UNIT 7 INTEGRATION AND APPLICATIONS IN ECONOMIC DYNAMICS

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7.0 OBJECTIVES

After going through this unit, you should be able to:

- identify the dynamics problems in economics; and
- use the mathematical tools of differential equation to solve problems related to economic theory.

7.1 INTRODUCTION

Events that change over time are put under the purview of dynamic analysis. In this unit, we introduce a framework for dealing with dynamic economic problems by introducing time explicitly into these. For that purpose, let us start with the mathematical techniques of integral calculus and differential equations.

7.2 DYNAMICS AND INTEGRATION

In a dynamic economics model, the basic objective is the identification of the time path of the variable on the basis of its rate of change. For example, national income y of a country changes overtime. To see the rate of change we need to see its change with respect to time and to find the time path followed by y. Thus, if we know the derivative $\frac{dy}{dt}$, it will be possible to get onto the function like $y = y(t)$ through the technique of integration which happens to be opposite of the process of differentiation. We will return to this process after a while.

7.3 THE TOOLS OF DYNAMICS

The dynamic exercises are usually carried on with the aid of differential and difference equations. By taking 'time' as the independent variable, an attempt is made to derive their solutions. The simple integrals, both indefinite and definite, help in many contexts to define some important concepts of economic dynamics. The discussion that follows begins by introducing the notion of the integral. Then we move on to discuss the technique of differential equations. From now on, all functions in this unit will be assumed to be continuous and real valued.

7.3.1 The Indefinite Integral and its Economic Applications

As we have pointed out above, basically indefinite integral is reverse differentiation. Recall that differential calculus gives the rate of change (derivative) of a given function. Indefinite integration reverses such a process and finds the unknown function whose rate of change (derivative) is given. If the function f(x) is written symbolically as

$$\int f(x)dx$$

which is read as "the integral of *f* (x) with respect to x" and *f* (x) is the 'integrand'. We can write

$$\int f(x)dx = F(x)$$

provided f(x) is the derivative of $F(x)$ i.e. f(x) = $\frac{d}{dx}$F(x). It is not difficult to see that the function F(x) is not unique, because if F(x) contains the derivative *f*(x), then so has F(x) + c where c is any arbitrary constant. For that reason the indefinite integral is always written with an arbitrary constant, called 'the constant of integration'. As functions differ only by an additive constant, the derivative remains the same.

Examples:

1) $\int (x^3 + 3)dx = \frac{1}{4}x^4 + 3x + c$

2) $\int e^x dx = e^x + c$

3) $\int \frac{1}{x} dx = \log(x) + c$

Through the above examples we may verify that in each case the derivative of the right hand side equals the corresponding integrand on the left-hand side. Thus we see that

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

We now state two useful rules of integration.

Rule 1: The integral of a constant time a function equals the constant times the integral of the function.

$$\int kf(x)dx = k\int f(x)dx$$

Example: $\int 2x^2 dx = 2\int x^2 dx = 2\left(\frac{x^3}{3} + c\right)$

If we recall that the derivative of a constant time a function is the constant times the derivative of the function, it will be easier for us to appreciate above rule. If k = –1, we have the result

$$\int [-f(x)]dx = -\int f(x)dx.$$

Rule 2: The integral of a sum of functions equals the sum of the integrals of the functions, viz.,

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$$

Example: $\int (5e^x - x^{-2} + \frac{3}{x})dx = \int (5e^x)dx - \int (x^{-2})dx + \int \frac{3}{x}dx$

$$= 5\int e^x dx - \int x^{-2}dx + 3\int \frac{1}{x}dx$$

$$= \left(5e^x + c_1\right) - \left(\frac{x^{-1}}{-1} + c_2\right) + \left(3\log x + c_3\right)$$

$$= 5e^x + \frac{1}{x} + 3\log x + c$$

The rule is directly related to that of the derivative of a sum of function is the sum of the derivatives of the functions.

Some Useful Formulae

We state without proof some useful formulae for integration.

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

2) $\int x^{-1}dx = \int \frac{dx}{x} = \log x + c$ for any x > 0

3) $\int e^{mx}dx = \frac{e^{mx}}{m} + c$ for any m.

4) $\int a^{mx}dx = \frac{a^{mx}}{m\log a} + c$

5) $\int \cos axdx = \frac{\sin ax}{a} + c$

6) $\int \sin ax dx = -\frac{\cos ax}{a} + c$

7) $\int [K_1 f(x) + K_2 g(x) dx] = K_1 \int f(x) dx + K_2 g(x) dx + c$ for $K_1, K_2 \geq 0$

Determination of the Constant of Integration using Initial or Boundary Conditions

We have seen above that the indefinite integral is assigned an arbitrary constant. Its value can be precisely determined if we know that the integral of the function y = *f* (x) + c obeys some prescribed **initial condition** ($y = y_0$ when x = 0) or more generally **boundary condition** ($y = y_0$ when $x = x_0$). To understand the underlying idea let us take the following example:

$$\int (x^3 + x + 1) dx = \left(\frac{x^4}{4} + c_1\right) + \left(\frac{x^2}{2} + c_2\right) + (x + c_3)$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + x + c$$

Here y = *f*(x) + c = $\frac{x^4}{4} + \frac{x^2}{2} + x + c$. Suppose, we are given the initial condition, y = 20 when x = 0. Then

$y_0 = f(0) + c = c = 20$

This fixes the integral as the unique function y = $\frac{x^4}{4} + \frac{x^2}{2} + x + 20$. We will return to the initial condition again in Section 7.4.

Some Computational Methods

The standard procedure of integration given above is sometimes inadequate for computational purposes. In such a situation, it becomes necessary to attempt certain kinds of manipulation before the function becomes amenable to integration. We have a few standard results to help initiate the process of integration. However, remember that these are not all that can be used and there are no routine manipulations that can be prescribed. Practice opens up the channels of finding a solution to the integral.

Method of Substitution

If F(x) = $\int f(x) dx$, the indefinite integral can be obtained by resorting to transformation. If we take x = g(y), then

$$\int f(x) dx = \int [g(y)] g'(y) dy.$$

See that $x = g(y) \Rightarrow dx = g'(y) dy$.

If $\phi(y) = \int f[g(y)] g'(y) dy$, then

$F(x) = \phi[g^{-1}$ (.) is the inverse of g(.)].

Example: i) Solve $F(x) = \int (1+5x)^{\frac{1}{2}} dx$.

Let y = 1 + 5x

dy = 5dx, or dx = $\frac{1}{5}$dy

$$\therefore F(x) = \frac{1}{5}\int y^{\frac{1}{2}} dy = \frac{1}{5}\cdot\frac{2}{3} y^{\frac{3}{2}} + c$$

$$= \frac{2}{15}\cdot y^{\frac{3}{2}} + c = \frac{2}{15}(1+5x)^{\frac{3}{2}} + c$$

ii) Solve $F(x) = \int \frac{dx}{(3x-1)^2}$

Let y = 3x – 1

So, dy = 3 dx, or, dx = $\frac{1}{3}$dy.

$$\therefore F(x) = \frac{1}{3}\int \frac{dy}{y^2} = \frac{1}{3}\left(-\frac{1}{y}\right) + c = \frac{1}{3(3x+1)} + c$$

Integration by Parts

Let us return to differentiation of product of two functions u = *f*(x) and v = *f*(x) which gives,

d(uv) = $u\frac{dv}{dx} + v\frac{du}{dx}$.

