UNIT 9 VECTOR ANALYSIS

Structure

9.0 OBJECTIVES

After going through this unit you will be able to:

- understand the presentation of variables with magnitude and direction;
- carry on the basic operations of addition, subtraction multiplication and division of vectors;
- deal with the problems in quadratic forms; and
- do the differentiation of vectors.

9.1 INTRODUCTION

Vectors as ordered sets

Generally elements of a set can be arranged in any manner. It does not matter whether a three element set is written as (a, b, c) or (a, c, b) or (b, c, a). But for certain purposes this is not helpful and we need to introduce the consideration of order. Sets are distinguished not only by their members but also by the order in which they appear. Thus, as an '*ordered*' set the pair (a, b) is different from (b, a) unless $b \equiv a$. An ordered set of this kind is called a '*vector*'.

Here we shall consider only vectors with a finite number of elements. Further, unless stated otherwise, we shall assume that each element in a vector is a real number. Thus, we are typically concerned with the n-type $(x_1, x_2, \ldots\ldots, x_n)$ of real number. A vector all of whose elements are zero is called the '*null vector*'. In economic analysis, quantities of various commodities consumed by an individual can be represented as a vector.

9.2 VECTOR ADDITION AND SCALAR MULTIPLICATION

A given vector, i.e., ordered set of finite number of elements, can be written as a row, e.g., (a, b, c...) or, as a column, e.g., $\begin{pmatrix} a \\ b \\ c \\ . \\ . \end{pmatrix}$.

These are referred as row and column vectors respectively.

Let us consider $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in which each $a_i \in \mathbb{R}$. **a** is then said to be a vector over the field $\mathbb{R}$. Like ordinary number, sum and difference of two vectors can be defined provided they have the same number of elements.

Thus, if, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_n)$

then $\mathbf{a} + \mathbf{b} = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$ and

$\mathbf{a} - \mathbf{b} = (a_1-b_1, a_2-b_2, \ldots, a_n-b_n)$

Note that here $\mathbf{a}=\mathbf{b}$ if and only if $a_1=b_1, a_2=b_2, \ldots, a_n=b_n$.

Further, if λ is a real number, we can define the scalar product $(\lambda.\mathbf{a})$ as follows: $(\lambda.\mathbf{a}) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$

Example 1: $\mathbf{a} = (1\ \ 3\ \ -2)$, $\mathbf{b} = (4\ \ 0\ \ 6)$

Then $\mathbf{a} + \mathbf{b} = (1+4\ \ 3+0\ \ -2+6) = (5\ \ 3\ \ 4)$;

$\mathbf{a} - \mathbf{b} = (1-4\ \ 3-0\ \ -2-6) = (-3\ \ 3\ \ -8)$ and $3\mathbf{a} = (3\ \ 9\ \ -6)$

Check Your Progress 1

1) List the Difference between vector and scalar presentation.

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2) If $\mathbf{a_1} = (2\ \ 3\ \ 4\ \ 7)$, $\mathbf{a_2} = (0\ \ 0\ \ 0\ \ 1)$ and $\mathbf{a_3} = (1\ \ 0\ \ 1\ \ 0)$, find $\mathbf{a_1} + 2\mathbf{a_2} + 3\mathbf{a_3}$.

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3) If $\mathbf{x_1} = (2\ \ 9\ \ 8)$, $\mathbf{x_2} = (0\ \ 1\ \ 0)$ and $\mathbf{x_3} = (1\ \ 0\ \ 1)$, find $\mathbf{2x_2 + 5x_3 - x_1}$.

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9.3 GEOMETRICAL AND PHYSICAL INTERPRETATIONS

Vector is often represented geometrically by a line with an arrowhead on it $\left(say, \vec{_x}\right)$. The length of the line indicates the magnitude of the vector, and the arrow denotes its direction. It should be noted that a vector is not a number. If we are considering vector lying in a plane, then two numbers are needed to describe any vector: one for its magnitude and another giving its direction (the angle it makes with one of the coordinate axes). If vectors in three-dimensional spaces are being studied, three numbers are needed to describe any vector: one number for its magnitude, and two to denote its orientation with respect to some coordinate system.

In general, a vector may originate at any point in space and terminate at any point. Physical quantities such as force are, of course, independent of where one places the vector in a coordinate system. They depend only on the magnitude and direction of the vector. For this reason, it is convenient to have a vector start always at the origin of the coordinate system as in Figure 9.1. Here we shall, for simplicity adopt the convention that all vectors begin at the origin of the coordinate system rather than at some other point in space.

