the term structure of interest rates.

Here the yield R_t^u is defined as the number that satisfies the equality

$$B(u,t) = 100e^{-R_t^u(u-t)},\ t < u \qquad (11.7.3)$$

or

$$R_t^u = \frac{\log B\cdot(u,t) - \log 100}{t - u} \qquad (11.7.4)$$

Examples of yield curves

In practice, there are two ways of proceedings. Often a market participant assumes a functional form for the yield curve and from there obtains the implied forward rates. A second method is to go the opposite direction. One can assume a dynamic behavior for the forward rates and from there obtain a yield curve. In most common case one assumes that the yields $R(\cdot)$ depends on one variable r_t which represents the short rate observed now, and they are given by the functional form.

$$R(r_t,\ u,\ t) = A(u,\ t) - C(u,t)\,r_t \qquad (11.7.8)$$

Functions $A(u,t)$ and $C(u,t)$ can be constructed in various ways, so that the yield curve can be upword, downward sloping or hump shaped.

11.8 INTRODUCTION TO STOCHASTIC INTEREST RATE MODELS AND ACTUARIAL APPLICATIONS

The interest rate that will apply in future years is of course not known. Thus it seems reasonable to ask why future interest rates have not been modelled as stochastic process. Two reasons are there: (1) Life insurance is particularly concerned with the long term development of interest rates and no commonly employed stochastic models exist for making long term predictions. (2) A reasonable assumption is that the remaining life times of the insured lives are essentially independent random variables. The probability distribution of aggregate loss can be obtained by convolution. The variance of the aggregate loss is the sum of the individual variances, which facilitates the use of normal approximation. Stochastic independence between policies would be lost with the introduction of stochastic interest rate, since all policies are affected by the same interest development.

The mathematics of derivative assets assume that time passes continuously. As a result, new information is revealed continuously and decision makers may face instantaneous changes in randomness. Hence technical tools for pricing derivative products require ways of handling random variables over infinitesimal time intervals. To have a clear understanding on stochastic interest rate, we must have clear concept on continuous payments.

Continuous Payments

Let us consider that payments are made continuously with an annual instantaneous rate of payment of $r(t)$. Thus the amount deposited to the fund during the infinitesimal time interval from $t + dt$ is $r(t)$ dt. Let $F(t)$ denote the balance of the fund at time t. We assume that interest is credited continuously, according to a time-dependent force of interest $\delta(t)$. Interest credited in the infinitesimal time interval from t *to* $t + dt$ is $F(t)\ \delta(t)$ dt.

The total increase in the capital during this interval is thus

$$dF(t) = F(t)\,\delta(t)\,dt + r(t)\,dt \qquad (11.8.1)$$

To solve the corresponding differential equation

$$F'(t) = F(t)\,\delta(t) + r(t) \qquad (11.8.2)$$

we write

$$\frac{d}{dt}\left[e^{-\int_0^t \delta(s)\,ds} F(t)\right] = e^{-\int_0^t \delta(s)\,ds}\, r(t) \qquad (11.8.3)$$

Integration with respect to t from 0 to h gives

$$e^{-\int_0^h \delta(s)\,ds} F(h) - F(0) = \int_0^h e^{-\int_0^t \delta(s)\,ds}\, r(t)\,dt \qquad (11.8.4)$$

Thus the value at time 0 of a payment to be made at time t (i.e., its present value) is obtained by multiplication with the factor $e^{-\int_0^t \delta(s)\,ds}$

From (11.8.4) we obtain

$$F(h) = e^{\int_0^h \delta(s)\,ds}\left[F(0) + \int_0^h e^{\int_0^h e(s)\,ds}\, r(t)\,dt\right] \qquad (11.8.5)$$

Thus the value at time h of a payment made at time $t < h$ (its accumulated value), is obtained by multiplication with the factor

$$e^{\int_0^h \delta(s)\,ds}.$$

In the case of a constant force of interest i.e., $\delta(t) = \delta$, the factors $e^{-\int_0^h \delta(s)\,ds}$ and $e^{\int_0^h \delta(s)\,ds}$ are reduced to the discount factors and accumulation factors.

Now we consider equation (11.7.1) where $B(\cdot)$ is the present value of 100 discounted at a rate 'r'. When interest rates become stochastic, the formula (11.7.1) would change. R_t would represent the risk free rate earned instantaneously, and to obtain the price of a discount bond with par value equal to 100, we would have to use

$$B(u,t) = 100\, E\left[e^{-\int_0^t r_s\,ds} \middle| I_t\right] \qquad (11.8.6)$$

First of all, we need to use the conditional expectations operator E[.|.], since r_s represents the instantaneous rate at some future time s > t and consequently is not known with certainty at time t, it can only be predicted.

Secondly, there is the issue of which probability measure is used to calculate these expectations. One can argue that Treasury bonds are risk – free assets and that there is no need to use the equivalent martingale measure. This would not be correct. As interest rates become stochastic, prices of Treasury bonds with larger maturities are subject to more shocks and, everything else being the same, longer maturities are riskier. In order to eliminate the corresponding risk premia, we need to use equivalent martingale measure in evaluating the expressions shown in (11.8.6).

