has the trivial solution $(x = y = z = 0)$ if $D \neq 0$ and has a non-trivial solution

if $a^3 + b^3 + c^3 - 3abc = 0$.

Again, find the solution treating x to be constant.

6) Find the inverse of the matrix $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix}$ and hence obtain the solution of the equations
$2x_1 - 3x_2 + 4x_3 = -4,\ x_1 + x_3 = 0,\ -x_2 + 4x_3 = 2.$

7) Consider the national income model given by

$$Y = C + I + G$$
$$c = a + b(Y - T) \qquad (a > 0,\ 0 < b < 1)$$
$$T = d + tY \qquad (d > 0,\ 0 < t < 1)$$

Solve this model for the variables Y, C and T by matrix inversion.

8) If $U = [1 \quad 0 \quad 1]$, $V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $X = \begin{bmatrix} 4 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then find a. UV; b.VU+X; c. XY

Use partitioned matrices to check the results.

9) If A is a non-zero (n x 1) column matrix, while B is a non-zero matrix, then show that P(BA) is unity and P(AB) is n.

10) Show that rank of the transpose of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix}$ is the same as that the original matrix.

UNIT 11 INPUT-OUTPUT ANALYSIS

Structure

11.0 OBJECTIVES

After studying this unit, you will be able to answer the following:

- with what proportions one sector of the economy are related to other sectors;
- how solution is attained in a framework of several variables production problems related to input-output;.
- how the system of equations input-output can be represented graphically by the linear programming; and
- how the system of input-output can be solved in a dynamic framework.

11.1 INTRODUCTION

The concept of Input-Output (I-O) analysis originated by the eighteenth century French economists Quesnay in the name of 'Tabelau Economique'. In its modern form, the I-O analysis has been developed by the American economist Wassily W. Leontief in his famous work 'Structure of the American Economy' in the year 1951.

I-O analysis is a method of analysing how an industry undertakes production by using the output of other industries in the economy and how the output of the given industries used up in other industries or sectors. Since various industries are interdependent, i.e., the output of one industry is an input for the others, their mutual relationship ultimately must lead to equilibrium between supply and demand in the economy consisting of 'n' industries. I-O analysis is also known as the inter-industry analysis as it explains the interdependence and interrelationship among various industries. Since the analysis is concerned with the relations among inputs, use of a commodity in the production by an industry, it is known as the input-output analysis. In other words, the I-O analysis explains the interdependence of inputs and outputs of various industries in the economy.

11.2 MAIN FEATURES OF INPUT-OUTPUT ANALYSIS

I-O analysis was an attempt made by Prof. Leontief to take account of '*general equilibrium*' phenomena in the '*empirical*' analysis of '*production*'. These three italicized elements are the main features of I-O analysis. First, the I-O analysis deals almost exclusively with production. The problem is essentially technological. Given the quantities of available resources and the state of technology, the analysis is concerned with the use of various inputs by the industries and outputs derived from them.

The second distinctive feature of I-O analysis is its devotion to empirical investigation. This is primarily what distinguishes it from the work of Walras and later general equilibrium theorists. I-O employs a model, which is more simplified and also narrower in the sense that it seeks to encompass fewer phenomena than does the usual general equilibrium theory. Its narrowness lies in its exclusive emphasis on the production side of the economy.

The third distinctive feature is its emphasis of general equilibrium phenomena where everything depends on everything else. Thus in the two-industry model coal is an input for steel industry and steel is an input for coal industry, though both are the output of respective industries. According to I-O analysis, it is not possible to find some industries as being in the 'earlier' stages of production and some other industries as being in the 'later' stages. For the production of coal, steel is needed; whereas, for the production of steel, coal is required. No one can say whether the coal industry or the steel industry is earlier or later in the hierarchy of production.

The basic problem, then, is to see what can be left over for final consumption and how much of each output will be used up in the course of the productive activities which must be undertaken to obtain these net outputs. Solution of this problem can be used in predicting future production requirements if usable demand estimates can somehow be obtained. Particularly, it can be used for economic planning including problems of economic development in 'backward areas' as well as problem of military mobilisation. A more modest purpose that it has already successfully begun to serve is the provision of a very illuminating detailed structure for national income accounting.

11.2.1 Assumptions

As it was stated earlier, the intransigence of empirical materials and the computational problems have forced on I-O analysis to adopt a number of simplifying assumptions even more extreme than those usually employed in our theoretical models.

The economy can be meaningfully divided into finite number of sectors (industries) on the basis of the following assumptions:

i) Each industry produces only one homogeneous output. No two products are produced jointly; but if at all there is such a case, then it is assumed that they (products) are produced in fixed proportions.

ii) Each producing sector satisfies the properties of linear homogeneous production function. In other words, production of each sector is subject to constant returns to scale so that k-fold change in every input will result in an exactly k-fold change in the output.

iii) A far stronger assumption is that each industry uses a fixed input ration for the production of its output; in other words, input requirement per unit of output in each sector remains fixed and constant. The level of output in each sector (industry) uniquely determines the quantity of each input, which is purchased.

iv) The final demand for the commodities is given from outside the system. The total amount of the primary factor is also given. These two are called open end of the system and for this, the model is called '*open model*'. In contrast in the '*closed model*' all the variables are determined within the system.

v) The model is static in the sense that all variables in it refer to the same period of time. The static model is also known as flow model.