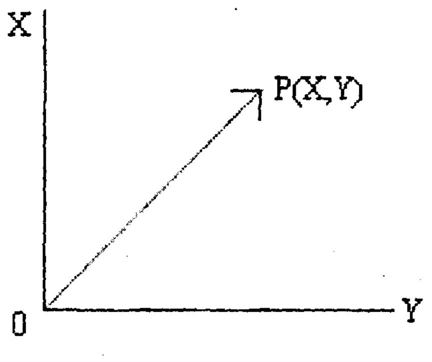

Fig. 9.1

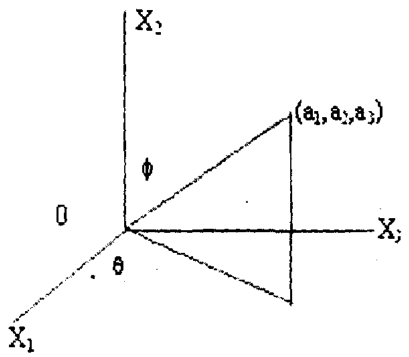

Fig. 9.2

Consider a vector in Figure 9.2. It will be observed that by specifying point (a_1 a_2 a_3), that is, the point where the head of the vector terminates, we have completely characterized the vector. Its magnitude (in some physical units) is $\sqrt{a_1^2+a_2^2+a_3^2}$ and its direction is characterised by the two angles θ and ϕ, where $\tan\theta = \frac{a_2}{a_1}$ and $\cos\phi = \frac{a_3}{a_1^2+a_2^2+a_3^2}$.

Thus, there is a one-to-one correspondence between all points in space and all vectors, which emanate from the origin. For any given point (a_1, a_2, a_3), a corresponding unique vector can be drawn from the origin to this point. Conversely, for any given vector **a**, there is a unique point (a_1, a_2, a_3), which is the point where the vector terminates. Because of this correspondence between vectors and points we can write **a** = (a_1, a_2, a_3).

9.4 NORM AND INNER PRODUCT

Both scalar multiplication and vector addition transform one or a pair of vectors into another vector. We can also define certain functions on one or a pair of vectors, which are useful in defining their lengths and angles between them. Unlike the geometrical representations defined in the preceding section, we are now dealing with functions, which transform a vector into a scalar, in the present case a real number. These are as follows: consider a vector **a** = (a_1, a_2, …, a_n).

We can define the '*norm*' of **a** as $\|\mathbf{a}\| = a_1^2 + a_2^2 + \ldots + a_n^2$

Example 2: If **a** = (5 -2 3) then $\|a\| = 5^2+(-2)^2+3^2=38$

The positive square root of the norm is called the length (or magnitude, as stated earlier) of **a**, and denoted by ℓ(**a**).

Thus, $\ell(\mathbf{a}) = \|\mathbf{x}\|^{1/2} = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$

For a pair of vectors (consisting of the same number of elements n):

a = (a_1, a_2, ..., a_n) and **b** = (b_1, b_2, …., b_n). We can define their inner product as

$$< a, b > = a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i-1}^{n} a_ib_i$$

Example 3: If **a** = (1, -1, 2); **b** = (-2, 3, 6),

then $< \mathbf{a}, \mathbf{b} > = 7$

Clearly, the norm defined above can be regarded as the '*inner product*' of vector **a** with itself.

Two vectors are said to be orthogonal to each other if their inner product is zero, i.e., <**a**, **b**> = 0

This would hold if, e.g., **a** = (1, 3), **b** = (-3, 1). It is easy to see that that null vector **0** is orthogonal to every other vector.

Using the distances and the inner product, we can now define

$$\cos\theta = \frac{< \mathbf{a}, \mathbf{b} >}{\{\|\mathbf{a}\| \cdot \|\mathbf{b}\|\}^{1/2}}$$

By Schwartz's inequality, it is shown that $\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$

$\Rightarrow \cos^2\theta \leq 1$ as it should be.

Two results follow right away from the above definition.

a) If $\mathbf{b} = \lambda\mathbf{a}$ i.e. **b** is a scalar multiple of a, then

$\|\mathbf{b}\| = \lambda^2 \|\mathbf{a}\|$ and $< \mathbf{a}, \mathbf{b} > = \lambda \|\mathbf{a}\|$

$\Rightarrow \cos\theta = 1$ or $\theta = 0$

b) If <**a**, **b**> = 0 i.e., **a** and **b** are orthogonal,

then $\cos\theta = 0 \Rightarrow \theta = \pi/2$. In other words, **a** and **b** are perpendicular to each other.

Check Your Progress 2

1) Given $x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, find 2y and x-y graphically.

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2) Find the norm of the following vectors: (i) (2, 5); (ii) (-2, 2).

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3) Find the inner product of the following pairs of vectors:

(i) (2, 3, 4) and (4, 5, 5); (ii) (-2, -3, 4) and (4, 5, -6).

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9.5 VECTOR SPACES AND SUBSPACES

Let us now define a set V such that it contains, as its elements, all n – component vectors that can be generated from the field of real numbers, i.e., $V=\{(x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R} ; i = 1, 2,..., n\}$

Properties:

i) If **x**, **y**, **z**, $\in V$, then $\mathbf{x} + \mathbf{y} \in V$

such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

and, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{Y}) + \mathbf{z}$.