Third, note a heuristic explanation for where the integral in (11.8.6) comes from. Consider the case of a 3 year bond. In discrete time frame-work the price of a 3-period discount bond will be given by

$$B(3,1) = E\left[\frac{100}{(1+r_1)(1+r_2)(1+r_3)} \,|I_1\right] \tag{11.8.7}$$

where r_1 is the current short rate, r_2 is the unknown short rate during year 2 and r_3 is the (unknown) short rate during year 3. According to equation (11.8.7) the bonds price is equal to the expectation under risk neutral probabilities of the discounted value.of the par value. The discount factor is obtained by multiplying the one period discounts during the present and the two future years. Since future discounts are unknown, an expectation operator needs to be used.

This example may be more useful in interpreting the continuous-time bond price given in (11.8.6). As time becomes continuous, discrete-time discount factors need to be replaced by the exponential function. But because interest rates are continuously changing an integral has to be used in the exponent.

Check Your Progress 2

1) Define annuity. Represent annuity as the difference of two perpetuities one at time 0 and the other at time n. Show that

$$\ddot{a}_n = \frac{1-v^n}{d}$$

....

....

....

....

2) Prove that in repayment of debt, each payment consists of two components, one is interest on running debt and the other is reduction in principal.

....

....

....

....

3) Discuss the method of continuous payments. Give an expression for yield.

....

....

....

....

11.9 LET US SUM UP

This unit discusses the Cash-Flows like interest, annuities, bonds, loans and other payments which arise due to insurance contracts. An important theme covered is the projection of interest in deterministic and stochastic framework. While dealing with the deterministic component, we have included the derivation of effective and nominal interest rates. These rates are extended to include the techniques of discounting to evaluate their present values. The general Cash-Flows in insurance is discussed taking various life insurance forms such as whole life and pure endowments.

11.10 KEY WORDS

Annuity: A sequence of payments of a limited duration.

Complete Expectation of Life or Complete Future Life Time: The average number of years a person of given age can be expected to live under the prevailing mortality conditions. It gives the number of years of life entirely completed and includes the fraction of the year survived in the year in which death occurs which on the average is taken as ½ year. So complete future life time = curtate future life time + ½ .

Conversion Period: The time interval at the end of which interest is credited or compounded.

Curtate Future Life Time: The average number of completed future years lived by the group attaining age x.

Discount Rate: When interest is credited of the beginning of conversion period.

Effective Interest Rate: When the conversion period and basic time unit are identical, interest rate is credited at the end of basic time unit.

Life Table: A conventional method of expressing the most essential facts about the age distribution of mortality in a tabular form and it gives the life history of a hypothetical population which is gradually diminished by deaths. It is a powerful tool for measuring the probability of life and death at various age sectors.

Net Single Premium: The expected value of present value of the payment.

Nominal Interest Rate: When the conversion period does not coincide with basic time unit.

Perpetuity: A sequence of payments of an unlimited duration.

11.11 SOME USEFUL BOOKS

Batten,R.W.(1978) Mortality Table Construction, Prentice-Hall Englewood diffs, New Jersey

Butcher, M.V., Nesbitt C.J. (1971) Mathematics of Compound Interest. Ulrich's Books, Ann Arbor, Michigan

Jordan, C.W. (1967) Life Contingencies, Second Edition, Society of Actuaries Chicago, III

Salih, N. Neftci (1996), An Introduction to the Mathematics of Financial Derivatives, Academic Press. San Diego-California, USA

11.12 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Read 11.2 and 11.3 and answer.

2) Use equation 11.3.4 and tabulate the result.

3) Read 11.5 with the example and answer.

Check Your Progress 2

1) Read 11.6 and use equation and 11.6.15 and 11.6.16 and answer.

2) Read 11.6 and use equation 11.6.32 and 11.6.33 and answer.

3) Read 11.8 and answer.

11.13 EXERCISES

1) Show that $i^{(m)} - d^{(m)} = \dfrac{i^{(m)} \, . \, d^{(m)}}{m}$.

2) A deposit of 100,000 is made into a newly established fund. The fund pays nominal interst of 12% convertible quarterly. At the end of each six months a withdrawal is made from the fund. The first withdrawal is X, the second is 2X, the 3rd is 3X and so on. The last is sixth withdrawal, which exactly exhausts the fund, Calculate X.

3) Given: (i) $A_{x:\,\overline{n|}} = 4$, (ii) $A^{1}_{x:\overline{n|}} = y$; (iii) $A_{x+n} = z$. Determine the value of A_x in terms of u, y and x.

4) Z_1 is the present value random variable for an n year continuous endowment insurance of 1 issued to x. Z_2 is the present value random variable for an n year continuous term insurance of 1 issued to x. Given

   (i) $Var(Z_2) = 0 \cdot 01$, (ii) $v^n = 0 \cdot 30$.

   (iii) ${}_n p_x = 0 \cdot 8$, (iv) $E(Z_2) = 0 \cdot 04$.

   Calculate $Var(Z_1)$.

5) A loan of 1000 at a nominal rate of 12% convertible monthly is to be repaid by six monthly payments with the first payment due at the end of one month. The first three payments are x each, and the final three payments are $3x$ each, calculate x.