Note on assumption (iii)

The fixed proportion assumption is far more restrictive and is a special case of an assumption of constant returns to scale (CRS). CRS is perfectly consistent with the substitution of one factor for another. A liner homogeneous production function (CRS) permits both labor-intensive and capital-intensive processes. The firm whose production function exhibits constant returns can, if it wishes say, to have 100 workers for every $1000 invested in machinery, or it may use machines which require only 10 workers per $1000 machine investment. A linear homogeneous production function requires only that if the firm decides to triple the scale of either of these types of operations, the result will be a tripling of output. Whereas Leontief fixed proportion premise requires that a manufacturing process, which is labor-intensive, offers no option of a capital-intensive alternative. If 53 men per $1000 of investment are required at any level of operation, it is assumed that the same ratio will be required no matter how much the size of the firm expands or contracts. Whether this assumption is relatively innocuous or does considerable violence to the input-output results is still under dispute. But the premise is certainly never absolutely true, even in those cases where chemistry and engineering dictate fixed proportions between some ingredients and outputs.

11.2.2 Input-Output Table

I-O table shows the disposition of the total products and total inputs among the different industries. Let us assume that an economy consists of 4 producing sectors only; and that the production of each sector is being used as the input in all the sectors and used for final consumption. Suppose,

i) X_1, X_2, X_3 and X_4 are total outputs of 4 sectors;

ii) F_1, F_2, F_3 and F_4 are the amounts of final demand, consumption, capital formation and exports for output of these sectors.

iii) X_{ij} be the amount of output of the i^{th} Industry used up as an intermediary input by the j^{th} industry ($i,j = 1,2,3,4$)

iv) L represents the given amount of primary factor (here, labor) and L_i is the amount of primary factor used in the i^{th} Industry.

Then the following table represents the I-O table for the simplified economy.

Producing sector number	Total output of the sector	Input requirements of producing sectors				Requirements for final uses
		X_1	X_2	X_3	X_4	
1	X_1	X_{11}	X_{12}	X_{13}	X_{14}	F_1
2	X_2	X_{21}	X_{22}	X_{23}	X_{24}	F_2
3	X_3	X_{31}	X_{32}	X_{33}	X_{34}	F_3
4	X_4	X_{41}	X_{42}	X_{43}	X_{44}	F_4
Primary input (labor)	Total primary input = L	L_1	L_2	L_3	L_4	

Here it should be remembered that all the items in the above table are flows, i.e. physical units per year. Since the entries in any row are all measured in the same physical units, it makes good sense to add across the rows. The 'total output' column gives the overall input of labor and output of each commodity. On the other hand, items in the same column are not measured in the same units, so that it would not be correct to add down the columns. But each column, thought as a whole (i.e., as a vector) does have a meaning. The third column describes the input or cost structure of the first industry: X_1 units of output of the first industry was produced with the use of X_{11} units of first good, X_{21} units of second good, X_{31} units of third good, X_{41} units of fourth good and L_1 units of labor. Similar meanings will follow for other columns, i.e., Columns 4, 5 and 6. The 'final demand' or 'requirement of final uses' i.e., Column (7) shows the commodity breakdown of what is available for consumption and government expenditure. It is, for convenience, assumed that labor is not consumed directly. Now suppose that each unit of output of each industry has a price of \$1 (say) and each unit of labor receives a wage rate of \$1. Then each entry of the above table can be expressed in terms of money value (rather than being a physical unit). It is then possible to add down the columns. The sum of each column gives the total cost of the corresponding industry. Thus, revenue of industry 1 is X_1 units (= $X_{11} + X_{12} + X_{13} + X_{14} + F_1$) and cost of that industry is ($X_{11} + X_{21} + X_{31} + X_{41} + L_1$) units. It is true for other industries also. Clearly, these prices are long-run competitive equilibrium prices. (In the long-run, competitive equilibrium price is equal to average cost and there is neither profit nor loss).

Those items in the above table that show the sales of the four industries to themselves and to each other might be described as 'non-GNP' items – because these transactions are intermediate transactions, which are not considered in the national income accounting. The 'final demand' column represents the output side of GNP, as final transactions are included in GNP accounting. The labor row represents the factor cost side of GNP. The inter-industrial sales have no welfare significance at all. Social benefits come from final consumption and social cost comes from use of labor. The economy can be viewed as a machine that used up labor and produces final consumption.

The above table represents a particular technology, a particular combination of inputs to get the outputs. If any element, along any row, changes, then other elements have to change accordingly so as to maintain the same total output of that industry. A particular technology is thus characterized by a column vector. Changing any elements of this vector results in a new vector and therefore represents a new technology.

From the above table, the production unction for the four industries can be written as follows: Para $X_1 = f_1 (X_{11}, X_{21}, X_{31}, X_{41}, L_1)$;

$X_2 = f_2 (X_{12}, X_{22}, X_{32}, X_{42}, L_2)$;

$X_3 = f_3 (X_{13}, X_{23}, X_{33}, X_{43}, L_3)$; and

$X_4 = f_4 (X_{14}, X_{24}, X_{34}, X_{44}, L_4)$.

In general terms, if there are 'n' number of producing sectors, then the production function of sector 'n' will be represented by $X_n = f_n (X_{1n}, X_{2n}, X_{3n}, X_{4n}, L_n)$. Further, we can always add across the rows, which, basically gives us the equality between the demand and the supply of each product. In general terms,

$X_1 = X_{11} + X_{12} + \ldots + X_{1n} + F_1;$

$X_2 = X_{21} + X_{22} + \ldots + X_{2n} + F_2;$

....;

$X_n = X_{n1} + X_{n2} + \ldots + X_{nn} + F_n$; and

$L = L_1 + L_2 + \ldots + L_n$

That is $X_i = \sum_{j=1}^{n} X_{ij} + F_i$ and $L = \sum_{i=1}^{n} L_i$.

Here, X_i = total output of the i^{th} sector, X_{ij} = output of i^{th} sector used as input in j^{th} sector, and F_i = final demand for i^{th} sector.