V contains a vector 0 such that $x + 0 = x$

ii) For any $\lambda, \theta, \delta \in \mathbb{R}$, it must be true that $\lambda.\mathbf{x} \in V$

such that $\lambda.(\delta.\mathbf{x}) = (\lambda.\delta).\mathbf{x}$, $\lambda.(\delta.\mathbf{x}) = \delta\ (\lambda.\mathbf{x})$ and $1.\mathbf{x} = \mathbf{x}$

iii) $\lambda(\theta + \delta)\mathbf{x} = (\lambda.\theta + \lambda.\delta)\mathbf{x}$ and $\lambda(\mathbf{x} + \mathbf{y}) = \lambda.\mathbf{x} + \lambda.\mathbf{y}$

The set V satisfying i, ii and iii is said to be a vector space or a linear space over the field $\mathbb{R}$. The aforesaid properties imply that V is closed under vector addition as well as under scalar multiplication of vectors. Both operations are commutative and associative. Scalar multiplication is distributive over vector addition. And V contains an element 0, the null vector, in which each component is equal to zero.

w is said to be a subspace of V if

a) w is a subset of V and,

b) w is a vector space in its own right, i.e., it satisfies all the axioms specified above for a vector space.

Example 4: Consider a two dimensional plane, V_2, and suppose the set w consisting of all points lying along the horizontal (or the vertical) axis. w is a subset of V_2 and it satisfies all the properties listed above. In fact, all points on any line drawn through the origin constitute a subspace. Algebraically,

$$w = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$

$$\text{or, } w = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$

or, $w = \{(x_1, x_2) | c_1x_1 + c_2x_2 = 0\}$

are subspaces of $v_2 = \{(x_1, x_2) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$

9.6 LINEAR DEPENDENCE OF VECTORS

A set of vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ are said to be linearly independent if there exists scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \ldots + c_n\mathbf{x}^n = 0$

On the contrary, if no such scalars exist, then the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent.

The term '*linearly*' in the above definition is important because only linear operations, i.e., scalar multiplication and vector addition, are permitted in obtaining the null vector.

Two important results regarding linear independence or lack of it are as follows:

a) If $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$, a subset of a set of vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ (n>k), are linearly dependent, then the entire set is linearly dependent.

b) If a set of vectors $x^1, x^2, \ldots, x^n$ are linearly independent, then any subset of vectors, say, $x^1, x^2, \ldots, x^k$ (k<n) are also linearly independent.

Example 5: Consider two vectors $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$

To put differently, $3\mathbf{x} - \mathbf{y} = 0$.

In other case, $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and $\mathbf{z} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

so, $\mathbf{x} - \mathbf{y} - \mathbf{z} = 0$

i.e., in each of these cases any of the given vectors can be obtained from the remaining vectors through scalar multiplication and vector addition (or subtraction); hence, vectors are linearly dependent.

9.7 GENERATORS AND BASIS

Consider the case of two-dimensional Euclidian space (E^2).

[Note: An n-dimensional Euclidian space (Euclidian vector space) is defined as the collection of all vectors (points) $a = [a_1, a_2, \ldots, a_n]$ and it holds the addition and multiplication property and the concept of distance between the vectors is also applicable here]

Now take any vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It may be generated from vectors such as $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ etc., as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c_2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + c_3\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

By specifying all possible values for c_1, c_2 and c_3 we may be able to obtain all possible two element vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belonging to the two dimensional space. If this is the case, then we say that u, v and w are the '*generators*' of the two dimensional spaces (E^2).

Example 6: Consider the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. This can be expressed as

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{or } 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, we can generate the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in a variety of ways by taking a set of vectors such as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then after a suitable choice of scalars c_1, c_2, etc., get the scalar product of these vectors and finally obtain the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ as the sum of vectors so obtained. So, we can say that $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is generated by vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and so on.

Let us discuss some important points that have to be kept in mind. First, an arbitrary choice of u, v and w may not be good enough to generate all vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Second, we need at last two vectors for this purpose. No single vector is, by itself, capable of giving us all elements of E^2. Third, the choice of vectors such as u, v and w can be made in a large number of alternative ways. In other words, such a set of vectors is not unique. Any set of such vectors is said to be a set of generators for E^2.

Now, it is clear that we need at least two vectors to generate the vectors in a plane. If these two vectors are linearly independent we can also assert that any additional vector would be unnecessary for our purpose. In fact, we need exactly two linearly independent vectors to generate all vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Any (two) vectors, which are linearly independent and are used to generate the elements of E^2 are said to constitute the '*basis*' of E^2.

To sum up, in general, we can say that a set of vectors $a_1, a_2, \ldots, a_k$ from E^n is said to span or generate E^n if every vector in E^n can be written as a linear combination of $a_1, a_2, \ldots, a_k$ and a basis for E^n is a linearly independent subset of vectors from E^n which spans the entire space.

