UNIT 8 DIFFERENCE EQUATIONS AND APPLICATIONS IN ECONOMIC DYNAMICS

Structure

8.0 OBJECTIVES

After going through this unit you should be able to:

- solve problems of economic dynamics where the time variable takes only discrete values.

8.1 INTRODUCTION

A **difference equation** is used to solve the values of an unknown function y(x) for different discrete values of x. We obtain a function y(x) such that it satisfies the equation for all values of x. In order to understand the process of formulation of the difference equation, you may recall the discussion on differential equation presented in the preceding unit. See that difference and differential equations are exactly analogous with the only difference that the former applies when the independent variable takes only discrete values, whereas the latter when it is continuous.

8.2 DIFFERENCE EQUATIONS IN ECONOMICS

To get an idea about how difference equations may come up in economics consider the case where it is known that the national income y of a particular country has been growing at a constant rate g over (say) a ten year period starting from some base year. The rate of growth of y at any period t may be represented as $\left(\frac{y_t - y_{t-1}}{y_{t-1}}\right)$. Note that this is the expression that gives the rate of growth of y at a particular point in time. In contrast, you have seen the expression $\frac{dy(t)/dt}{y}$ when y was treated as a continuous variable. Treating y as discrete, we find the numerator as the increment of income in current period (period t) over the level attained in the immediately preceding period (period t – 1). The ratio of this to the income y_{t-1} of the preceding period gives the current rate of growth. Since we have said that rates of growth of income, g, is constant (i.e., independent of t) over ten year's interval, it can be written as:

$$\frac{y_t - y_{t-1}}{y_{t-1}} = g, \; t, = 1, 2, 3, 4, \ldots, 10$$

or, $y_t = (1 + g)\, y_{t-1}$, t = 1, 2, …, 10…………………………………….. (1)

Equation (1) relates the values of the variable y at two distinct periods t and (t – 1). It is an example of a difference equation.

There is a one-period lag in the values of the relevant variable (y_t and y_{t-1}). Therefore, it is an example of a **first order** difference equation. The **order** of a difference equation is determined by the **maximum** number of periods lagged. Some examples of difference equations are given below with the orders noted.

$y_{t-3} - 3y_{t-4} = 0$	order 1.
$y_t = a(y_{t-1} - y_{t-2}) + 10$	order 2.
$\log y_{t+9} - y_{t+7}\,(y_{t+6})^3 + 6y_t = 0$	order 9.
$18\, y_{t+4} - y_t = 2^t - 5^{5+1}$	order 4.
$y_{t+3} + ay_{t+1} = by_{t-1} + c$	order 4

Consider a difference equation of the following form:

$y = a_1 y_{t-1} + a_2 y_{t-2} + \ldots. + a_n y_{t-n} + b$

where $a_1, a_2, \ldots, a_n$ and b are constants.

This is called an **n^{th} order linear constant coefficient** difference equation (assuming $a_n \neq 0$, otherwise the order will be less than n). It is linear because the dependent variable y is not raised to any power and there are no product terms, constant coefficients because $a_1, \ldots, a_n$ are constants and do not change with t. This equation will be **homogeneous** if b = 0. If b ≠ 0, then it is **non-homogenous**. In this unit, we shall work only with difference equations of this special type of orders one and two (n = 1, 2).

8.3 SOLVING FIRST ORDER DIFFERENCE EQUATIONS

In solving a difference equation, we find a time path y(t) from a given initial condition. As pointed out above, a first order difference equation takes the form

$y_t = f(t, y_{t-1})$ for all *t*.

We can solve such an equation by successive calculation,also called recursive method , taking the initial value of y (say y_0) as given. Thus,

$y_1 = F(1, y_0)$
$y_2 = F(2, y_1) = f(2, f(1, y_0))$
and so on.

Note that given any value y_0, there exists a unique solution path y_1, y_2, ...

However, resorting to calculation of the solution through such a method doesn't tell us much about the properties of the solution. We should have a general formula, which exists if the form of f is simple.

Let us start with a first-order linear difference equation with constant coefficient. It takes the form

$y_t = ay_{t-1} + b_t$ where b_t for $t = 1$, ... are constants.