The above identity states that all the output of a particular sector could be utilised as an input in one of the producing sectors of the economy and/or as a final demand. Basically, therefore, I-O analysis is nothing more than finding the solution of these simultaneous equations.

11.2.3 Technological Coefficient Matrix

From the assumption of fixed input requirements, we see that in order to produce 1 unit of j^{th} commodity, the input used of i^{th} commodity must be a fixed amount, which we denote by a_{ij}; thus $a_{ij} = \frac{X_{ij}}{X_j}$. If X_j represents the total output of the j^{th} commodity (or j^{th} producing sector) the input requirements of the i^{th} commodity will be equal to $a_{ij}X_j$ or $X_{ij} = a_{ij}X_j$.

So the production function can be written as follows:

X_1 = min. $(X_{11}/a_{11}, X_{21}/a_{21}, \ldots, L_1/a_{L1})$; ....; X_4 = min.$(X_{14}/a_{14}, X_{24}/a_{24}, \ldots, L_4/a_{L4})$.

In other words, this means that X_i (i =1, 2, 3, 4) will be equal to the minimum of the five ratios within the bracket.

It can be seen that if each of the X_{ij} is multiplied by a constant, the corresponding X_j is multiplied by the same constant so that we have a constant returns to scale. If any a_{ij} is zero, we can ether omit the corresponding term from the right hand side or we can think of X_{ij}/a_{ij} as infinity in which case it will certainly never be the smallest ratio.

An alternative way of writing the production function is to note that since X_1 equals the smallest of X_{11}/a_{11}, X_{21}/a_{21}, X_{31}/a_{31}, X_{41}/a_{41}, L_1/a_{L1}, it must be less than or equal to all five of the ratios, i.e., $X_{11}/a_{11} \geq X_1$, i.e., $X_{11} \geq a_{11}X_1$. Similarly, $X_{21} \geq a_{21}X_1$...and $L_1 \geq a_{L1}X_1$.

In general, $X_{ij} \geq a_{ij}X_j$ and $L_j \geq l_jX_j$.

It should be noted that equality would hold at least once in each row. In fact, if none of the commodities concerned is free good, the equality will hold everywhere. Assuming that no goods are free, we can consider all the above equalities. Then putting the values of X_{ij} in equation (1), (2),..., we get,

$X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + F_1$;

$X_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + F_2$;

....;

$X_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + F_n$; and

$L = l_1X_1 + l_2X_2 + l_3X_3 + l_4X_4$.

In general, $X_i = \sum_{j=1}^{n} a_{ij}X_j + F_j$, (i=1, 2, ..., n) and $L = \sum_{i=1}^{n} l_iX_i$.

The equation may be put in matrix notations:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

$X = AX + F$ and, $L = \sum_{i=1}^{n} l_iX_i$.

Check Your Progress 1

1) What is an input coefficient matrix?

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2) What is an open input-output model?

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3) The input coefficient matrix is given as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}.$$ Get the technology matrix.

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11.3 CLOSED AND OPEN INPUT-OUTPUT MODELS

In the above example, besides n industries, our model contains exogenous sector of final demand, which supplies primary input factors (labor services-which are not produced by these n industries) and consumes the output of n producing industries (not as input). Such an I-O model is known as '*open model*'. It includes exogenous sectors in terms of 'final demand bill' along with the endogenous sectors in terms of n producing sectors. I-O model, which has endogenous final demand vector, is known as '*closed model*'.

11.4 COEFFICIENT MATRIX AND OPEN MODEL

Our open model in matrix notation is given by, $X = AX + F$, where A is the input coefficient matrix, F is the final demand vector and X is the total output matrix.

The input coefficient matrix or 'technology matrix' represented by $[a_{ij}]$ is of great importance. Each element must be non-negative, i.e., we rule out the possibility of negative inputs. But to maintain complete interdependence among the industries each element of $[a_{ij}]$ matrix must be positive and no element can exceed unity, i.e., we rule out the possibility of negative outputs. Each column of this matrix specifies the input requirements for the production of 1 unit of a particular commodity; thus, sum of the elements in each column must be less than unity. Symbolically, this fact may be stated as:

$\sum_{i=1}^{n} a_{ij} < 1$, (j = 1, 2,..., n), and each a_{ij} is non-negative, i.e., either zero or greater than zero. The cost of the primary inputs (which is also termed as '*value added*') needed in producing a unit of j^{th} commodity would be $(1 - \sum_{i=1}^{n} a_{ij})$. [Note here that a_{ij}'s are in value terms]. If this were not true, it would mean that the total value of intermediate products used by an industry exceeded the value of its input. This in turn would mean that the value added by that industry was negative. Now, this is not impossible, but, if we assume that the wage bill cannot be negative, it means that the industry must be making losses (indeed, losses greater in absolute value than its wage bill). An industry in which value added is negative is not covering variable costs (intermediate inputs plus the wage bill), and we know from the elementary micro theory that in such a case, losses will be reduced by closing down. Thus, we do not want to describe such an industry in our technology matrix at all. Given the assumption of CRS, we've described the technology by a constant coefficient matrix. We should notice that we have also built in the assumption that there are no externalities. An externality in production would exists if, for example, a factory discharged waste into a river so that a factory further downstream had to use the resources to clean the water before it could use it. In this case, the resource requirement of the later factory would not depend solely on its output but would, also depend on the activity of the former.

11.5 SOLUTION TO OPEN MODEL

Let us consider an economy with n industries. If producing sector 1 is to produce an output just sufficient to meet the input requirements of the n industries as well as the final demand of the exogenous sector, its output level X_1 must satisfy the following equations:

$X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + F_1$, or,

$(1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = F_1$

For the entire set of n industries, the correct output levels, therefore, can be symbolized by the following set of n linear equations:

$(1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = F_1$
$- a_{21}X_1 + (1-a_{22})X_2 - \ldots - a_{1n}X_n = F_2$

....