Example 7: The vector **a** = [2, 3, 4,] can be written uniquely in terms of the vectors

e_1 = [1, 0, 0], e_2 = [0, 1, 0], and e_3 = [0, 0, 1] which form a basis for E^3: a = $2e_1 + 3e_2 + 4e_3$

9.8 VECTORS AND MATRICES

Before discussing the relationship between vectors and matrices, let us first introduce the concept of matrix. Though detail discussion will be given in the next unit, here we'll just introduce the basic concepts of matrices.

A system of 'm' linear equations in 'n' variable (x_1, x_2,..., x_n) can be arranged into the following format:

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = d_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = d_2$

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$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = d_m$

Here variable x_1 appears only within the leftmost column, and in general, the variable x_j appears only in the j^{th} column on the left side of the equals sign. The double – subscript parameter symbol a_{ij} represents the coefficient appearing in the i^{th} variable. For example, a_{21} is the coefficient in the second equation, attached to the variable x_1. The parameter $d_{i,\text{ which}}$ is unattached to any variable, on the other hand, represents the constant term in the i^{th} equation. For instance, d_1 is the constant term in the first equation. All subscripts are, therefore, keyed to the specific locations of the variables and parameters in the above set of equations.

There are three types of elements in the above equation system. The first is the set of coefficients a_{ij}; the second is the set of variables x_1, x_2,..., x_n and the last is the set of constant terms d_1, ..., d_m. If we arrange the three sets as three rectangular arrays and label them, respectively, as A, X and d (without subscripts), then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mxn} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{nx1} \quad d = \begin{bmatrix} d \\ d \\ \vdots \\ d_m \end{bmatrix}_{mx1}$$

Example 8: Consider the following linear – equation system

$$6x_1 + 3x_2 + x_3 = 22$$
$$x_1 + 4x_2 - 2x_3 = 12$$
$$4x_1 - x_2 + 5x_3 = 10$$

We can write,

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

Each of the term given within [ ] constitutes a matrix. In particular, a matrix is defined as a rectangular arrow of numbers, parameters or variables. The members of the array, referred to as the elements of the matrix and the location of each element in a matrix is unequivocally fixed by the subscript, every matrix is an ordered set. The number of rows and the number of columns in a matrix together define the dimension of the matrix. Since matrix A contains m row and n columns, it is said to be of dimension m x n (read: 'm by n'). Matrix with only one column is called column vector and with only one row is called row vector.

So, there exists pronounced similarity between matrices of one row or column and vector. Indeed, even the notation is the same. Let us point out the complete equivalence between vectors and matrices having a single row or column. First, if one examines the definition of equality, addition and multiplication by a scalar for both vectors and matrices, it will be observed that they are equivalent when a matrix has a single row or column. Furthermore, the scalar product of a row vector and a column vector corresponds precisely to matrix multiplication, when the pre-multiplier is a row matrix, and the post multiplier a column matrix. We shall see later that the notation indicating the scalar product of two column vectors or two row vectors will again be identical with the appropriate matrix notation. Hence, here is a complete equivalence between matrices of one row or one column and vectors.

9.9 CHARACTERISTIC VALUE PROBLEM

A problem which arises frequently in application of linear algebra is that of finding values of a scalar parameter λ for which there exists vector $x \neq 0$ (i.e., non-null vector) satisfying $Ax = \lambda x$, where A is a given nth order square matrix. Such a problem is called a characteristic value (*'Eigen Value'*, or proper value) problem. If $x \neq 0$ satisfies the equation $Ax = \lambda x$ for a given x, then A operating on x yields a vector which is a scalar multiple of x.

Thus, λ is called the characteristic value (or equivalently, characteristic root, Eigen value or proper value) and x is called the characteristic vector (or *'Eigen Vector'*) associated with A if $Ax = \lambda x$

9.9.1 Characteristic Equation

To solve simultaneously for n and for λ, the system of equation $Ax = \lambda x$ can be written as $(\mathbf{A} - \lambda I)\mathbf{x} = 0$, where λ is a unit matrix of order n. Now, if the system has to have a non-trivial solution for x then rank$[\mathbf{A} - \lambda \mathbf{I}] < n$.

[Note: the concept of rank of the matrix will be discussed in detail in the next chapter.]

See that $[\mathbf{A} - \lambda \mathbf{I}] = 0$

i.e, if and only if $$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

This gives us an n^{th} degree polynomial equation in λ. This is called the *'characteristic equation'* of A and the left hand side is called the *'characteristic polynomial'*.

Example 9: Characteristic polynomial for second order matrix is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \quad \begin{array}{l} \Rightarrow \quad (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \\ \Rightarrow \quad \lambda^2 - \lambda\,(a_{11} - \lambda) + (a_{11}a_{22} - a_{12}a_{21}) = 0 \end{array}$$

which is a quadratic equation in λ.

[Note: if A is of order n, then the equation $[A - \lambda I] = 0$ has n roots which may be real or complex, distinct or multiple, and zero or non-zero.]