From this, we can obtain

$$\int (uv)'dx = \int uv'dx + \int vu'dx$$

$$\Theta \int (uv)'dx = uv$$

$$\therefore uv'dx = uv - \int vu'dx$$

Example: Solve F(x) = $\int xe^{-x}dx$

Let u(x) = x, v(x) = $-e^{-x}$

So that u′(x) = 1, v′(x) = e^{-x}

$$\therefore f(x) = \int u.v'dx$$

$$= uv - \int vu'dx$$

$$= -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + c = -e^{-x}(1+x) + c$$

Economic Applications of Indefinite Integration

Consider the following two examples as a part of your exercise to apply the tool of indefinite integration to often cited problems of economics.

a) **Investment and the Stock of Capital**

Let net investment I is the rate of change of the stock of capital K. If time is treated as a continuous variable, we can express this as

$$I(t) = \frac{dK(t)}{dt}.$$

Thus, if the rate of investment I(t) is known, the capital stock K(t) can be estimated through the formula,

$$K(t) = \int I(t)dt$$

Example: The rate of net investment is given by $I(t) = 12t^{\frac{1}{3}}$ and the initial stock of capital at t = 0 is 25 units. Find the equation for the stock of capital.

$$K(t) = \int 12t^{\frac{1}{3}}dt = 12\left(\frac{3}{4}\right)t^{\frac{4}{3}} + c$$

$$= 9t^{\frac{4}{3}} + c$$

As K(0) = c = 25 given,

$$K(t) = 9t^{\frac{4}{3}} + 25$$

b) **Obtaining the Total from the Margin**

Integration helps us recover the total function from the marginal function if the concerned variable varies continuously. Thus, it will be possible to derive the total functions such as cost, revenue, production and saving from their marginal functions. We will examine a simple application to see the procedure involved.

Example: If the marginal revenue function of a firm in the production of output is $MR = 40 – 10q^2$ where q is the level of output and total revenue is 120 at 3 units of output, find the total revenue function.

Since $MR = \frac{dTR}{dq}$, we can write

$$TR = \int MRdq$$

$$= \int (40 - 10q^2)\, dq$$

$$= 40q - \frac{10}{3}q^3 + c$$

At q = 3, TR = 30 + c = 100 given. So c = 90. The required total revenue function is

TR(q)= $40q - \frac{10}{3}q^3 + 90$.

7.3.2 The Definite Integral and its Economic Applications

The definite integral of the function *f*(x) over the interval [a, b] is expressed symbolically as $\int_a^b f(x)dx$, read as "the integral of *f* with respect to x from a to b".

The smaller number a is termed the **lower limit** and b, the **upper limit**, of integration. Geometrically, this definite integral denotes the area under the curve representing *f*(x) between the points x = a and x = b.

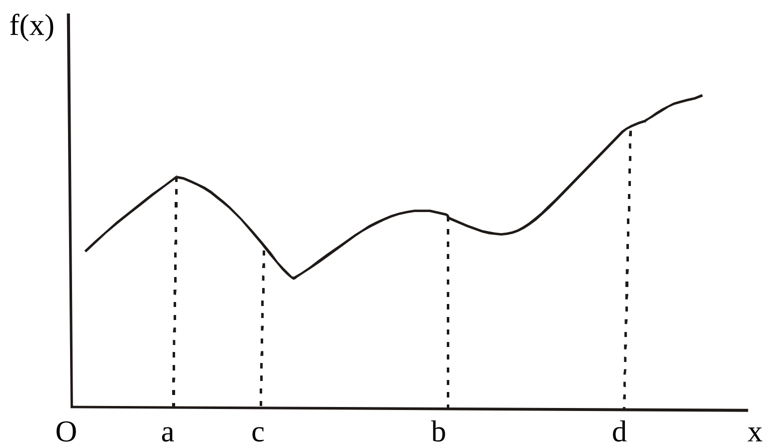

Fig. 7.1

It should be noted that the indefinite integral *f*(x) dx is a **function** of x, whereas the definite integral $\int_a^b f(x)dx$ is a **number**. The numerical value of the definite integral depends on the two limits of integral also changes. This is clear from Figure 7.1 where if we change the interval (a, b) to (c, d) the value of the area under the curve will, in general, change.

Another feature of the definite integral is that its value does not depend on the particular symbol chosen to represent the independent variable so long as the form of the function is not changed. That is,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \text{etc.}$$

The following theorem establishes the connection between indefinite and definite integration and supplies the method for evaluating definite integrals.

The Fundamental Theorem of Calculus

If $\int_a^b f(x)dx$ = *f*(x) + c, then

$\int_a^b f(x)dx$ = *f*(b) – *f*(a).

Examples: 1) To evaluate $\int_1^5 x^2 dx$.

$$\int_1^5 x^2 dx = x^3 + c$$

So, $\int_1^5 x^2 dx = 5^3 - 1^3 = 124.$

2) To evaluate $\int_{-1}^{1} (ax^2 + bx + c)dx.$

$$\int (ax^2 + bx + c)dx = a\frac{1}{3}x^3 + b\frac{1}{2}x^2 + cx + c'.$$

So, $\int_{-1}^{1} (ax^2 + bx + c)dx = \left(a\frac{1^3}{3} + b\frac{1^2}{2} + c.1\right) - \left(a\frac{(-1)^3}{3} + b\frac{(-1)^2}{2} + c.(-1)\right)$

$$= (\frac{1}{3}a + \frac{1}{2}b + c) - (-\frac{1}{3}a + \frac{1}{2}b - c)$$
$$= (\frac{1}{3}a - \frac{1}{2}b + c)$$
$$= \frac{2}{3}a + 2c$$
$$= 2(\frac{1}{3}a + 1)$$

Definite integrals are subject to certain rules of operation.

Rule 1: If the two limits are equal, the value of the integral is zero.

$$\int_a^b f(x)dx = 0.$$

Rule 2: Reversing the limits of integration changes the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Rule 3: The definite integral can be expressed as the sum of subintegrals.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

where b is a point within the interval (a, c).

We now discuss briefly one special type of definite integral, the **improper integral**. When one of the limits of integration is + ∞ or (–∞) a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits according to the following rules:

i) $\int_{a}^{\infty} f(x)dx = \underset{b\to\infty}{Lt} \int_{a}^{b} f(x)dx$.

ii) $\int_{-\infty}^{b} f(x)dx = \underset{a\to\infty}{Lt} \int_{a}^{b} f(x)dx$

Example:

Evaluate $\int_{1}^{\infty} \frac{dx}{x^2}$

Since $\int_{1}^{b} \frac{dx}{x^2} = -\frac{1}{b} + 1$, the desired integral is

$$\underset{b\to\infty}{Lt} \int_{1}^{b} \frac{dx}{x^2} = \underset{b\to\infty}{Lt} \left(-\frac{1}{b}\right) = 1.$$

Economic Applications of the Definite Integral

a) **Consumer's Surplus**

Consumer's surplus (CS) measures the net benefit that a consumer enjoys from the purchase of a particular commodity in the market. To measure CS, we take (i) the demand function of a consumer P = *f*(q) representing the highest price a consumer is willing to pay (her 'demand price') for any specified quantity, (ii) the actual price paid for the quantity purchased and (iii) get the difference between (i) and (ii). In the figure below, a consumer is willing to pay a price of p_1 per unit for q_1 units, p_2 per unit for q_2 units, and so on. Suppose the market price is $\bar{p}$. At this prices she purchases $\bar{q}$ units and her actual expenditure is $\overline{pq}$, represented by the rectangle $O\bar{p}E\bar{q}$. Her total willingness to pay for $\bar{q}$ is obtained as the sum of her demand prices for all the units from 0 to $\bar{q}$.

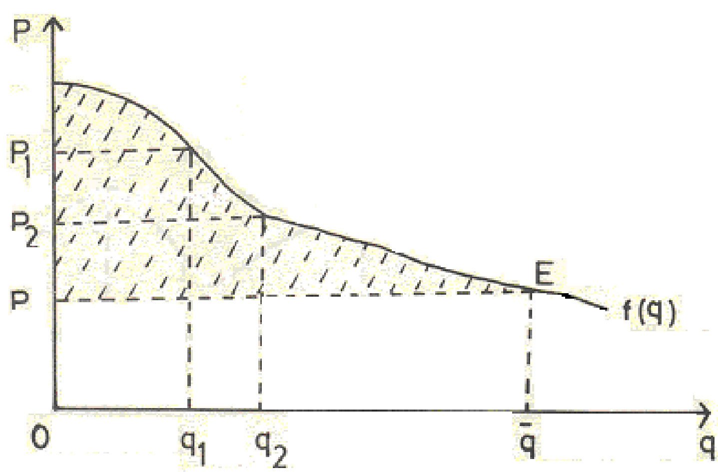

Fig. 7.2

Mathematically, this is the definite integral of the demand function up to $\bar{q}$, or the area under the demand curve up to $\bar{q}$. The excess of this total willingness to pay in units of money over her actual expenditure is her Consumer's surplus.

$$CS = \int_0^{\bar{q}} f(q)dq - \overline{pq}\,.$$

It is represented by the crossed area in the diagram.