$-a_{n1}X_1 - a_{n2}X_2 - \ldots\ldots + (1-a_{nn})X_n = F_n$

In the matrix notation this may be written as:

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}$$

$[I-A]\, X = F$, i.e. $X = [I-A]^{-1} F$

Here A is the given matrix of input coefficients, while X and F are the vectors of output and final demand of each producing sector. If $|I-A| \neq 0$, then $[I-A]^{-1}$ exists, we can then estimate for either of the two matrices X and F by assuming one of them to be given exogenously. It is to be observed that, the assumptions made in I-O analysis go a long way in making the problem simplified. For example, with the assumption of linear homogenous function, it is possible to write a linear equation of each producing sector, which then can be easily transformed into matrix notation. On the other hand, as long as the input coefficients remain fixed (as assumed), the matrix A will not change or [I-A] will not change. Therefore, in finding the solution of $X = [I-A]^{-1} F$, only one matrix inverse needs to be performed even if we are to consider thousand of different final demand vectors according to alternative development targets. Hence, such an assumption of fixed technical coefficient has meant considerable savings in computational effort.

11.5.1 Determination of Gross Output

Suppose, X stands for the column matrix $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, F stands for the column matrix $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, and a stands for the matrix of a_{ij} coefficients $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then from $X = [I-A]^{-1} F$ we can get the value of each element of X. Before this, we need to evaluate $[I-A]^{-1}$.

$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}$, matrix of co-factors $= \begin{pmatrix} 1-a_{22} & a_{21} \\ a_{12} & 1-a_{11} \end{pmatrix}$;

Transpose of this matrix or adjoint matrix of $[I-A] = \begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}$. Now let D $= |I-A|$ = determinant of [I-A] matrix. Assume that $D \neq 0$, i.e., [I-A] is nonsingular. Then the inverse matrix can be written as $[I-A]^{-1} = \begin{pmatrix} \frac{1-a_{22}}{D} & \frac{a_{12}}{D} \\ \frac{a_{21}}{D} & \frac{1-a_{11}}{D} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}$

Therefore, $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

Or, $X_1 = \frac{(1-a_{22})F_1 + a_{12}F_2}{D}$, $X_2 = \frac{a_{21}F_1 + (1-a_{22})F_2}{D}$ where, $D = \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} = (1-a_{11})(1-a_{22}) - a_{12}a_{21}$

11.5.2 The Hawkins-Simon Conditions

Many a times I-O solution may give output expressed by negative numbers. If our solution gives negative outputs, it means that more than one unit of a product is used up in the production of every one unit of that product; it is definitely an unrealistic situation. Such a system is not a viable. Hawkins-Simon condition guards against such eventualities. Our basic equation is $X = [I-A]^{-1} F$, in order that this does not give negative numbers as a solution, the matrix [I-A], which in fact is:

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix}$$ should be such that

i) the determinant of the matrix must always be positive, and

ii) the diagonal elements: $(1-a_{11}), (1-a_{22}), \ldots, (1-a_{nn})$ should all be positive or, in other words, elements, $a_{11}, a_{22}, \ldots, a_{nn}$ should all be less than one. Thus, one unit of output of any sector should use not more than one unit of its own output. These are called Hawkins-Simon conditions. Further, the first condition, that implies $D > 0$, implies that (for 2 industry case) $\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} > 0$, or, $(1-a_{11})(1-a_{22}) - a_{12}a_{21} > 0$. This condition implies that the direct and indirect requirement of any commodity to produce one unit of that commodity must also be less than one. On the other hand, the interpretation is always that all subgroups of commodities should be 'self sustaining', directly and indirectly.

Example 1: Suppose $[A] = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$.

Then $[I-A] = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$ and the value of the determinant [I-A] = (-) 8.12, which is less than zero. As the Hawkins-Simon conditions are not satisfied, no solution will be possible in this case.

Check Your Progress 2

1) Describe the features of a closed input-output model.

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2) Discuss the importance of Hawkins-Simon conditions in an input-output model.

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3) Suppose $[A] = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$, then check whether any solution will be possible for the system or not.

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11.5.3 Consumption Possibility Locus

So long we assumed that the final demands F_1 and F_2 are given and our main problem was to determine the gross outputs of the two industries to support the final demands and also to know whether these gross outputs can be produced with the given labor force or not. Consider now a different type of problem. Let us suppose that the total labor supply is fully employed while F_1 and F_2 are not given. Let us now consider different combinations of F_1 and F_2 which can be obtained from the full employment of the given labor force. The different combinations of F_1 and F_2 which can be obtained from the full employment of the given labor force is known as consumption possibility locus. Let us see how we can derive this consumption possibility locus. Consider, for simplicity, 2 industry cases. Then equations represent the level of outputs can be written as $X_1 = a_{11}X_1 + a_{12}X_2 + F_1$ and $X_2 = a_{21}X_1 + a_{22}X_2 + F_2$. Therefore, $X_1 = \frac{(1-a_{22})F_1 + a_{12}F_2}{D}$, $X_2 = \frac{a_{21}F_1 + (1-a_{11})F_2}{D}$.