9.9.2 Sum and Product of Roots

Consider the equation $\lambda^2 - \lambda(a_{11} - \lambda) + (a_{11}a_{22} - a_{12}a_{21}) = 0$ and let us write it as $c_0 \lambda^2 + c_1 \lambda + c_2 = 0$ where, $c_0 = 1$, $c_1 = -(a_{11} + a_{22})$, $c_2 = a_{11}a_{22} - a_{12}a_{21}$

Applying the knowledge of determining roots of the quadratic equation, we can write,

$$\lambda_1 + \lambda_2 = -\frac{c_1}{c_0} = a_{11} + a_{22}$$

and $\lambda_1 \lambda_2 = \frac{c_2}{c_0} = a_{11}a_{22} - a_{21}a_{12} = |A|$

In general, if A is of order n, then the characteristic equation $[A - \lambda I] = 0$ can be written as $c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_{n-1}\lambda + c_n = 0$

where $c_0 = 1, c_1 = -\sum_{i=1}^{n} a_{ii}$, and $c_n = (-1)^n |A|$

C_2 = Sum of all principal minors of A of order i.

$C_j = (-1)^j \sum$ all principal minor or order j

Thus, if characteristic roots are $\lambda_1, \lambda_2, ..., \lambda_n$,

then $\sum_{i=1}^{n} \lambda i = \sum_{i=1}^{n} a_{ii} = \text{Trace}(A)$ and $\lambda_1.\lambda_2. .... .\lambda_n = \prod^n_{i=1} \lambda_i = |A|$

Example 10: Let, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is

$$[A - \lambda I] = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = 0$$

or, $4 - 4\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} = 3, 1 \quad \text{i.e. } \lambda_1 = 3, \lambda_2 = 1$$

Properties:

i) If A is singular, i.e., $|A| = 0$, then at least one characteristic root of A is zero.

ii) If A is symmetric all roots are real, i.e., a real symmetric matrix cannot have complex characteristic roots.

iii) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the roots of a non-singular matrix A, then the roots of A^{-1} are $1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n$

iv) If A and B are two square matrices of the same order such that AB=BA and their roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$, then the roots of AB are $\lambda_1\mu_i$; i =1, 2...., n.

v) The eigen values of a diagonal or a triangular matrix coincide with the diagonal elements.

vi) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of A and k is an integer, then the roots of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.

9.9.3 Characteristic Vector

Let us now consider the determination of **x** of the equation $\mathbf{A}x = \lambda\,\mathbf{x}$. Note that λ is chosen in such a way that $[\mathbf{A} - \lambda\mathbf{I}]$ is singular, i.e., of rank less than n. This ensures that $[\mathbf{A} - \lambda\mathbf{I}]\,\mathbf{X} = 0$ is a homogeneous system of equations having a non-trivial solution. But this solution will be non-unique. We may make it unique through some additional restrictions. Further, one set of non-trivial solution x^i will correspond root of λ_i.

Example 11: Let us find the characteristics vector for the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

As we derived earlier, $\lambda_1 = 3,\ \lambda_2 = 1$.

$\therefore$ For $\lambda_1 = 3$, $(A - \lambda_i I) = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

To get $x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$ we have $\left.\begin{aligned} -x_1^1 + x_2^1 &= 0 \\ x_1^1 - x_2^1 &= 0 \end{aligned}\right\} \Rightarrow x_1^1 = x_2^1$

So, the number of solutions is infinite. Let us choose one by setting $x_1^1 = 1$,

so that $x^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next, take $\lambda_2 = 1$, and proceeding in the same way, we get, $x_2^2 = -x_1^2$

Again, setting $x_2^2 = 1$, we get $x^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

we could normalize x^1 and x^2, i.e., define these such that $\|x^1\| = \|x^2\|$. This would give us $x^1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $x^2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Note: x^1 and x^2 are orthogonal to each other.

9.9.4 Diagonalisation

It is a general property of real symmetric matrices that their roots are always real and that the associated characteristic vectors are mutually orthogonal. This implies that such a matrix can always be diagonalized by P, the matrix of n characteristic vectors.

It we define a 2 x 2 matrix as $P \equiv (x^1 x^2) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Then P'P = I, i.e., P is an orthogonal matrix.

$$\text{and } P'AP = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

i.e., $P'AP = \Lambda$, where Λ is the diagonal matrix consisting of the characteristic values of A as its diagonal elements.

9.10 LINEAR INDEPENDENCE OF EIGEN VECTORS

Another important property in this context is that, if $\lambda_{1,} \lambda_{2,} ......\lambda_{n,}$ are distinct, then the corresponding characteristic vectors of **A**, $x_1, x_2, ..., x_n$, are linearly independent. The condition (that roots are distinct) is sufficient but not necessary. Even when some root (or roots) is repeated, it may be possible to find a set of n characteristic vectors, which are linearly independent.

Moreover, if two roots are distinct then the characteristic vectors corresponding to them will be orthogonal.

9.11 QUADRATIC FORMS

Consider the expression $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ where a_{11}, a_{12} and a_{22} are given constants and x_1 and x_2 are real variables. Expressions such as this, being homogeneous of degree 2 in x_1 and x_2, is called quadratic forms.