Example: Suppose the demand function of a consumer is given by p = 80 – q. If the price offered is p = 60, find the consumer surplus.

For p = 60, we get q = 20 from the demand equation. Actual expenditure pq = 1200.

Now $CS = \int_0^{20} (80 - q)dq - pq$

$= 1400 - 1200 = 200\,.$

Thus the consumer's surplus is Rs.200.

b) **Capital Accumulation Over a Specified Period**

Since $\int I(t)dt = K(t) + c$, we may use the definite integral

$\int_a^b I(t)dt = K(b) - K(a)$ to find the total capital accumulation during the time interval [a, b].

Example: Given the rate of net investment I(t) = $9t^{1/2}$, find the level of capital formation in (i) 16 years and (ii) between the 4^{th} and the 8^{th} years.

i) K = $\int_0^{16} 9t^{1/2}dt = 6(16)^{3/2} - 0 = 384$

ii) K = $\int_4^{8} 9t^{1/2}dt = 6(8)^{3/2} - 6(4)^{3/2} = 135.76 - 48 = 87.76.$

c) **Present Value or Discounted Value Under Continuous Compounding of Interest**

A basic concept in capital theory is the present or discounted or capital value of a specified sum of money that will be available at a future date. If the annual rate of interest is 100r percent, then the present value Y of Rs. x available next year is Y = $\frac{x}{1+r}$, because Rs. $\left(\frac{x}{1+r}\right)$ now will become Rs.x after one year at the stipulated annual rate of interest of 100r per cent. Similarly, the present value of Rs.x available t years hence is

$$Y = \frac{x}{(1+r)^t}$$

If interest is compounded n times a year at 100r per cent per year then the present value is

$$Y = \frac{x}{\left(1+\frac{r}{n}\right)^{nt}} = x\left(1+\frac{r}{n}\right)^{nt} \quad \ldots (1)$$

If interest is compounded continuously, then n $\to \infty$ and the continuous counterpart of (1) becomes

$Y = x\, e^{-rt}$

using the result: $\underset{n\to\infty}{Lt}\left(1+\frac{k}{n}\right)^{nx} = e^{kx}$.

Now consider a project that yields an income x(t) at future period t for t = 1, 2, …, T. That is, the income stream associated with the project for T years is : x(1), x(2), …, x(T). The present or discounted value of this income stream at annual compounded is:

$$Y = \sum_{t=1}^{T} \frac{x(t)}{(1+r)^t} \quad \ldots (2)$$

When income flows continuously at the rate of x(t) per period up to period T and interest is compounded continuously the expression for present value becomes

$$Y = \int_0^T x(t)e^{-rt}\,dt \quad \ldots (3)$$

Note that the magnitude of present value depends on the size of the income stream, the number of years it flows (the time horizon) and the rate of interest (the discount factor).

You should keep in mind the distinction between the present value of the **sum** x(T) available T periods hence and the present value of the **stream of income** x(t) per period up to period T. In the figure the former is the ordinate at t = T, whereas the latter is the shaded area under the curve upto t = T.

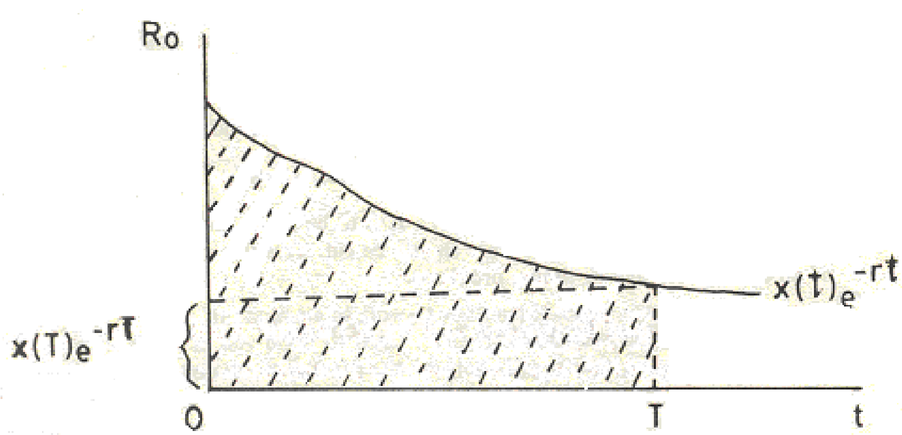

Fig. 7.3

A particular case of interest is the valuation of an asset (a bond or a piece of land) yielding a fixed income Rs. R for ever. The market value Y of such an asset is the present value of the perpetual yield.

$$Y = \int_0^\infty \text{R}e^{-rt}\, dt = \text{R} \int_0^\infty e^{-rt}\, dt$$

Remembering the rule for evaluating improper integrals.

$$\int_0^\infty e^{-rt} dt = \underset{b\to\infty}{Lt} \int_0^b e^{-rt} dt = \underset{b\to\infty}{Lt}\left(-\frac{1}{r}e^{-rb} + \frac{1}{r}\right) = \frac{1}{r}.$$

Hence, the market value is:

$$Y = \frac{R}{r}.$$

To illustrate further, the use of the concept of present value, we consider the following more complex problem of **optimal timing**.

The value of timber planted on a plot of land grows over time according to the function V(t) = $2^{\sqrt{t}}$. Assuming zero cost of maintenance and a discount factor of r, find the optimal time to cut the timber for sale.

Since cost of production (upkeep) is zero, profit maximisation here is equivalent to the maximisation of sales revenue V. Due to the interest factor, r, different V values, however, are not comparable because they accrue at different points of time. The solution involves discounting each V value to its present value (the value at t = 0). The process of discounting puts them on comparable footing.

Assuming continuous compounding, the present value R(t) can be written as

$$R(t) = V(t)e^{-rt} = 2^{\sqrt{t}}\, e^{-rt}.$$

The optimal time of cutting is the value of t that maximises R(t). Since *f*(x) and log *f*(x) attain their maximum at the same value of x, the problem can equivalently be restated as finding the value of t that maximises log R(t).

$$\ln R(t) = \sqrt{t}\,\log 2 - rt$$

Differentiating with respect to t and setting the derivative equal to zero, we get

$$\frac{1}{R} - \frac{dR}{dt} = \left(\frac{\ln 2}{2\sqrt{t}} - r\right) = 0$$

$$\text{or, } \frac{dR}{dt} = R\left(\frac{\ln 2}{2\sqrt{t}} - r\right) = 0 (R \neq 0)$$

$$\text{or, } \sqrt{t} = \frac{\ln 2}{2r}, \quad \text{since } R \neq 0.$$

$$\text{or, } t = \left(\frac{\ln 2}{2r}\right)^2$$

We leave it to you to check that at this value of t the second order condition for maximisation $\frac{d^2R}{dt^2} < 0$ is also satisfied. Thus, the expression for the optimum time for cutting the timber is $(\log 2/2r)^2$. It is to be noted that the higher the rate of discount r, the sooner the timber should be cut. This is a general characteristic of all optimal storage or timing problems.

7.4 DIFFERENTIAL EQUATIONS AND ITS ECONOMIC APPLICATIONS

We deal with many economic models which have temporal dimensions involving relationships between the values of variables at a given point of time and the changes in these values over time. As an expample we may consider a model of economic growth that often postulates a functional relationship between the change in the capital stock and the value of output. When time is modelled as a continuous variable, differential equations are formulated by involving the derivatives (or differentials) of unknown functions.

7.4.1 Solving Differential Equations

Solving a differential equation means finding a function that satisfies that equation.

Let us start with some basic ideas behind these equations. If y = f(x) is a function for which derivatives of adequate order exist, then $\frac{dy}{dx} = f'(x)$. Suppose that we know f'(x) and would like to go back to the function y. Therefore, we try to solve the problem.

$$dy = f'(x)dx$$

$$\Rightarrow y = \int f'(cx)dx .$$

Through differential equations, we attempt to solve the problems, which are related to change over time, i.e., dynamic variables. For example, suppose that a hypothetical economy's income (y) is related to time (x). It is given in functional form: $y(x) = 2x^{1/2}$. If the income changes over time, we find the rate of change as $\frac{dy}{dx} = x^{-\frac{1}{2}}$. Let us work to find the time path of the income change, so that we write y = y(x). The derivative of this function, however, will be same as that of y = y(x) + c, where c is any arbitrary constant. In such a situation, we cannot determine a unique time path of the income change. It is necessary, therefore, to work out a definite value of c. Additional information required for that purpose is to have the *initial condition*. If we know the initial income of the economy, say, y(0), i.e., value of y at x = 0, then the value of the constant c can be determined.