Let us denote the coefficients of F_1 and F_2 in these solutions by A_{ij}, i.e., let, $A_{11} = \frac{1-a_{11}}{D}$, $A_{12} = \frac{a_{12}}{D}$, $A_{21} = \frac{a_{21}}{D}$, and $A_{22} = \frac{1-a_{22}}{D}$. Then the solution can be written as: $X_1 = A_{11}F_1 + A_{12}F_2$; and $X_2 = A_{21}F_1 + A_{22}F_2$. These A_{ij} coefficients have economic meanings: A_{ij} is the direct and indirect input requirement of the i^{th} commodity to support one unit of final demand of the j^{th} commodity. Note that, $A_{ij} = \frac{\delta X_i}{\delta F_j}$ ($\forall$ i, j = 1, 2). The derivative $\frac{\delta X_i}{\delta F_j}$ represents the change in X_i (i=1, 2) if F_j (j=1, 2) is changed by one unit. Thus, if F_j (j=1, 2) is to be changed by 1 unit, we require A_{1j} units of X_1 and A_{2j} units of X_2. Therefore, if we like to get F_1 units of the first commodity and F_2 units of the second commodity, then to meet these final demand the total direct and indirect requirement of the first commodity is equal to $A_{11}F_1 + A_{12}F_2$. Therefore, the equation $X_1 = A_{11}F_1 + A_{12}F_2$ represents the equality between the supply and demand in the market for the first commodity. Similar analysis will be followed for the second commodity. In the same way the equation $L = l_1X_1 + l_2X_2$ can be interpreted as the supply-demand equation for the labor market. Leontief's model can thus be regarded as a general equilibrium system of three markets (two commodity markets + one labor market).

Let us put the values of X_1 and X_2 as obtained from the solution of the equations:

$X_1 = a_{11}X_1 + a_{12}X_2 + F_1$; $X_2 = a_{21}X_1 + a_{22}X_2 + F_2$; and $L = l_1X_1 + l_2X_2$.

Therefore, $L = l_1 (A_{11} F_1 + A_{12} F_2) + l_2 (A_{21} F_1 + A_{22} F_2)$. [Substituting the values of X_1 and X_2]

Or, $L = A_{L1}F_1 + A_{L2}F_2$, where $A_{L1} = l_1A_{11} + l_2 A_{21}$; $A_{L2} = l_1A_{12} + l_2 A_{22} F_2$.

The coefficients A_{L1} and A_{L2} can be given economic interpretations: A_{Li} (i=1, 2) represents the direct and the indirect requirement of labor to support 1 unit of final demand of the i^{th} commodity. Alternatively, A_{Li} (i=1, 2) is the total labor cost of 1 unit of final demand of the i^{th} commodity. A_{Li} coefficients are derived from a_{ij} coefficients, which are technologically given. Hence A_{Li}'s are also technologically given. Since A_{Li}'s are given, the equation $L = A_{L1}F_1 + A_{L2}F_2$ is the equation of a straight line. This is the equation of the '*consumption possibility locus*' (CPL). This equation gives us different combinations of F_1 and F_2, which can be obtained from the full employment of the given labor force. The equation can be written as $A_{L2}F_2 = - A_{L1}F_1 + L$; or, $F_2 = - (A_{L1}/A_{L2}) F_1 + L/A_{L2}$

If we plot the above equation on the (F_1, F_2) plane then, we will get a downward sloping straight line with absolute slope (A_{L1}/A_{L2}).

A consumption possibility locus can be regarded as a social transformation curve. If the society desires to consume only F_i (i=1, 2), it can consume an amount equal to L/A_{Li}. Because the CPL is a straight line, substitution of F_1 and F_2 is possible and it takes place at a constant cost. The rate at which F_2 is substituted for F_1 is called the marginal rate of substitution (MRS). The MRS is equal to the absolute value of the slope of the CPL, i.e., (A_{L1}/A_{L2}). This means that giving up of 1 unit of F_1 the society can get (A_{L1}/A_{L2}) units of F_2. This straight-line transformation curve reflects not only the linearity of the technology but also the presence of only one primary factor and the absence of joint production.

We know that under the assumption of perfect competition, the MRS between any two commodities will be equal to the price ratio. Let p_i (i=1, 2) represents the price of the i^{th} commodity. Then we get $(p_1/p_2) = (A_{L1}/A_{L2})$. This represents that the price ratio is equal to the ratio of labor costs of the two commodities. This is nothing but an expression of the Ricardian labor theory of value which states that the value of a commodity determined by the amount of labor embodied in that commodity. It also shows that the relative price is determined by technological coefficients and is independent of demand. In other words, whatever may be the composition of the final demand, the price ratio (p_1/p_2) remains the same.

Check Your Progress 3

1) In a two-industry input-output model we have the following information:

$[I-A]^{-1} = \frac{1}{0.84}\begin{bmatrix}0.9 & 0.4\\0.3 & 0.8\end{bmatrix}$, $l_1 = 0.2$, $l_2 = 0.3$ and $L = 10$. Find the equation of the consumption possibility locus.

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11.5.4 Determination of Equilibrium Prices

Let the prices of commodities 1, 2, 3,... be p_1, p_2, p_3,...respectively, and the price of the primary factor inputs be w (here, primary factor is labor; so w represents wage rate), then the technology matrix or transaction matrix in quantity may be converted into that in value terms. The problem can be posed as follows:

Sector	1	2	Final demand
1	$a_{11}X_1p_1$	$a_{12}X_2p_1$	F_1p_1
2	$a_{21}X_1p_2$	$a_{22}X_2p_2$	F_2p_2
Primary Input	$l_1X_1 w$	$l_2X_2 w$	
Total Costs	$a_{11}X_1p_1+ a_{21}X_1p_2 + l_1X_1 w$	$a_{12}X_2p_1+ a_{22}X_2p_2+ l_2X_2 w$	

With pure competition and free entry, profit in each industry must be zero, i.e., revenues equal costs. Hence, for the first industry receipts are (output x price) and cost is $a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1 w$. Same is true for the second industry.