Generally, the sign of the expression would depend not only on a's but also on the values of x_1 and x_2. But some times given the values of a's, it is possible to determine the sign of the expression, whatever x_1 and x_2 may be.

For example, $Q(x_1, x_2) = x_1^2 - 4x_1x_2 + 4x_2^2 = (x_1 - 2x_2)^2$

So, the sign of this is positive for all x_1 and x_2.

Similarly, $Q(x_1 x_2) = -10x_1^2 + 6x_1x_2 - x_2^2 = -x_1^2 - (3x_1 - x_2)^2$

Therefore, the sign is negative for all (x_1, x_2). Such quadratic forms, which have a definite sign for all (x_1, x_2) are said to be either positive definite or negative definite depending on the sign. Others, which could vary in sign, are said to be indefinite.

In general, with variables $x_1, x_2,..., x_n$, we may have a quadratic form as:

$$Q(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j$$

Thus, Q(.) is positive definite if $Q(x) > 0 \ \forall \ x \in \mathbb{R}_n$

Or, positive semi-definite if $Q(x) \geq 0 \ \forall \ x \in \mathbb{R}_n$

Identical definitions hold for negative definite and negative semi-definite quadratic forms.

Consider the equation: $Q(x_1,x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

$$= a_{11}\left[(x_1 + \frac{a_{12}}{a_{11}}x_2)^2 + \left(\frac{a_{22}}{a_{11}} - \frac{a_{12}^2}{a_{11}^2}\right).x_2^2\right]$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \left(\frac{a_{11}a_{22} - a_{12}^2}{a_{11}}\right)x_2^2$$

Clearly, $Q > 0$ for all (x_1, x_2) if $a_{11} > 0$ and $\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} > 0$

Thus, $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$ are sufficient conditions for $Q > 0$. These are necessary also, as for $a_{11} < 0$, Q can be less than zero with high x_1 and low x_2 or as for $(a_{11}a_{22} - a_{12}^2) < 0$. Q can be less than zero if $a_{11} < 0$.

9.12 DEFINITENESS AND EIGEN VALUES

A necessary and sufficient condition for

$Q \geq 0$ is that $|A| = 0$ and $a_{11} \geq 0$.

If $a_{11} > 0$, then $a_{22} = 0$.

$$\left[Here\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right]$$

In terms of characteristic roots, the necessary and sufficient conditions are that $[A - \lambda I] = 0$ $\lambda_1 \geq 0$, $\lambda_2 \geq 0$. At least one root must be zero.

For Q to be negative semi definite, the corresponding conditions are

$a_{11} \leq 0$ and $|A| = 0$

And in terms of characteristic roots $\lambda_1 \leq 0$, $\lambda_2 \leq 0$, equality must hold in at least one case.

9.13 VECTOR DIFFERENTIATION

In some maximisation and minimisation problems, expressions involving vectors and matrices must be differentiated. The derivatives of simple expressions involving vectors and matrices can be obtained directly from the definitions. The vector derivatives of linear functions, quadratic forms and bilinear forms are defined and illustrated in this section. Second and higher order derivatives are not discussed explicitly, but can be obtained by successive differentiation in the usual manner.

Vector Differentiation of a Linear Function

An expression of the form **a'x**, where **a** is nx1 and **x** is nx1 is a linear function. For the linear function $\mathbf{a'x} = [a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

the partial derivative of **a'x** with respect to the scalar x_i, i= 1, 2, … , n, are

$$\frac{\partial}{\partial z}(x'Bz) = B'x$$

that is, the partial derivatives are the elements of the vector **a**. Thus, if the n partial derivatives are arranged as a vector **a**, the process of vector differentiation is defined by $\frac{\partial(a'x)}{\partial x} = a$ where $\frac{\partial(a'x)}{\partial x}$ indicate the operation of differentiating **a'x** with respect to the elements of the vector **x**.

Check Your Progress 3

1) Define the terms: eigen value, eigen vector and characteristic equations.

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2) If $a = \begin{bmatrix} 2a \\ -a \\ 3a \\ a \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then prove that $\frac{\partial}{\partial x}(a'x) = a$

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Vector Differentiation of a Vector of Functions

If **y** is an n dimensional column vector each element of which is a function of the m elements of **x**, that is, if $y_i = f(x_1, x_2, \ldots, x_m)$, i=1, 2, … , n, then each y_i can be differentiated partially with respect to each x_j and these partial derivatives can be arranged in an m x n matrix as follows:

$$\frac{\partial y}{\partial x} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_m} & \frac{\partial y_2}{\partial x_m} & \cdots & \frac{\partial y_n}{\partial x_m} \end{vmatrix}$$

Example: If $y = \begin{bmatrix} x_1^2 + 2x_2^2 - 3x_1x_3 \\ 3x_1x_2^2 - x_2^2 + 4x_2x_3^2 \end{bmatrix}$,

then $$\frac{\partial y}{\partial x} = \begin{bmatrix} 2x_1 - 3x_3 & 3x_2^2 \\ 4x_2 & 6x_1x_2 - 2x_2 + 4x_3^2 \\ -3x_1 & 8x_2x_3 \end{bmatrix}$$