When the recursive method is used, you will see a pattern as follows:

$$y_t = ay_0 + \sum_{k=1}^{t} a^{t-k} b_k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

and such an equation has a unique solution path.To check that we get the unique solution from the above formulation, verify that it satisfies the original equation.

Since we have

$$ay_{t-1} + b_t = a\left(a^{t-1}y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k\right) + b_t$$

$$= a^t y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k + b_t$$

$$= a^t y_0 + \sum_{k=1}^{t} a^{t-k} b_k$$

$$= y_t \,,$$

so that the solution obtained is correct.

Taking the equation (2), we can examine the special case of

$b_k = b$ for all $k = 1$……..

We have

$$y_t = a^t y_0 + b\sum_{j=0}^{t-1} a^{t-1}$$

Making use of the result of geomtric series summation, the term $\sum_{j=0}^{t-1} a^{t-1}$ may be expanded as $1 + a + a^2 + ... + a^{t-1}$ to give

$1 + a + a^2 + ... + a^{n-1} = (1-a^n)/(1-a)$.

if $a \neq 1$. Thus we have

$y_t = a^t y_0 + b\cdot(1 - a^t)/(1 - a)$

if $a \neq 1$.

For any given value y_0, the unique solution of the difference equation

$y_t = ay_{t-1} + b$,

where $a \neq 1$, is

$y_t = a^t(y_0 - b/(1 - a)) + b/(1 - a)$.

Equilibrium or Stationary Value

For a given value y_0, the value of y_t changes with *t*. But there may be some value of y_0 for which y_t doesn't change. Such a solution exists if

$y^* = b/(1 - a)$

and y_t is constant, equal to $b/(1 - a)$.

We call y* the **equilibrium** value of y and rewrite the solution as

$y_t = a^t(y_0 - y^*) + y^*$.

Example: Solve $y_{t+1} = \alpha y_t + \beta$, ……………………..…..(3)

where α and β are constants.

Look for a stationary or equilibrium value of y_t over time which can be repeated for any t consistently satisfying the above equation. May be you consider $\bar{y}$ as an equilibrium value of y_t such that

$$\bar{y} = \alpha \bar{y} + \beta$$
$$or, \bar{y} = \frac{\beta}{1-\beta}$$

To understand the above example, we need to remember the dynamic multiplier.

Write

$C_t = \alpha y_{t-1} + \beta$ ……………………………………..(4)

Let the investment be fixed at $\bar{I}$ for every t so that we have

$$Y_t = C_t + I_t = C_t + \bar{I}$$
$$= \alpha Y_{t-1} + (\beta + \bar{I})$$
$$= \alpha Y_{t-1} + \beta^{'}$$

where $\beta^{'} = \beta + \bar{I}$

Use the above relation (4) we have

$$Y_{t+1} = \alpha Y_t + \beta^{'}$$

If an equilibrium income $\bar{Y}$ is found, the solution can be written as

$$\bar{Y} = \alpha \bar{Y} + \beta^{'}$$
$$= \frac{\beta^{'}}{1-\alpha} = \frac{\beta + \bar{I}}{1-\alpha}$$

Note that $\frac{1}{1-\alpha}$ is the Keynesian multiplier.

It is important to remember that we have solved equation (3) for the stationary level of y_t i.e., $\bar{y}$. There is no guarantee that the actual path of y converges to $\bar{y}$. In case y_t approaches $\bar{y}$, then

$$(y_t - \bar{y}) \to 0.$$

If these values of y_t and y_{t+1} hold, we can write

$g_t = y_t - \bar{y}$ ………………..(5)

Since y_t and $\overline{\mathrm{y}}$ satisfy (3), we have

$$y_{t+1} = \alpha y_t + \beta \text{ and}$$
$$\overline{\mathrm{y}} = \alpha \overline{y} + \beta$$

Thus,

$$y_{t+1} - \overline{y} = \alpha\left(\mathrm{y}_{\mathrm{t}} - \overline{y}\right).$$

From (5),

$g_t = y_t - \overline{y}$

or, $\mathrm{g}_{\mathrm{t+1}} = y_{t+1} - \overline{y}$

$or,\ \mathrm{g}_{\mathrm{t+1}} = y_{t+1} - \overline{y}$

$or, g_{t+1} = \alpha g_t$..................(6)

Since

$g_{t+1} = \alpha g_t$

$g_t = \alpha g_{t-1}$

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$g_1 = \alpha g_0$

Substituting backward,

$g_{\&+1} = \alpha^2 g_{t-1} = \alpha^3 g_{t-2}$ ....

we get

$g_{t+1} = \alpha^{t-1}\mathrm{g}_0$

$or\ \mathrm{g}_{\mathrm{t}} = \alpha^t \mathrm{g}_0$ for t = 0, 1, 2 ....