Thus, from $y(x) = 2x^{1/2} + c$, when x = 0, we get $y(0) = 2(0)^{1/2} + c = c$.

See that constant c is no longer arbitrary as if y(0) = 10,000, c = 10,000 and

$y(x) = 2x^{1/2} + 10{,}000$. More generally, for any given initial income, y(0), the time path will be

$y(x) = 2x^{1/2} + y(0)$.

Note that the income example, its dynamic form, consists of the sum of initial condition and another term with time variables.

Remember a general principle on the initial value problem: The differential equation that involves only the first derivative, has a unique solution if it has one initial condition. In addition, the differential equation that involves only the first and second derivatives, has a unique solution if it has two initial conditions.

Differential Equation: Equilibrium and Stability

In a difference equation, if the initial value has a solution that is a constant function and hence independent of *t*, then the value of the constant is called an **equilibrium state** or **stationary state** of the differential equation.

Example:

Consider the differential equation

$$y'(t) + y(t) = 2.$$

The general solution of this equation, as we shall below, is

$$y(t) = Ce^{-t} + 2.$$

Thus for the initial condition y(0) = 2, the solution of the problem is y(*t*) = 2 for all *t*. Thus the equilibrium state of the system is 2.

The **order** of a differential equation is the order of the highest derivative appearing in the equation. Its **degree** is the highest power to which the highest order derivative is raised. A differential equation is **linear** if the dependent variable and derivatives are raised to the first power only and no product term $y\,\dfrac{dy}{dx}$ occurs.

Examples: In all the examples that follow the unknown function is y = *f*(x).

1) $\dfrac{dy}{dx} = 4x - 9$ First order, first degree

2) $\left(\dfrac{dy}{dx}\right)^4 - x^2 = 0$ First order, fourth degree

3) $\dfrac{d^2y}{dx^2} - 2y = 0$ Second order, first degree

4) $\frac{d^4y}{dx^4}+x^3\frac{d^3y}{dx^3}-\log x\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}+x^2y+10=0$ Four order, first degree

5) $\left(\frac{d^3y}{dx^3}\right)^4+\left(\frac{d^2y}{dx^2}\right)^7=8+2y$ Third order, fourth degree.

Equation (1), (3) and (4) are linear since they are all of the first degree.

Please note that depending upon the complexity of the equations that we use in course of the following discussion, the notations adopted will be in the form of either $\frac{dy}{dx} or, f'(x)$.

First Order Differential Equation

Solution to first order differential equation in specific instances can be worked out with (1) separation of variables. Consider the equation,

$$\frac{dy}{dx}=f'(x)$$

or, dy = f'(x)dx

If we integrate both the sides,

$\int dy=\int f'(x)dx$, so that variables x and y are separated, it becomes easy for applying appropriate technique of integration.

Example: $\frac{dy}{dx}=x^2$

or, dy = x^2dx

So, $\int dy=\int x^2dx$

or, y = $\frac{x^3}{3}+c$

Exact equations

From $\frac{dy}{dx}=f'(x)$, we get

y = $\int f'(x)dx+c$ which can be written as g(x) + c.

Remember that the addition of a constant term, ‘c’ to the function does not affect its derivative. However, it shifts the function parallelly. Depending upon the different values acquired by the constant term, we get a family of curves for the function. Take, for example, the above solution, y = $\frac{x^3}{3}+c$ with c= 0. Then $\frac{dy}{dx}=x^2$, or, dy = x^2dx

or, dy – x^2dx = 0

Note that '0' in the right side of this solution is due to a choice like c = 0. So we write

dy – x^2dx = c, (where c = constant)

Generalising the above result, we can write

u(x, y) = c.

So, du = $u_x d_x + u_y d_y = 0$

If u = $\frac{-x^3}{3} + y$, then du = $-x^2$dx + dy = 0

or, $\frac{dy}{dx} = x^2$

Since u = $\frac{-x^3}{3} + y$, $\frac{\partial u}{\partial x} = -x^2$ and $\frac{\partial u}{\partial y} = 1$.

Therefore, dx = $\frac{\partial u}{\partial x}$ and dy = $\frac{\partial u}{\partial y}$. From these we write the generalised form of differential equation as P(x, y) dx + Q(x, y) dy = 0.

If we can find a function g(x, y) such that P(x, y) = $\frac{\partial g}{\partial x}$ and Q(x, y) = $\frac{\partial g}{\partial y}$, we write

d [g(x, y)] = $\frac{\partial g}{\partial x}$ dx + $\frac{\partial g}{\partial x}$ dy

= P(x, y) dx + Q(x, y) dy.

Then g(x, y) = c are integral curves of the above differential equation. This class of differential equations is called exact differential equations.

Example:

For xdx + ydy =0, set g= $\frac{1}{2}(x^2+y^2)$. Then g_x = xdx and g_y = ydy. The solutions are x^2+y^2 = c, or circles.

To determine if a differential equation is exact or not, check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Example:

From, $(3x^2+y^2)$dx + 2xydy = 0, we get

P(x, y) = $3x^2+y^2$

Q(x, y) = 2xy, so that

$P_y = 2y = Q_x$. The equation, therefore, is exact.

Exercise:

Solve the exact differential equation

$2yxdy + y^2dx = 0$

In this equation

P(x, y) = 2yx

$Q(x, y) = y^2$

Now, $F(y, x) = \int 2yxdy + \phi(x) = y^2 x + \phi(x)$

$$\frac{dF}{dx} = y^2 + \phi'(x)$$

As $Q = \frac{dF}{dx}$, we equate $Q(x, y) = y^2$ and $\frac{dF}{dx} = y^2 + \phi'(x)$ to get $\phi'(x) = 0$

$\phi'(x) = \int \phi'(x)dx = \int 0dx = k$, which give the specific form of $\phi'(x)$.

So, $F(y, x) = y^2x + K$

The solution of the exact differential equation should then be F(y, x) = c. K being a constant can be merged with c, so that $y^2x = c$, or $yx = cx^{-1/2}$ where c is arbitrary.

A first order linear differential equation is general written as

$$\frac{dF}{dx} = P(x)y = Q(x) \qquad \text{…. (4)}$$

where P and Q are two functions of x, and that of y. P and Q may also be expressed in other forms as x^2 and e^x. These may also be constants. In the following, we will discuss homogenous differential equations.

Example: Solow's model of economic growth

Consider a production function

$$q = f\left(K, L\right)$$

where q= output, K= capital and L= labour. It is specified that the production function takes the form: $q = AL^{\alpha}K^{1-\alpha}$, where *A* is a positive constant and 0 < *a* < 1. A constant fraction *s* of output is "saved" (with 0 < *s* < 1), and used to augment the capital stock. Thus, the capital stock changes according to the differential equation

$$K^{'}(t) = sAL(t)^{\alpha} K(t)^{1-\alpha}$$

and takes the value K_0 at $t = 0$. The labour force is $L_0 > 0$ at $t = 0$ and grows at a constant rate λ, so that

$$\frac{L'(t)}{L(t)} = \lambda$$

You can solve this model by solving for L, then substitute the value into the equation for $K'(t)$ to get K.

Note that the equation for L is separable, and we can write

$$\frac{dL}{L} = \lambda dt\ .$$

Integrating it we get

$$\log L = \lambda t + C$$
$$or, L = Ce^{\lambda t}$$

Given the initial condition, we have $C = L_0$.