Hence, for equilibrium; $p_1X_1 = a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1w$; and $p_2X_2 = a_{12}X_2p_1 + a_{22}X_2p_2 + l_2X_2w$, which simplify to $p_1 = a_{11}p_1 + a_{21}p_2 + l_1w$; and $p_2X_2 = a_{12}p_1 + a_{22}p_2 + l_2 w$, which can be put in matrix form as under:

$\begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix}$. Notice that the set of coefficients here are transposed, this matrix is transposed of [I-A].

Therefore $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix}^{-1} \begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix} = [I-A]^{-1} \begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix}$.

Therefore, $p_1 = \frac{1}{D}(A_{11}l_1 + A_{12}l_2)w$, and $p_2 = \frac{1}{D}(A_{21}l_1 + A_{22}l_2) w$, where A_{11}, A_{12}, etc., are the cofactors of the matrix [I-A] as in the preceding cases.

Check Your Progress 4

1) Consider the following technology matrix

Steel	*Coal*	*Final*	*Demand*
Steel	0.4	0.1	50
Coal	0.7	0.6	100
Labor	5	2	

Determine the equilibrium prices, if the wage rate is Rs. 10 per man day.

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11.6 LINEAR PROGRAMMING FORMULATION

Leontief's I-O model (in case of 2 industries) can be interpreted as a simplified linear programming model. Suppose that the final demand F_1 and F_2 are given. The society likes to get these final demands. Now if X_1 is the total output of the first industry, then out of this $a_{11}X_1$ will be used up in the first industry itself and $a_{12}X_2$ will be used up in the second industry. The amount left is, therefore, $(X_1 - a_{11}X_1 - a_{12}X_2)$ and it must be equal to the final consumption F_1 if this final consumption is to be achieved. From this relation we get one constraint $(1-a_{11}) X_1 - a_{12}X_2 \geq F_1$, which will be utilized in the linear programming formulation. Similarly, for the second industry we can get another constraint $-a_{21}X_1 + (1-a_{22}) X_2 \geq F_2$. The gross output levels X_1 and X_2 cannot be negative so that $X_1 \geq 0$ and $X_2 \geq 0$ are the non-negativity conditions. Consider now the objective function. Here our problem is that the final demands F_1 and F_2 are specified. To get these specified final demands we have to choose gross outputs X_1 and X_2 in such a manner that the total labor cost is minimized. The labor cost (in money terms) of X_1 and X_2 is equal to $l_1X_1w + l_2X_2w$. This is the objective function to be minimised.

Thus, in the Leontief model we get a typical linear programming problem, which can be stated as follows:

Minimise $W = l_1X_1w + l_2X_2w$

Subject to $(1-a_{11}) X_1 - a_{12}X_2 \geq F_1$

$-a_{21}X_1 + (1-a_{22}) X_2 \geq F_2$

$X_1 \geq 0$ and $X_2 \geq 0$.

By following the rules regarding the primal problem and the dual problem, the dual of the above primal problem can be written as follows:

Maximise $R = p_1F_1 + p_2F_2$

Subject to $(1-a_{11}) p_1 - a_{21}p_2\ p \leq l_1w$

$-a_{12}p_1 + (1-a_{22}) p_2 \leq l_2w$

$p_1 \geq 0$, and $p_2 \geq 0$.

Here it is to be noted that minimum value of the objective function in the primal problem is equal to the maximum value of the objective function in the dual problem. This means that total value of net output is just imputed as wages to the scarce factor, labor.

11.7 COEFFICIENT MATRIX AND CLOSED MODEL

We shall now examine whether we will be able to estimate F or X if the model is closed one. If the exogenous sector (final demand bill) of the open model is absorbed into the system of endogenous sectors, the model would turn into a closed one. In such a model final demand bill and primary inputs will not appear any more; rather in their place, we shall have the input requirements and output of this newly conceived industry, the 'household industry' producing the primary input labor. Final demand sector would now be considered as one of the endogenous sectors. As such now we shall have (n+1) industries in place of n industries and all producing for the sake of satisfying the requirements. This will mean, for example, that households will consume each commodity in fixed proportion to the labor services they supply.

Let us assume that, there are four industries only – including the new one (of final demand) designated b subscript 0(zero).We shall, therefore, have the following set of equations:

$X_0 = a_{00}X_0 + a_{01}X_1 + a_{02}X_2 + a_{03}X_3$; $X_1 = a_{10}X_0 + a_{11}X_1 + a_{12}X_2 + a_{13}X_3$;

$X_2 = a_{20}X_0 + a_{21}X_1 + a_{22}X_2 + a_{23}X_3$; and $X_3 = a_{30}X_0 + a_{31}X_1 + a_{32}X_2 + a_{33}X_3$.

This gives us a homogenous set of equation system

$$\begin{pmatrix} 1-a_{00} & -a_{01} & -a_{02} & -a_{03} \\ -a_{10} & 1-a_{11} & -a_{12} & -a_{13} \\ -a_{20} & -a_{21} & 1-a_{22} & -a_{23} \\ -a_{30} & -a_{31} & -a_{32} & 1-a_{33} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the four rows of the input coefficient matrix happen to be linearly dependent, $|I-A|$ will turn out to be zero. Hence the solution is indeterminate.

This means that in a closed model no output-mix of each sector exists. We can at most determine the output levels of endogenous sectors in proportion to one another, but cannot fix their absolute levels unless additional information is made available exogenously.

11.8 DYNAMIC INPUT-OUTPUT MODEL

The Leontief I-O model has appeared in a number of modified forms. We have discussed above how the model can be changed to an open or close one by including or ignoring the exogenous sectors. Here we shall discuss a dynamic form of I-O model in which specific account is taken of inter-relationship of the current and past outputs.