Thus, any difference equation of the form $y_t = \alpha y_{t-1}$ has a solution $y_t = \alpha^t y_0$, where y_0 is the value of y at some chosen initial point.

General Solution

Suppose we intend to solve the equation

$y_{t+1} + ay_t = c$................ (7)

Its general solution will be consisting of particular solution $\left(y_p\right)$ and complementary function $\left(y_c\right)$, i.e., $\mathrm{y_g} = y_p + y_c$. In this approach, the y_p component represents the inter-temporal equilibrium level of y while that of y_c gives the deviations if the time path from that equilibrium. The solution is called general solution due to the presence of an arbitrary constant. In order to get a definite solution, we need an initial condition.

Let us work with complementary function. From (7), we get its reduced from as

$y_{t+1} + ay_t = 0$ ……………………………..(8)

It is seen above that $y_t = \alpha^t y_o$ is a solution to the difference equation. In that case we have $y_{t+1} = a^{t+1} y_0$ as well. We modify this and rewrite

$y_t = Ab^t$ and $y_{t+1} = Ab^{t+1.}$

Substitution of these into (8) gives

$$Ab^{t+1} + aAb^t = 0$$

$$or, Ab^t\left(b+a\right) = 0$$

$$or, (b+a) = 0$$

$$or, b = -a$$

We must have $b = -a$ in the trial solution such that the complementary solution can be written as

$$y_c = Ab^t = A\left(-a\right)^t.$$

Particular solution needs to be recasted such that it is in agreement with the general solution. Consider the simplest value of y. If y_t has an equilibrium value k such that it remains constant overtime, we have $y_t = k$ as well as $y_{t+1} = k.$ Substitution of these values to the trial solution gives

$$k + a_k = c$$

$$or,\ \mathrm{k} = \frac{\mathrm{c}}{1+\mathrm{a}}$$

Since the value, k, satisfies the equation, the particular solution can be written as

$$y_p = k = \frac{c}{1+a} \quad \text{for} \quad a \neq 0$$

In case $a = -1$, however, the particular solution is not defined and some other solution of (7) must be searched for.

Substituting k into (7), we get

$$k\left(t+1\right) + ak_t = c.$$

$$or,\ \mathrm{k} = \frac{\mathrm{c}}{\mathrm{t+1+a_t}} = C \text{ and } \mathrm{y_p} = C_{t.}.$$

The general solution can now be written in one of the following forms:

$$y_{tt} = A(-a)^t + \frac{C}{1+a} \quad \text{if} \quad a \neq -1$$

$$or,\ y_t = A(-a)^t + C_t = A + C_t \ \text{if } a = -1.$$

Notice that the solution above still remains indeterminate. This is due to the presence of arbitrary constant A. We have to take the help of initial condition $(y_t = y_o)$ for eliminating it. Thus, taking t =0, we have

$$y_0 = A + \frac{C}{1+a}$$

$$or, A = y_9 - \frac{C}{1+a}$$

The definite solution therefore, becomes

$$y_t = \left(y_0 - \frac{C}{1+a}\right)(-a)^t + \frac{C}{1+a} \quad \text{for } a \neq -1$$

$$or, y_t = y_o + C_t \quad \text{for } a = -1$$

8.3.1 Behaviour of Solutions of First Order Equations

The solution of a difference equation gives an expression for the relevant variable as an explicit function of time. In other words, a time path of the variable is obtained. To investigate the nature of this time path of a solution of the first order equation, we write the solution for a ≠ 1.

The behavior of the solution path depends on the value of *a*.

$|a| < 1$

y_t converges to y* and the solution is **stable**. There are two subcases:

$0 < a < 1$,

Monotonic convergence.