Substituting this result into the equation for $K'(t)$ yields

$$K'(t) = sA(K(t))^{1-\alpha}(L_0 e^{\lambda t})^{\alpha} = \mathrm{sA}\left(\mathrm{L}_0\right)^{\alpha} e^{\alpha\lambda t}\left(K\left(t\right)\right)^{1-\alpha}$$

This equation is separable, and may be written as

$$\mathrm{K}^{\alpha\text{-}1} dK = sA(L_0)^{\alpha} e^{\alpha\lambda t} dt\ .$$

Integrating both sides, we obtain

$$\frac{\mathrm{K}^{\alpha}}{\alpha} = sA(L_0)^{\alpha}\frac{e^{\alpha\lambda t}}{\alpha\lambda} + C,$$

so that

$$\mathrm{K}\left(\mathrm{t}\right) = \left(sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}}{\lambda} + C\right)^{\frac{1}{\alpha}}.$$

Given $\mathrm{K}\left(0\right) = K_0$, we conclude that $C = (K_0)^{\alpha} - \frac{sA(L_0)}{\lambda}$.

Thus, $\mathrm{K}\left(\mathrm{t}\right) = \left[sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}-1}{\lambda} + (K_0)^{\alpha}\right]^{\frac{1}{\alpha}}$ for all t.

An interesting feature of the model is the emergence of capital-labor ratio. We have

$$\frac{K(t)}{L(t)} = \frac{\left[sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}-1}{\lambda} + (K_0)^{\alpha}\right]^{\frac{1}{\alpha}}}{L_0 e^{\lambda t}}$$

for all t.

As $t \to \infty$, $\frac{K(t)}{L(t)}$ converges to $\left(\frac{sA}{\lambda}\right)^{\frac{1}{\alpha}}$.

Homogenous Case

If P and Q are constant functions and if Q is identically equal to zero, equation (4) becomes $\frac{dy}{dx}$ + ay = 0 where a is some constant … (5)

Please note that the constant term '0' can be regarded as in the first degree in terms of y because 0y = 0.

Equation (5) can be written as

$$\frac{1}{y}\frac{dy}{dx} = -a \qquad \text{…. (6)}$$

For solution, we write $\frac{dy}{y} = cdx$ (with c = –a) and integrate both the sides, such that $\int \frac{dy}{y} = \int cdx$. The left side of the above gives log y + c_1 for y ≠ 0.

Whereas right side becomes

cx + c_2

Bringing together the result of the left and right sides,

log y + c_1 = cx + c_2

or, log y = cdx + c_3 (combining c_1 and c_2 of both sides)

or, $e^{\log y} = e^{(cx+c_3)}$

or, y = $e^{cx}.e^{c_3}$ = A e^{cx} where A = e^{c_3}

Putting back c = –a,

we get y(x) = Ae^{-ax} where A is arbitrary … (7)

To get rid of the arbitrary constant, set x = 0 in the equation y(x) = Ae^{-ax}, so that

y(0) = Ae^{0} = A.

Thus, y(x) = y(0)e^{-ax} … (8)

In (7), A is an arbitrary constant. The solution, therefore, is a **general solution**. When a particular value is substituted for A, we derived the **particular solution** in (8). There are an infinite number of particular solutions, value of y(0). However, y(0) is important since it can alone satisfy the initial condition. From the feature of giving a definite value to the arbitrary constant, we refer the result in (8) as the definite solution of the differential equation.

Non-homogenous case

When we have a non-zero constant in place of the zero in equation (5) above, it is called a non-homogenous linear differential equation.

Thus, $\frac{dy}{dx} + ay = b$ … (9)

is a non-homogeneous differential equation. The solution of this class of equations has two parts, (a) complementary function (y_c) and (b) particular integral (y_p). Before we proceed to solve equation (9), it will be useful to point out that homogeneous equation (5) is called a reduced equation of (9) and the non-homogenous equation (9) itself is categorised as the complete equation. Moreover, the complementary function (y_c) is the general solution of the reduced equation, whereas the particular integral (y_p) is any of the particular solution of the complete equation.

Solution to non-homogenous differential equation is seen as a sum of the complementary function and the particular integral.

Thus, $y(x) = y_c + y_p$.

We have noted above that y_c is the general solution of the reduced equation. We take the general solution of the homogeneous differential equation (5) above, which was Ae^{-ax}. Thus, $y_c = Ae^{-ax}$.

Let us came to particular integral. Recall that it is any particular solution of the complete equation. Perhaps the simplest possible type of solution we can think of is to take it being some constant (y = k). Taking of as a constant, we get $\frac{dy}{dx} = 0$. Therefore, equation (9) becomes ay = b, or y = $\frac{b}{a}$ where $a \neq 0$.

In that case $y_p = \frac{b}{a}$, we get the general solution to the equation as

$y(x) = Ae^{-ax} + \frac{b}{a}$ … (10)

See that the solution remains general. The presence of arbitrary constant A is responsible for it. In order to make it definite, we need to take an initial condition. Setting x = 0, y can be assigned the value y(0) and we get

$y(0) = A + \frac{b}{a}$

or, $A = y(0) - \frac{b}{a}$

Putting this value in (10*)*, the solution becomes

$$y(x) = \left[y(0) - \frac{b}{a}\right] e^{-ax} + \frac{b}{a} \qquad \ldots (11)$$

which is the definite solution as long as a ≠ 0.

Exercise:

Solve the equation $\frac{dy}{dx} + 2y = 6$ with the initial condition y(0) = 10.

We have a = 2 and b = 6. Hence, according to (11), the solution is

$y(x) = [10 - 3]e^{-2x} + 3 = 7e^{-2x} + 3$

Solution when a = 0

If $\frac{dy}{dx} + ay = b$ has a = 0, then

$$\frac{dy}{dx} = b \qquad \ldots (12)$$

Its general solution is found by integration, i.e., y(x) = bx + c where c = arbitrary constant.

Complementary function: with a = 0

$y_c = Ae^{-ax} = Ae^0 = A$ (A = an arbitrary constant)

Particular Integral

As a = 0, the constant solution y = k does not work and some non-constant solution needs to be tried. Take y = kx so that

$$\frac{dy}{dx} = k.$$

From the complete equation (12) above k = b.

$\therefore y_P = bx$

General solution: $y(x) = y_c + y_P = A + bx$ ………. (13)

Example:

Solve the equation $\frac{dy}{dx} = 2,$ with the initial condition y(0) = 5. 0

From (13) above, y(x) = 5 + 2x.

Verification of the Solution

You can check the correct answer of your solution to a differential equation by taking its differentiation. Follow the following two steps:

1) Test that the derivative of the time path is consistent with the given differential equation.

2) Test the definite solution to find that the solution satisfies the initial condition.

Check Your Progress 1

1) Solve the following differential equations.

   a) $(y(t))^2 y'(t) = t + 1;$

   b) $y'(t) = t^3 - t.$

   c) $y'(t) = te^t - t.$

   d) $y(t) = e^{y(t)} y'(t) = t + 1.$

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2) Solve the following differential equation for the given initial value.

   a) $ty'(t) = y(t)(1 - t), (t, y) = (1, 1/e).$

   b) $(1 + t^3)y'(t) = t^2 y(t), (t, y) = (0, 2).$

   c) $y(t)y'(t) = t, (t, y) = (\sqrt{2}, 1).$

   d) $e^{2t} y'(t) - (y(t))^2 - 2y(t) - 1 = 0, (t, y) = (0, 0).$

3) Find y_c, y_p , the general solution and definite solution of the equation and check its validity:

   $$\frac{dy}{dx} + 4y = 12; y(0) = 2$$

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7.4.2 Solving Linear First-order Differential Equations

A **linear first-order differential equation, in general,** takes the form

$y'(t) + a(t)y(t) = b(t)$ for all t and with a and b representing functional forms.

Coefficient of *y*(*t*) constant

Consider the case in which $a(t) = a \neq 0$ for all t, so that

$y'(t) = a(t)y(t) = b(t)$ for all t.

If the left-hand side were the derivative of some function and we could find the integral of b then we could solve the equation by integrating each side. If we multiply both sides by $g(t)$ for each t, then

$g(t)y'(t) + ag(t)y(t) = g(t)b(t)$ for all t.

See that the left-hand side of *this* equation to be the derivative of a product of the form $f(t)y(t)$ provided we have $f(t) = g(t)$ and $f'(t) = ag(t)$. See that if $f(t) = e^{at}$, then $f'(t) = ae^{at} = a f(t)$.