In our static open model, we assumed that current output can be used for current consumption and/or as current input in production of some other outputs. But in a dynamic model it is assumed that the current output can also be used (along with these two purposes) as an addition to industry's inventory stock and fixed capital formation.

The balance equation will, therefore, be

$$X_i = X_{i1} + X_{i2} + \ldots + X_{in} + \Delta \sum_{j=1}^{n} S_{ij} + F_i$$

$$(X_i)_t = (X_{i1})_t + (X_{i2})_t + \ldots + (X_{in})_t + \Delta(\sum_{j=1}^{n} S_{ij})_t + (F_i)_t$$

Here X_i is the total output of the i[th] (endogenous) producing sector in period t, which is used for three purposes:

i) as an input for production in economy's n industries, i.e., $(X_{i1})_t + (X_{i2})_t + \ldots + (X_{in})_t$

ii) as remaining final demand: (Fi) t (which now includes purchases by government sector, household and foreign sector).

iii) as an addition to the stock of n industries: $\Delta\left(\sum_{j=1}^{n} S_{ij}\right)_t$. If depreciation is ignored, the net addition to stock of i[th] output in the n producing sectors will be equal to the difference between the current year's and last year's accumulated capital

$$\left[\Delta\left(\sum_{j=1}^{n} S_{ij}\right)_t\right]_{net} = \left(\sum_{j=1}^{n} S_{ij}\right)_t - \left(\sum_{j=1}^{n} S_{ij}\right)_{t-1}.$$

Given the fixed relation between capital and output: $b_{ij} = \frac{S_{ij}}{X_j}$,

so that $b_{ij}(X_j)_t = (S_{ij})_t$ and $b_{ij}(X_j)_{t-1} = (S_{ij})_{t-1}$

Therefore, the balance equation of dynamic model turns out to be:

$$(X_i)_t = \sum_{j=1}^{n} a_{ij}(X_j)_t + (F_i)_t + \sum_{j=1}^{n} b_{ij}\left[(X_j)_t - (X_j)_{t-1}\right]$$

This is a basic equation of dynamic model representing a set of first order linear difference equations with constant coefficients: a_{ij}'s and b_{ij}'s.

Another way of formulating dynamic model is as follows:

Let the investment demand by sector j be $I_{jt} = b_j(X_{j,t} - X_{j,t-1})$, where b_j is the acceleration coefficient. In matrix notation $I = b'(X_t - X_{t-1})$. In the continuous case the above relation will be $I = b'\left(\frac{dX}{dt}\right)$

Now in order to transform these aggregate figures of capital formation (C.F) by sector of use to sector of origin or to demand for sectoral outputs, a capital formation matrix has to be defined. Let it be, $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$

where b_{ji}=(demand for j^{th} sector's output by i^{th} sector for C.F)/(C.F in sector i)

Hence the vector of investment by origin is $= Bb'(X_t - X_{t-1})$ or, $Bb'\frac{dX}{dt}$ (in continuous case).

Hence the Leontief consistency conditions are

$X_t = AX_t + F_t + Bb'(X_t - X_{t-1})$ [in discrete case]

$X = AX + F + Bb'\frac{dX}{dt}$ [in continuous case]

Thus, the system may be described in first order difference or differential equations. However, in a dynamic (model) situation it is better to use the inequalities to account for over production and excess capacity. Planning for long-term cannot be reduced to simple matter of solution of a system of simultaneous equations as in the input-output case.

11.9 LIMITATIONS OF INPUT-OUTPUT ANALYSIS

i) Errors in forecasting final demand will have grave consequence.

ii) Current relative prices of inputs may not be same as the ones implied in the table.

iii) The assumption of linear homogenous production function may not be valid. The technical coefficients will not remain constant even if input price ratios are held constant in such circumstances.

iv) The constant coefficient formulation also ignores the possibility of industry outputs reaching capacity, changing prices and input proportions in the table.

v) The assumption of constant technical coefficients goes counter to the possibility of substitutions of inputs and factors.

vi) Sectoral division is, for practical purposes, limited. Such a sectorisation is not good enough for many forecasting purposes.

vii) Sectorisation (grouping of commodities in sectors) is often arbitrary. The intra-sectoral heterogeneity with respect to technologies, efficiency and demand is not invariant over time.

viii) Input-output model building is highly costly in terms of time and money.

ix) Regional input-output analysis involves many more assumptions and difficulties in construction of such tables.

11.10 LET US SUM UP

This unit tells us about the interrelationship among different industries in the market. It also shows the way of determining output and price of the product for each industry, which is the most important thing for this kind of inter linkage among the industries. Relation between input-output analysis and linear programming problem reflects the important linkage of two real life applications of matrix algebra.

11.11 KEY WORDS

Closed and Open Input-Output Model: The I-O model that considers 'final demand bill' as exogenous factor is said to be as open I-O model and in closed I-O model 'final demand bill' is considered as endogenous factor.

Consumption Possibility Locus: The different combinations of F_1 and F_2 (final demands) which can be obtained from the full employment of the given labor force is known as consumption possibility locus.

Dynamic Input-Output Model: In dynamic I-O model it is assumed that the current output can also be used (along with current consumption and/or current input in production) as an addition to industry's inventory stock and fixed capital formation.

Hawkins-Simon Conditions: It basically states that more than on unit of a product cannot be used up in the production of every unit of that product. If A is the technological coefficient matrix then, according to Hawkins-Simon condition, determinant of [I-A] must be positive and all principal minors of [I-A] must also be positive.