$-1 < a < 0$

Damped oscillations.

$|a| > 1$

Divergence:

$a > 1$

Explosion.

$a < -1$

Explosive oscillations.

To understand these features see Figure 8.1.

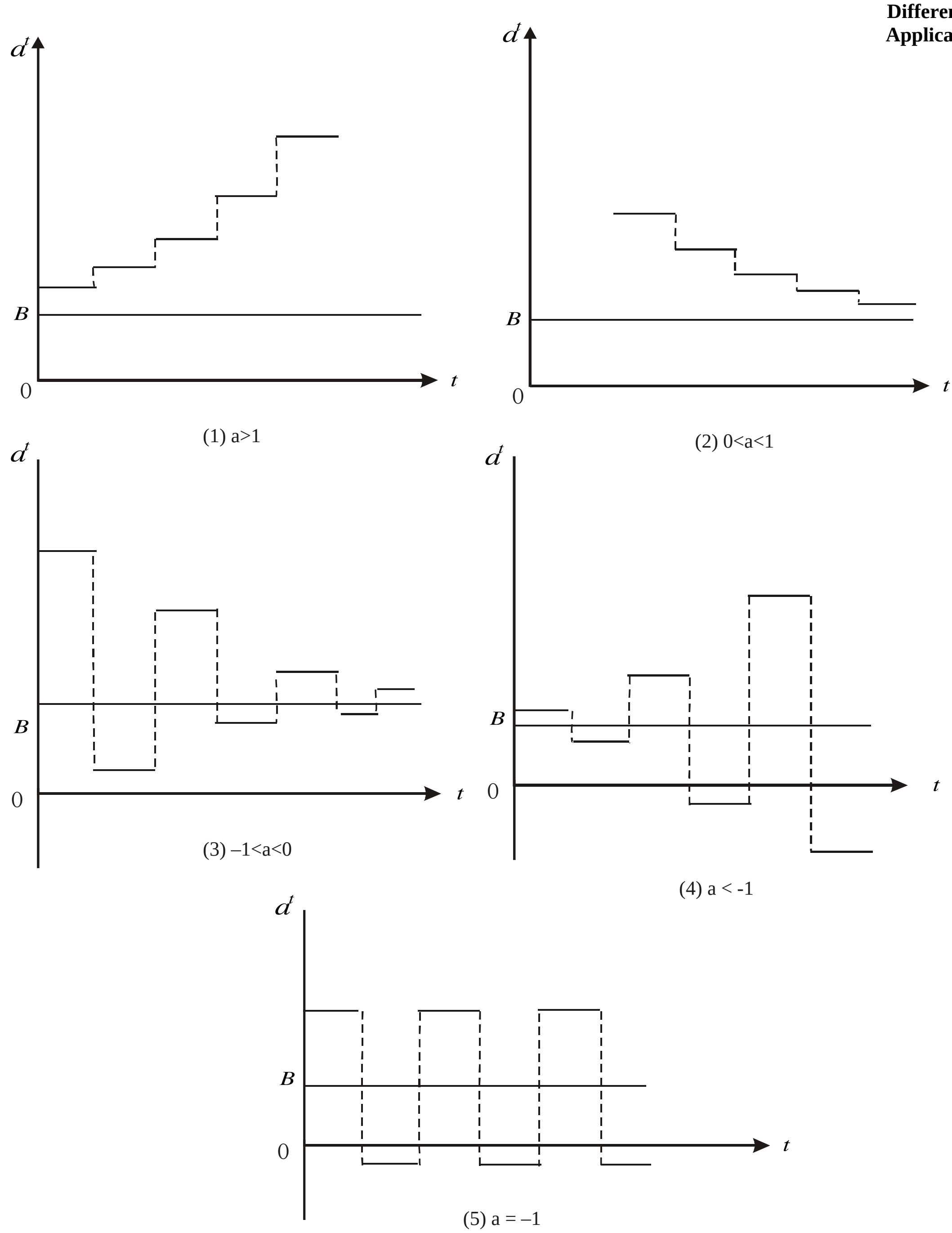

Fig. 8.1

In short,

$|a| > 1$ time path explodes (diverges)

$|a| < 1$ time path converges

$a > 0$ time path non-oscillating

$a < 0$ time path oscillating.