Thus if we set $g(t) = e^{at}$, so that we have

$e^{at}y'(t) + ae^{at}y(t) = e^{at}b(t)$ and the integral of the left-hand side is $e^{at}y(t)$. We get the solution of the equation as

$e^{at}y(t) = C + \int e^{as}b(s)ds$

or, $y(t) = e^{-at}\left(C + \int e^{as}b(s)ds\right)$

So, the general solution of the differential equation

$y'(t) + ay(t) = b(t)$ for all t,

where a is a constant and b is a continuous function, is given by

$y(t) = e^{-at}\left(C + \int e^{as}b(s)ds\right)$ for all t.

Because multiplying the original equation by e^{at} allows us to integrate the left-hand side, we call e^{at} an **integrating factor**.

If $b(t) = b$ for all t then the solution simplifies to

$y(t) = Ce^{-at} + b/a$

Looking at the original equation we see that $y'(t) = 0$ if and only if $y(t) = b/a$. Thus $y = b/a$ is an equilibrium state.

For the initial condition $y(t_0) = y_0$ we have $y_0 = Ce^{-at_0} + b/a$ so that $C = \left(y_0 - b/a\right)e^{at_0}$. The solution of the difference equation is given by

$$y(t)=\left(y_0-\frac{b}{a}\right)e^{a(t_0-t)}+\frac{b}{a}.$$

As $t\to\infty$, $y(t)$ converges to b/a if $a > 0$, and grows without bound if $a < 0$ and $y_0 \neq b/a$. That is, the equilibrium is stable if $a > 0$ and unstable if $a < 0$.

Example:

The demand function is D(p) = a – bp and that of the supply is S(p) = α + β*p*, where *a*, *b*, α, and β are positive constants. If the speed at which the price changes is proportional to the difference between supply and demand, find the equilibrium price and examine its stability.

Since $p'(t)=\lambda(D(p)-S(p))$ with $\lambda>0$ from the supply and demand functions we have

$p'(t)+\lambda(b+\beta)p(t)=\lambda(a-\alpha)$. Consequently, the general solution of this differential equation is

$p(t)=Ce^{-\lambda(b+\beta)t}+(a-\alpha)/(b+\beta)$ and the equilibrium price is $(a-\alpha)/(b+\beta)$. Since $\lambda(b+\beta)>0$, the equilibrium derived is stable.

Check Your Progress 2

1) Find the general solution of $y'(t)+(1/2)y(t)=1/4$. Determine the equilibrium state and examine its stability.

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2) Find the general solution of the differential equation $y(t)-3y(t)=5y'(t)-3y(t)=5$ if the initial value is given as $y(0) = 1$.

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3) Solve the differential equation, $ty'(t)+2y(t)+t=0$ for $t\neq 0$.

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7.4.3 Solving Second-order Differential Equations

General form

A **second-order ordinary differential equation** consists of time as the independent variable with the dependant variable y with its first and second derivatives. Consider for example an equation $G(t, y(t), y'(t), y''(t)) = 0$ for all t such that we can write it in the form

$$y''(t) = F(t, y(t), y'(t)) .$$

Equations of the form y″(t) = F (t, y′(t))

Take an equation of form

$$y''(t) = F(t, y'(t)),$$

in which $y(t)$ does not appear. See that can be reduced to a first-order equation if we take $z(t) = y'(t)$.

Example:

Consider *Arrow-Pratt measure of relative risk aversion,*

$\rho(w) = \dfrac{-wu''(w)}{u'(w)}$ where $u(w)$ is postulated a function for wealth w.In such a formulation, if we consider two utility functions, u and v, then greater risk-aversion of the former is assumed whenever $\rho_u(w) > \rho_v(w)$.

Find the utility function that has a degree of risk-aversion independent of the level of wealth? Or, for what utility functions u do we have an equation

$a = \dfrac{-wu''(w)}{u'(w)}$ for all w?

Note that we have a second-order differential equation in which the term $u(w)$ does not appear. If we define $z(w) = u'(w)$, then

$$a = \frac{-wz'(w)}{z'(w)}$$

or, $az(w) = -wz'(w)$.

The equation becomes separable and we can write as

$$a\frac{dw}{w} = -\frac{dz}{z}.$$

Consequently, its solution is given by

$$a\ln w = -\ln z(w) + C$$

or, $a \ln w = \ln z(w) + C,$

or, $z(w) = Cw^{-a}$

Since we have taken $z(w) = u'(w)$, to get u by integrating

u(w)=Clnw+B if a =1

and

$\frac{Cw^{-a}}{a-1} + B$ if $a \neq 1.$

Thus, when a takes this form we have a utility function with a constant degree of risk-aversion.

Linear second-order equations with constant coefficients

A **linear second-order differential equation with constant coefficients** takes the form

$$y''(t) + ay'(t) + by(t) = f(t)$$

for constants a and b and a function f. The above equation is **homogeneous** when if $f(t) = 0$ for all t.

Let us call $y''(t) + ay'(t) + by(t) = f(t)$ as the "original equation" and assume that y_1 as its solution. For any other solution of this equation y, define $z = y - y_1$.

Since $y - y_1$ can be writen as

$$\left[y''(t) + ay'(t) + by(t)\right] - \left[y''_1(t) + ay'_1(t) + by_1(t)\right] = f(t) - f(t) = 0,$$

z is a solution of the homogeneous equation

$$y''(t) + ay'(t) + by(t) = 0$$

Further, for every solution z of the homogeneous equation, $y_1 + z$ is also a solution of original equation. Therefore, as has been discussed above in first order non-homogenous equation case, solutions of the original equation may be found by

- a particular solution of the equation and
- adding to it the general solution of the homogeneous equation.

Finding the general solution of a homogeneous equation

Recall that the solution derived in case of first-order homogenous equation was of the form $y(t) = Ae^{rt}$. Therefore, we can write $y'(t) = Ae^{rt}$ and $y''(t) = Ae^{rt}$. Substituting these into $y''(t) + ay'(t) + by(t)$

We get

$r^2 Ae^{rt} + arAe^{rt} + bAe^{rt}$

$= \mathrm{Ae}^{\mathrm{rt}}(r^2 + ar + b).$

Thus, for $y(t)$ to be a solution of the equation we need

$r^2 + ar + b = 0.$

This equation is knows as the **characteristic equation** of the differential equation.

Let us look at the solutions offered by the characteristic equaion. If

- $a^2 > 4b$, then there are two distinct real roots, say r_1 and r_2. We have both $y(t) = A_1e^{r_1t}$ and $y(t) = A_2e^{r_2t}$ as solutions the equation for any values of A_1 and A_2. Hence, also $y(t) = A_1e^{r_1t} + A_2e^{r_2t}$ is a solution. It can be shown that *every* solution of the equation takes this form;

- $a^2 = 4b$, then the characteristic equation has a single real root. The general solution of the equation is $\left(A_1 + A_2\right)e^{rt}$, where $r = -\frac{1}{2}a$;

- $a^2 < 4b$, then the characteristic equation has complex roots. Derivation of the results on these roots will be taken up in greater details in Unit 8 and we will present here the general solution of the equation as

  $\left(\mathrm{A}_1\cos(\beta \mathrm{t}\right) + A_2 \sin(\beta t))e^{\alpha t},$

where $\alpha = -\frac{a}{2}$ and $\beta = \sqrt{\left(b - \frac{a^2}{4}\right)}$. We will express this solution alternatively as $\mathrm{C}e^{\alpha t}\cos(\beta t + \omega)$, where the relationships between the constants C, ω, A_1, and A_2 are A_1=C cosω and $\mathrm{A}_2 = -$C sinω.

Solution of a second order nonhomogeneous equation

We follow a procedure similar to the one in case of first order equation. In the second order equation take a linear combination of $f(t)$ and its first and second derivatives to try for solution that satisfies the equation. If, for example,

- $f(t) = 3t - 6t^2$, then examine the values of *A*, *B*, and *C* such that $A + Bt + Ct^2$ is a solution;

- $f(t) = 2\sin t + \cos t$, find values of *A* and *B* such that $f(t) = A\sin t + B\cos t$ is a solution;

- $f(t) = 2e^{Bt}$ for some value of *B*, find a value of *A* such that Ae^{Bt} is a solution.

Stability of solutions of second order homogeneous equation

Consider the above homogeneous equation

$$y''(t) + ay'(t) + by(t) = 0.$$

If $b \neq 0$, this equation has a single equilibrium, viz., 0. That is, the only constant function that is a solution is equal to 0 for all *t*. We will consider three possible forms of the general solution of the equation to evaluate the stability of such an quilibrium.