Vector Differentiation of a Quadratic Form

An expression of the form **x'Ax**, where **a** is an n x n symmetric matrix, is a quadratic form. The quadratic form **x'Ax** can be expanded as follows

$$\mathbf{x'Ax} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + 2a_{13}x_1x_3 + \ldots + 2a_{1n}x_1x_n + a_{22}x_2^2 + 2a_{23}x_2x_3 + \ldots + 2a_{2n}x_2x_n + \ldots + a_{nn}x_n^2.$$

Taking partial derivatives with respect to the elements of **x**

$$\frac{\partial}{\partial x_1}(x'Ax) = 2(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)$$

$$\frac{\partial}{\partial x_2}(x'Ax) = 2(a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)$$

....

$$\frac{\partial}{\partial x_n}(x'Ax) = 2(a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n)$$

Apart from the factor 2, the right hand side of the set of equations contains the elements of **Ax**, which is an n dimensional column vector; alternatively. The right hand side contains the elements of **x'A**, which is an n dimensional row vector. Thus, $\frac{\partial}{\partial x}(\mathbf{x'Ax}) = 2\mathbf{Ax}$ or $\frac{\partial}{\partial x}(\mathbf{x'Ax}) = 2\mathbf{x'A}$

In practice, the choice between these two forms usually depends on the context in which differentiation occurs, since matrices can be equated only if they are of the same order and, specifically, a row vector cannot be set equal to a column vector.

Check Your Progress 4

1) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix}$ then show that $\frac{\partial}{\partial x}(x'Ax) = 2x'A$

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2) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 2b_1 & b_1 + b_2 \\ b_2 + 2b_3 & 3b_2 \\ b_1 + b_3 & 4b_3 \end{bmatrix}$ $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

then $\frac{\partial}{\partial x}(x'Bz) = Bz$ and $\frac{\partial}{\partial z}(x'Bz) = B'x$

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Vector Differentiation of a Bi-linear Form

An expression of the form **x'Bz**, where **x'** is 1xm, **B** is mxn, and **z** is nx1, is a bilinear form. The following derivatives for a bilinear form can be verified using the procedures above.

$$\frac{\partial}{\partial x}(x'Bz) = Bz \text{ and } \frac{\partial}{\partial z}(x'Bz) = B'x.$$

9.14 LET US SUM UP

This unit tells us the way of presenting a variable with magnitude and direction. It also gives the idea about the basic operations of vector such as addition, subtraction, multiplication and division along with the concepts such as norm, inner product, generator, basis, characteristic equation, eigen value, and eigen vector. The relationship between vector and matrices is covered through this unit. We have learnt the way of testing the definiteness of an equation of quadratic forms. It gives a method of finding the solutions of an equation system by applying differentiation of vectors.

9.15 KEY WORDS

Basis: Any (two) vectors, which are linearly independent and are used to generate the elements of E^2 are said to constitute the basis of E^2.

Characteristic Equation: To solve simultaneously for n and for λ, the system of equation $Ax = \lambda x$ can be written as $(A - \lambda I)x = 0$, where λ is a unit matrix of order n. Now, if the system has to have a non-trivial solution for x then $\text{rank}[A - \lambda I] < n$. $\Rightarrow [A - \lambda I] = 0$ i.e., if and only if

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

This gives us an n^{th} degree polynomial equation in λ. This is called the 'characteristic equation' of A and the left hand side is called the 'characteristic polynomial'.

Eigen Value and Eigen Vector: If there exist a vector $\mathbf{X} \neq \mathbf{0}$ (i.e. non-null vector) satisfying $\mathbf{AX} = \lambda \mathbf{X}$, where **A** is a given nth order square matrix and λ is a scalar parameter, then such a problem is called a eigen value problem and x is called the eigen vector associated with A if $Ax = \lambda x$.

Generator: Any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + c_3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \text{ where}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{ etc.,}$$

By specifying al possible values for c_1, c_2 and c_3 we may be able to obtain all possible two element vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belonging to the two dimensional space. If this is the case, then we say that u, v and w are the '*generators*' of the two dimensional spaces (E^2).

Inner Product: For a pair of vectors (consisting of the same number of elements n): **a** = ($a_1, a_2, ..., a_n$) and **b** = ($b_1, b_2, ..., b_{an}$), their inner product can be defined as $< a, b > = a_1b_1 + a_2b_2 + ... + a_nb_n = \sum_{i-1}^{n} a_ib_i$

Norm: For a single vector **a** = ($a_1, a_2, ..., a_n$), the 'norm' of **a** can be defined as $\|a\| = a_1^2 + a_2^2 + ... + a_n^2$

Quadratic Forms: The expression $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ where a_{11}, a_{12} and a_{22} are given constants and x_1 and x_2 are real variables, being homogeneous of degree 2 in x_1 and x_2, is called quadratic forms.