Thus, the condition for stability is $|a| < 1$.

The different cases are shown in Figure 8.1.

8.3.2 Economic Applications of First Order Equations

We consider three applications of the type of equations discussed in the previous section. The first is an analysis one sector Harrod-Domar model while the second is of price dynamics. The last one deals with the amortisation problem of hire purchase of consumer durables.

a) **Harrod-Domar One Sector Model**

An economy produces one good Q with capital K through a production function $Q_t = bK_t$, where b = constant productivity of capital. Accumulation of capital between t and t+1 is given by

$I_t = K_{t+1} – K_t$, where I_t = investment in t.

Saving $S_t = sQ_t$.

Equilibrium level of income is determined at the equality of savings and investment. So,

$S_t = I_t$

or, $sQ_t = K_{t+1} – K_t$.

Since $Q_t = bK_t$, we have $sbK_t = K_{t–1} – K_t$

or, $K_{t+1} = (1 + sb)K_t$, a homogeneous first order linear difference equation. Therefore, solution to this equation is given by

$K_t = (1 + sb)^t K_0$.

Since b is productivity of capital in the model, we write $\frac{1}{b}$ = capital output ratio = v (say).

Now $K_t = \left(1+\frac{s}{v}\right)^t K_0$ and

$Q_t = \left(1+\frac{s}{v}\right)^t Q_0$

Remember that $\frac{s}{v}$ = warranted rate of growth and constituted by two basic parameters s and v. We can find out the output growth rate given s and v.

b) **The Cobweb Model**

The essential feature of this model is that production or supply responds to price with a one-period lag. This type of lagged supply response is often observed for agricultural products.

We assume: 1) The market demand and supply functions are linear and do not change over time, 2) demand in any period t responds to price prevailing in the same period t, but supply in t depends on price that prevailed in the last period, (t – 1) and 3) the market is competitive in the sense that the price that prevails in each period is the price that equates demand and supply. Thus, the model can be set out as consisting of the following equations.

$D_t = a – b P_t$; a, b > 0

$S_t = \alpha – \beta P_{t-1;}$ $\alpha, \beta > 0, \alpha < a$

$D_t = S_t$ for all t.

The first equation gives us the simple demand curve in period t. The second displays the lag in supply. Supply in t, S_t, is determined by prices of the immediately preceding period, P_{t-1}. The last equation is the condition of market clearing in each period. The three equations together yield a first order constant coefficient non-homogeneous difference equation in price.

$$P_t = \left(-\frac{\beta}{b}\right)P_{t-1} + \frac{a-\alpha}{b} \qquad \dots (9)$$

With a, b α, β known, a specification of the initial price P_0 allows us to solve the equation as:

$$P_t = \left(P_0 - \frac{a-\alpha}{b+\beta}\right)\left(-\frac{\beta}{b}\right)^t + \frac{a-\alpha}{b+\beta} \qquad \dots (10)$$

From our previous discussion, it is clear that the behaviour of P over time depends crucially on the term $\left(-\frac{\beta}{b}\right)$.

Since this term is negative (b, β > 0) the time path will always be oscillatory. Let us denote the constant $\frac{a-\alpha}{b+\beta}$ by P^*.

Then

$\left(\frac{\beta}{b}\right) > 1$ Price diverges

$\left(\frac{\beta}{b}\right) = 1$ Price oscillates uniformly

$\left(\frac{\beta}{b}\right) < 1$ Price converges to P^*.

Only in the last case (P_t approaches P^* as t increases), the system is **stable**. Thus, the condition for stability is $\left(\frac{\beta}{b}\right) < 1$. Since graphically $\left(\frac{1}{\beta}\right)$ is the slope of the supply curve and $\left(\frac{1}{b}\right)$ that of the demand curve in absolute value, the stability condition states that the slope of the supply curve must be steeper than the absolute value of the slope of the demand curve.

At this point, we pause to note the significance of the value $\frac{a-\alpha}{b+\beta}$. This is the constant value of price that is a solution of the equation (9). To check, substitute $P_t = P_{t-1} = P^*$ (a constant) in (9).

$$P^* = \left(-\frac{\beta}{b}\right)P^* + \frac{a-\alpha}{b}$$

or, $P^* = \frac{a-\alpha}{b+\beta}$.