Case I: Characteristic equation has two real roots

If r_1 and r_2, are the two roots of the characteristic equation, then the general solution of the equation is $y(t) = Ae^{r_1 t} + Be^{r_2 t}$. The equilibrium is stable if and only if $r_1 < 0$ and $r_2 < 0$.

Case II: Characteristic equation has a single real root

With a single root (say *r*), the characteristic equation is in stable equilibrium if and only if this root is negative. Note that if $r < 0$ then for any value of *k*, $t^k e^{rt}$ converges to 0 as $t \to \infty$.

Case III: Characteristic equation has complex roots

When the characteristic equation has complex roots, the form of the solution of the equation is $\mathrm{Ae}^{\mathrm{at}} \cos(\beta t + \omega)$, where $\alpha = -\frac{a}{2}$, the *real* part of each root. The equilibrium will be stable if and only if the real part of each root is negative.

On the basis of the above results we can sya that the stability of the equilibrium is ensured if and only if the real parts of both roots of the characteristic equation are negative. A bit of algebra shows that this condition is equivalent to $a > 0$ and $b > 0$.On the other hand, if $b = 0$, then every number is an equilibrium, and none of these equilibria is stable.

Check Your Progress 3

1) Solve the differential equation $y''(t) + y(t)' - 2y = -10$ with initial conditions $y(0) = 12$ and $y'(0) = -2$.

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2) Solve the differential equation $y''(t)+6(t)+9y=27$ with initial conditions $y(0)=5$ and $y'(0)=-5$.

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7.4.4 Economic Applications: Examples

a) **The Harrod-Domar Analysis of Steady Growth**

Consider a macroeconomic model consisting of the following equations:

$$S(t) = sY(t),\ 0 < s < 1$$

$$I(t) = v\frac{dy}{dx},\ v > 0$$

$$S(t) = I(t)$$

where Y, S, I stand for the rates of flow of national income, planned saving and planned investment at any point of time t. The first equation says that a constant fraction s of income is saved in each time period. The second equation represents the acceleration theory of investment in which induced investment is proportional to the rate of change of income (v is a constant of proportionality). There is no autonomous investment. For dynamic equilibrium, we need equally between saving and investment as each period. This is the significance of the final equation. The three equations lead to a simple differential equation

$$\frac{dY}{dt} - \frac{s}{v}Y = 0 \qquad \dots (14)$$

By formula (10), the general solution is

$$Y(t) = Ae^{(s/v)t}$$

Suppose it is known that $Y = Y_0$ at $t = 0$. Then $A = Y_0$ so that the equation becomes

$$Y(t) = Y_0\, e^{(s/v)t} \qquad \dots (15)$$

This solution gives the behaviour of national income over time in dynamic equilibrium. For any variable x that changes with time, its **rate of growth** at any point in time is defined to be $\left(\frac{1}{x}\frac{dx}{dt}\right)$, the rate of change of x divided by (as a proportion of) the value of x at that point of time. It is clear from equation (14) that in this model the rate of growth of national income $\left(\frac{1}{Y}\frac{dY}{dt}\right)$

assumes the constant value $\frac{s}{v}$. In other words, national income grows at the constant rate $\frac{s}{v}$ in dynamic equilibrium.

It follows as a corollary from our discussion that if a variable x is changing at a constant rate g over time (growing if g > 0, decaying if g < 0), then its time path is given by $x = x_0 e^{gt}$ where x_0 is the value of x at initial or base period (t = 0).

Our next application is taken from microeconomics. It will help us to understand the notion of stability.

b) **The Dynamics of Price in a Single Market**

Suppose the demand and supply functions for a particular commodity are given by

$$D_t = a - bP_t$$

$$S_t = c + bP_t;\ a, b, c, d > 0,\ a > c$$

The first equation tells us that demand D_t in a particular period t is a decreasing linear function of price prevailing in that period P_t. The second equation, the supply function, has a similar interpretation.

The equilibrium price in this context has the property that it (i) clears the market in each period and (ii) does not change over time. Let us denote it by P^*. (Note that since it is constant through time there is no t-subscript). Writing P^* for P_t in the demand and supply equations and setting $D_t = S_t$ we obtain the expression for the equilibrium price

$$P^* = \frac{a-c}{b+d}$$

The restriction a> c ensures that P^* is positive. Now comes an important point. Knowledge of the equilibrium price tells us nothing about the behaviour of price out of equilibrium. In other words, although we know P^* we do not know what happens when in any period t the price Pt is not equal to P^*. Does price rise or fall or fluctuate in some unpredictable manner? To answer questions of this type precisely we have to introduce a dynamic **adjustment rule** for price. It seems natural and sensible to assume that price will tend to rise if demand exceeds supply and fall if demand falls short of supply and stay unchanged if demand and supply just balance in any period. This type of adjustment is incorporated in the analysis through a simple liner relationship.

$$\frac{dP}{dt} = \theta(D_t - S_t),\ \theta > 0 \qquad \dots (16)$$

Since θ is a positive constant, this tells us that

i) $\frac{dP}{dt}$ = 0 or P rises if $Dt > S_t$

ii) $\frac{dP}{dt}$ = 0 or P falls if $Dt < S_t$ and

iii) $\frac{dP}{dt}$ = 0 or P stays unchanged if $Dt = S_t$

Substituting the demand and supply functions we obtain the differential equation for price

$$\frac{dP}{dt} = \theta(b + d)\, P = \theta(a - c).$$

This, being of the form (9), has the solution

$$P = Ae^{-\theta(b + d)t} + \frac{a-c}{b+d}.$$

Suppose it is known that $P = P_0$ at t = 0. Then from the above solution

$A = \left(P_0 - \frac{a-c}{b+d}\right)$. But $\frac{a-c}{b+d} = P^*$, the equilibrium price. So the solution can be written as

$$P = (P_0 - P^*)\, Ae^{-\theta(b + d)t} + P^* \qquad \ldots (17)$$

This completely describes the time path of price in the market characterised by the given demand and supply curves and the adjustment rule (16). Note that the time path (for any given initial price P_0) is determined by the demand and supply parameters (a, b, c, d) and the coefficient of adjustment θ. The dependence of the solution on θ brings out clearly the importance of adjustment rules in dynamics.

Now let us take up the question of **stability**. Suppose that the initial price P_0 is not the equilibrium price P^*, that is $(P_0 - P^*) \neq 0$. The system is stable if price tends to approach the equilibrium price P^* as time passes, that is if $\underset{t\to\infty}{Lt}\, P(t)=P^*$.

It is clear that since $\theta > 0$ and (b + d) > 0 the term $e^{-\theta(b + d)t}$ (and hence the first term of (16) will tend to zero as t tends to infinity, so that P will indeed converge to P^* and we have a system that is dynamically stable.

7.5 LET US SUM UP

Economic models with a temporal dimension involve relationships between the values of variables at a given point in time and the changes in these values over time. Solution to such problems are attempted by taking time is as a continuous (or discrete) variable. In this unit, we have discussed some of the basic tools of dynamic analysis – the indefinite integral, the definite integral and differential equations. In the process, we have learnt the application of such tools in solving problems related to economic dynamics.

7.6 KEY WORDS

Consumer's Surplus: This notion was introduced by Alfred Marshall to measure the net benefit that a consumer enjoys from his act of purchasing a particular commodity in the market. It is defined in terms of the excess of the consumer's total willingness to pay in units of money over his actual expenditure.

Definite Integral: The definite integral of the function *f*(x) over the interval (a, b) is expressed symbolically as $\int_a^b f(x)dx$, read as "integral of *f* with respect to x from a to b. The smaller number a is termed the **lower limit** and b, the **upper limit** of integration. Geometrically, this definite integral denotes the area under the curve representing *f* (x) between the points x = 0 and x = b. Note that the definite integral $\int_a^b f(x)$ is a number.

Differential Equations: Differential equations are equations involving the derivatives (or differentials) of unknown functions. Solving a differential equation means finding a function that satisfies that equation.

Economic Dynamics: Dynamics is essentially concerned with change and the effects of change on the behaviour of variables over time. Economic dynamics deals with economic variables like national income, price, etc. The task of dynamics is to consider the actual process of transition from the initial pre-change position to the final equilibrium.