Sub Space: w is said to be a subspace of $\vee$ if

a) w is a subset of $\vee$ and,

b) w is a vector space in its own right, i.e., it satisfies all the axioms specified above for a vector space.

Vector Space: Define a set $\vee$ such that it contains, as its elements, all n – component vectors that can be generated from the field of real numbers, i.e., $\vee = \{(x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R} ; i = 1, 2, ..., n\}$. Now if $\vee$ satisfies the following properties

i) $x, y, z, \in \vee$, then $x + y \in \vee$

such that $x + y = y + x$ and,

$x + (y + z) = (x + Y) + z$.

$\vee$ contains a vector 0 such that $x + 0 = x$

ii) For any $\lambda, \theta, \delta \in \mathbb{R}$,

it must be true that $\lambda. x \in \vee$

Such that $\lambda.(\delta.x) = (\lambda.\delta).x$, $\lambda.(\delta.x) = \delta(\lambda.x)$ and $1.x = x$

iii) $\lambda(\theta + \delta)x = (\lambda.\theta + \lambda.\delta)x$ and $\lambda(x + y) = \lambda.x + \lambda.y$

Then the set $\vee$ is said to be a vector space or a linear space over the field $\mathbb{R}$.

Vector: An ordered set expressed in a particular form like (a, b) with a, b as its elements is called a 'vector'.

9.16 SOME USEFUL BOOKS

Chiang Alpha C.(1984), *Fundamental Methods of Mathematical Economics*, Third edition, McGraw-Hill Book Company.

Chakravorty, J.G & P.R. Ghosh (1996), *Higher Algebra including Modern Algebra*, Twelfth Edition, U.N. Dhur & Sons Pvt. Ltd. Calcutta.

Hadley, G. (2000), *Linear Algebra*, Narosa Publishing House, New Delhi, 2000.

Mukherji, B. & V. Pandit (1982), *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd., New Delhi.

9.17 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) $a_1 + 2a_2 + 3a_3 = (5\ \ 3\ \ 7\ \ 9)$

2) $2x_2 + 5x_3 - x_1 = (3\ -11\ \ -3)$

Check Your Progress 2

1) Do yourself

2) Answer: (i) 9; (ii) 8.

3) Answer: (i) 43; (ii) -47.

Check Your Progress 3

1) See the text and answer

2) [Hint: **a'x** = $2ax_1 - ax_2 + 3ax_3 + ax_4$ then find

$$\frac{\partial}{\partial x_1}(a'x) = 2a,\ \frac{\partial}{\partial x_2}(a'x) = -a,\ \frac{\partial}{\partial x_3}(a'x) = 3a \text{ and } \frac{\partial}{\partial x_4}(a'x) = a\,]$$

Check Your Progress 4

1) [Hint: **x'Ax** $=3x_1^2 + 2x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$ then find

$$\frac{\partial}{\partial x_1}(x'Ax) = 6x_1 + 2x_2 - 4x_3,\ \frac{\partial}{\partial x_2}(x'Ax) = 2x_1 + 6x_3 \text{ and}$$

$$\frac{\partial}{\partial x_3}(x'Ax) = -4x_1 + 6x_2 + 4x_3\,]$$

2) Follow the same logic as given above.

9.18 EXERCISES

1) Derive the norm of the following vectors:

(i) **a**=(0 2 1); **b**=(6 3 2); **c**=(-5 4 3); **d**=(0 0 1); **e**=(-3 -4 -2).

[Answer: (i) 5; (ii) 49; (iii) 50; (iv) 1; (v) 29.]

2) If $b = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, find $\frac{\partial}{\partial x}(b'x)$. [Answer: $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$]

3) If $c = \begin{bmatrix} c^2+1 \\ 3c \\ 4c-5 \\ c^3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, find $\frac{\partial}{\partial x}(c'x)$. [Answer: $\begin{bmatrix} c^2+1 \\ 3c \\ 4c-5 \\ c^3 \end{bmatrix}$]

4) If $y = \begin{bmatrix} x_1^2 + 3x_2 \\ 2x_1x_2 - x_2 \\ 2x_1 + x_1x_2 - 3x_2^2 \\ 3x_1^2 - x_1^2x_2 - x_2^3 \end{bmatrix}$, find $\frac{\partial y}{\partial x}$.

[Answer: $\begin{bmatrix} 2x_1 & 2x_2 & 2+x_2 & 6x_1 - 2x_1x_2 \\ 3 & 2x_1 - 1 & x_1 - 6x_2 & -x_1 - 3x_2^2 \end{bmatrix}$]

5) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$,

find $\frac{\partial}{\partial x}(x'Ay)$ and $\frac{\partial}{\partial y}(x'Ay)$.

[Answer: $y = \begin{bmatrix} 3y_1 - y_3 \\ 2y_1 + y_2 + 4y_3 \\ -y_1 + 3y_3 \end{bmatrix}$]