Technological Coefficient Matrix: The matrix $[a_{ij}]$, which basically represents input requirement from ith industry to produce one unit output of jth industry, is known as technological coefficient matrix.

11.12 SOME USEFUL BOOKS

Chiang Alpha C. (1984), *Fundamental Methods of Mathematical Economics*, Third edition, McGraw-Hill Book Company.

Archibald, G.C & R.G. Lapse (1976), *Introduction to Mathematical Economics: Methods and Applications*, Harper & Row, New York.

Dorman, Samuelson & Sallow, *Linear Programming and Economic Analysis*, McGraw-Hill Book Company, New York, 1958.

Mukherji, B. & V. Pandit, *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd., New Delhi, 1982.

11.13 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 11.2 and answer.
2) See Section 11.2 and answer.
3) $A = \begin{pmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{pmatrix} = \begin{pmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & -0.8 \end{pmatrix}$

Check Your Progress 2

1) See Section 11.3 and answer
2) See Sub-section 11.5.2 and answer
3) Follow Example 1.

Check Your Progress 3

1) Let F_1 and F_2 be the final demands for the output of the two industries X and Y respectively. Then, $X_1 = 0.9X_1 + 0.4X_2 + F_1$; $X_2 = 0.3X_1 + 0.8X_2 + F_2$; solving these equations for X_1 and X_2 with the help of Cramer's rule we get $X = [I-A]^{-1}F$, i.e.$X_1 = \frac{1}{0.84}[0.9F_1 + 0.4F_2]$ and $X_2 = \frac{1}{0.84}[0.3F_1 + 0.8F_2]$

   Putting these values of X_1 and X_2 in the labor supply equation we get and $10 = 0.2X_1 + 0.3X_2$

   i.e., $10 = (0.2)\frac{1}{0.84}[0.9F_1 + 0.4F_2] + (0.3)\frac{1}{0.84}[0.3F_1 + 0.8F_2]$

   $8.4 = 0.27F_1 + 0.32F_2$. This is the required equation of the consumption possibility locus.

Check Your Progress 4

1) Here $I-A = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$, $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 5\text{x}10 \\ 2\text{x}10 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

   Therefore, price of steel = Rs. 200 per ton and price of coal = Rs. 100 per ton.

11.14 EXERCISES

1) The input coefficient matrix is of an open input-output system is given as $A = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$. If the final demand vector in thousand rupees happens to be $d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$, solve the system for output production.

   [Hints: Primary input requirement is Rs. 21,000.]

2) Given the input-output matrix and final demand vector as

   $A = \begin{pmatrix} 0.05 & 0.25 & 0.34 \\ 0.33 & 0.10 & 0.12 \\ 0.19 & 0.38 & 0 \end{pmatrix}$ and $d = \begin{bmatrix} 1800 \\ 200 \\ 900 \end{bmatrix}$, find the solution for output levels of three industries .

[Answer $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 3091 \\ 1638 \\ 2114 \end{bmatrix}$]

3) Consider the following inter-industry transactions table:

Industry	1	2	Final Consumption	Total
1	500	1,600	400	2,500
2	1,750	1,600	4,650	8,000
Labor	250	4,800	...	5,050
Total	2,500	8,000	5,050	15,500

Construct technology coefficient matrix showing direct requirements. Does a solution exist for this system?

[Hint: Determine $a_{ij} = X_{ij} / X_j$, then construct $[A] = [a_{ij}]$. Then evaluate $|I-A|$.]

Ans: Find the demand vector D in the equation AX + D = X consistent with the output vector X = $\begin{bmatrix} 25 \\ 21 \\ 18 \end{bmatrix}$ and the coefficient matrix A $= \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$. Test whether the Hawkins-Simon conditions for the viability of the system are satisfied.

4) Given A = $\begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$ and final demands are F_1, F_2 and F_3. Find the output levels consistent with the model. What will be the output levels if $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$?

[Answer: $X_1 = 1.11F_1 + 0.42F_2 + 0.28F_3$, $X_2 = 1.25F_2 + 0.36F_3$, and $X_3 = 1.43F_3$]

5) In the above example if final demand changes by 10, 10, 10, then what will be the change in sector outputs?

[Hint: $X = [I-A]^{-1} F$, so, $\Delta X = [I-A]^{-1} \Delta F$]

6) A three sector input-output matrix [I-A] is given as: $\begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$ with labor coefficients (per unit of output) as 0.4, 0.7, 1.2, if the household demand for the outputs of the 3 sectors is 1000, 5000 and 4000, determine the level of output and employment.

If the wage rate is Rs. 10 per labor day, find the equilibrium prices.

7) Determine the consumption possibility locus, given the total available labor supply = 1000 units and the technology matrix is given by A $= \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}$ and L = $\begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$.

UINIT 12 LINEAR PROGRAMMING

Structure

12.0 OBJECTIVES

The objectives of this unit is to:

- enable you to grasp the basic idea of linear programming principles;
- explain the difference between different processes of solution and their application;
- introduction to the game theory and basic methods of solution of a game; and
- enable to get the idea about how these mathematical tools are applied in our basic problems.

12.1 INTRODUCTION

Linear programming (LP) is a technique used for deriving optimum use of limited resources. Specifically, it deals with maximising a linear function of variables subject to linear constraints. Applications range from economic planning and environmental management to the diet problem.

12.2 LINEAR PROGRAMMING: BASIC CONCEPT

LP deals with a class of programming problems where both the objective function to be optimised is linear and all relations among the variables corresponding to resources are linear.

Remember that optimisation problems consist of two basic features, viz.,

- An objective function that you want to minimise or maximise.
- The objective function describes the behavior of the measure of effectiveness and captures the relationship between that measure