Equilibrium: If an initial value has a solution that is a constant function (i.e., independent of *t*), then the value of the constant is called an equilibrium state or **stationary state** of the equation.

Improper Integral: It is a special type of definite integral. When one of the limits of integration is + ∞ or – ∞, a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits.

Indefinite Integral: The indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function. Indefinite integration reverses the process and finds the unknown function whose rate of change (derivative) is given.

Initial Value: To solve the differential (or difference) equation by specifying the value of y or the value of its derivatives at any value of *t*.It may not necessarily be the "first" value.

Stable Solution : If, for all initial conditions, the solution of the differential (or difference) equation converges to the equilibrium as *t* $t \to \infty$, then the equilibrium is **stable**.

7.7 SOME USEFUL BOOKS

Archibald, G.C. and R.G. Lipsey, 1983, *An Introduction to a Mathematical Treatment of Economics* (Third Edition), ELBS London, Chapters 12, 13 and 14.

Baumol, W.J., 1974, *Economic Dynamics* (Second Edition), Macmillan, New York, Chapter 14.

Chiang, Alpha C., 1983, *Fundamental Methods of Mathematical Economics* (Third Edition) McGraw Hill, International Students Edition Chapters 13, 14.

IGNOU, 1990, *MTE-01: First Elective Course in Mathematics* (Block 3: Integral Calculus).

7.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) a) $y(t)=(\frac{3}{2})t^2+3t+3C)^{\frac{1}{3}}$

b) $y(t)=\frac{t^4}{4}-\frac{t^2}{2}+C$

c) $y(t)=te^t-e^t+\frac{t^2}{2}+C$

d) $y(t)=\log(\frac{1}{2})t^2+t+C$

e) The equation is separable:

b $\int ye^y dy=\int\frac{1}{t}dt$. So integrating by parts on the left to get ye^y-e^y = ln t + C. Thus the solution is defined by the condition $\left(y(t)\text{-}1\right)e^{y(t)}$ = ln t + C.

f) The equation is separable:

$\int(1/(4y+1))dy=\int t\,dt$,

so that $(1/4)\ln(4y+1)=(1/2)t^2+C$,

or, $y(t)=C\exp(2t^2)-1/4$.

2) a) $y(t) = Cte^{-1}$; C = 1.

b) $y(t) = C(1 + t^3)1/3$; C = 2.

c) $y(t) =\sqrt{(t^2+C)}$; C = -1.

d) $y(t) =\frac{(2 - C - e^{-2t})}{(C + e^{-2t})}$; C = 1.

3) $y(x) = e^{-4x}+3$

Check Your Progress 2

1) $y(t) = Ce^{-\frac{t}{2}} + \frac{1}{2}$. Equilibrium: y^* = 1/2; stable.

2) $y(t)=Ce^{3t} - \frac{5}{3}$; C=8/3

3) $y(t)=\left(\frac{1}{t^2}\right)\left[C - \frac{t^3}{3}\right] = \frac{C}{t^2} - \frac{t}{3}$

Check Your Progress 3

1) $y(t) = 4e^t + 3e^{-2t} + 5$

2) $y(t) = 2e^{-3t} + te^{-3t} + 3$

7.9 EXERCISES

1) If the rate of change of y with respect to x is 2x and y = 4 when x = 1, find y as a function of x.

2) The rate of change of y with respect to x is $(0.8x - 0.6x^2)$ and y = 0 and x = 0. Find y as a function of x.

3) Let the consumer's demand function be P = 20 – 2q.

Calculate the consumer's surplus for P = 8. Is it larger or smaller than the CS for P = 47?

4) At the rate of interest of 4 per cent a year, what is the present value of Rs.1000 available 2 years later?

5) A piece of land yields a constant rent of Rs.1000 per year. Find its market value if the rate of interest is 10 per cent per year.

6) Solve the equation $\frac{dy}{dx} - 5y = -25$ with y(0) = 6

7) Explain the dynamics of price adjustment process in a single market. What happens if $\theta < 0$?

8) Solve the equation $\frac{dy}{dx} + 4y = 0$, with initial condition y(0) = 1.

9) Find y_c, y_p, the general solution and definite solution of the equation and check its validity $\frac{dy}{dx} + 4y = 12$; y(0) = 1.

10) Solve $\frac{dy}{dx} - 5y = 0$; y(0) = 6 and check its validity.

11) Solve the differential equation $y''(t)+3y'-4y(t)=12$ with initial condition $y(0)=4$, $y'(0)=2$. Check the stability of the solution.

12) Find the particular solution of the defferential equation

$$y''(t)+y'(t)-2y=-10.$$

13) Solve the differential equation $y''(t)-4y'(t)+4y(t)=5$ with the initial conditions $y(0)=4$ and $y'(0)=6$.

Answer or Hints to Exercises

1) $\frac{dy}{dx}=2x$

or $\int dy=\int 2xdx=2\int xdx=\frac{2x^2}{2}+c=x^2+c$

or y = x^2 + c

Now y = 4 when x = 1 so that 4 = 1 + c or c + 3.

∴ y = *f*(x) = x2 + 3

2) $\frac{dy}{dx}=0.8x-0.6x^2$

or $\int dy=\int(0.8x-0.6x^2)dx$

= $\int 0.8xdx-\int 0.6x^2dx$

$=0.8\int xdx-0.6\int x^2dx$

∴ y = $0.8\frac{x^2}{2}-0.6\frac{x^3}{3}+c$

or y = $0.4x^2 - 0.2x^3$ + c

Now y = 0 for x = 0 ⇒ c = 0

∴ y = $0.4x^2 - 0.2x^3$

3) P = 20 – 2q = *f*(q)

For P =8, 12 = 2q or q = 6.

∴ Pq = 48 = total expenditure when P = 8.

$$\int_0^6 f(q)dq=\int_0^6(20-2q)dq$$

$$= 20\int_0^6 dq - \int_0^6 2q\,dq$$

$$= 20\int_0^6 dq - 2\int_0^6 q\,dq$$

$$= 20[q]_0^6 - 2\left[\frac{q^2}{2}\right]_0^6 = 120 - 36 = 84.$$

$\therefore$ Consumer's surplus $= \int_0^8 f(q)dq - Pq = 84 - 48 = 36.$

Now, for P = 4, q = 8.

$$\text{CS} = \int_0^8 (20 - 2q)dq - 32 = 64$$

$\therefore$ Consumer's surplus increases as the price of the commodity falls.

4) Let A = amount; P = principal, r = rate of interest, n = number of years.

Then applying the compound interest formula,

$A = P(1 + r)^n$,

Here A = 100, r = 0.04, n = 2, P = ?

Hence $P = \dfrac{100}{(1+0.04)^2} = \dfrac{100}{(1.04)^2}$

5) Let Y = market value of land

$$Y = \int_0^\infty \text{R}e^{-rt}dt$$

$= \dfrac{R}{r}$ (For the steps which you have to do, refer to Example of Economic Applications of the Definite Integral). Here R = 1000, r = 0.1

$$Y = \frac{1000}{0.1} = 10{,}000$$

6) $\dfrac{dy}{dx} - 5y = -25$

Here m = –5, k = –25

$\therefore$ $y = Ae^{-mx} + \dfrac{k}{m}$ becomes

$y = Ae^{5x} + \frac{-25}{-5} = Ae^{5x} + 5.$

Now, y(0) = 6 ⇒ $Ae^0 + 5 = 6$ or A.1 = 1 or A = 1.

∴ $y = e^{5x} + 5$, is the answer.

7) See Section 7.4.1 Example.

   If θ < 0, the system cannot attain a stable equilibrium, i.e., the system is unstable. You reason why.

8) $y(x) = [1 - 0]e^{-4x} + c = e^{-4x}$

9) $y(x) = -e^{-4x} + 3$

10) $y(x) = 6e^{5x}$

11) The roots of the characteristic equation are 1 and –4. A particular integral is $y(t) = -3$. Thus, the general solution is

$$Ae^{t} + Be^{-4t} - 3.$$

   For the given initial conditions we have A = 6 and B = 1. The general solution is unstable.

12) $y_p = \frac{b}{a}t$

13) $y(t) = A_1e^{2t} + A_2te^{2t} + \frac{5}{4} with A_1 = \frac{11}{4} and A_2 = \frac{1}{2}$