Thus, $P_t = \dfrac{a-\alpha}{b+\beta}$ is a solution of (9). This type of constant solution is called **Stationary solution**. The price P^* may be called the **equilibrium price** because it equates demand and supply and stays unchanged over time.

Example: We want to investigate the behaviour of price in a market with the demand and supply functions:

$D_t = 86 - 0.8 P_t$

$S_t = -10 + 0.2 P_{t-1}$

Assuming market clearing in each period ($D_t = S_t$) we have

$(-0.8) P_t = 0.2 P_{t-1} - 96$

or, $P_t = (-0.25)P_{t-1} + 120$

The solution is

$$P_t = \left(P_0 - \frac{120}{1+0.25}\right)(-0.25)^t + \frac{120}{1+0.25}$$

$= (P_0 - 96)\,(-0.25)^t + 96$

Since $|-0.25| = 0.25 < 1$, the time path of P is oscillating but converges. The market is stable and with the passage of time price approaches the equilibrium value 96.

c) **The Amortisation Problem**

We are all familiar with the practice of hire purchase or purchase by instalments of consumer durables like refrigerators, cars or T.V. sets. The buyer pays a part of the price at the time of purchase (the down payment) and pays the rest in monthly or annual instalments over a specified period. Because the payments are spread over a period of time, an interest cost is included in the value of instalments. **Amortisation** is the term associated with this method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Let the value of the article purchased by V and P the down payment. Then the initial debt of the buyer is $D_0 = V - P$. The contract states that the debt, D_0 is to be paid off over T periods. The rate of interest is r (100 r%). The question we are interested in is: how is the magnitude of periodic instalment to be determined?

Let us denote the value of the instalment (still unknown) by B. This value stays constant over time. The outstanding debt D_t at the end of the period t obeys the equation

$D_t = (1 + r)D_{t-1} - B$ ……………………….….(11)

This simply says that to find the outstanding debt at the end of the t^{th} period you take the debt outstanding at the end of the previous ($(t-1)^{th}$) period D_{t-1}, add the interest charge on it, rD_{t-1}, but subtract the payment B made in that period. Given the initial debt of D_0 the solution of (11) is

$$D_t = \left(D_0 + \frac{B}{r}\right)(1 + r)^t + \frac{B}{r} \qquad \text{…. (12)}$$

The value of B is to be selected so that the debt disappears at the end of period T, that is, $D_t = 0$. From (12) we get

$$\left(D_0 + \frac{B}{r}\right)(1 + r)^t + \frac{B}{r} = 0$$

or, $B = \dfrac{rD_0}{1-(1+r)^{-T}}$

Thus, we have the exact relationship between the magnitude of the periodic payment and the rate of interest, the magnitude of the initial debt and the time horizon of the contract. The expression:

$$\left(\frac{1-(1+r)^{-T}}{r}\right)$$

is referred to as the **amortisation factor** and value of this factor has been extensively tabulated for different values of r and T.

Check Your Progress 1

1) What is a difference equation? Distinguish it from a differential equation.

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2) Discuss the nature of the following time paths

i) $y_t = 3^t + 1$ (ii) $y_t = 5\left(-\frac{1}{10}\right)^t + 3$

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3) Suppose you find the following the path of y.

$y_t = Aa^t + B$; A < 0, B > 0.

Draw the different cases of the behaviour of y_t for different values of a.

4) Solve the following equations:

i) $y_{t+1} - \frac{1}{3}y_t = 6$ for $y_0 = 1$

ii) $y_{t+1} - y_t = 3$ for $y_0 = 5$

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8.4 SOLVING SECOND ORDER DIFFERENCE EQUATIONS

A general **second-order difference equation** which we have already mentioned at begniging outset of this unit takes the form

$$y_{t+2} = f(t, y_t, y_{t+1}).$$

Just as in the case of first-order equation, a second-order equation will have a unique solution and can be derived by successive (recursive) calculation. We will show that given y_0 and y_1 there exists a uniquely determined value of y_t for all $t \geq 2$. Note that for a second-order equation we need two starting values, y_0 and y_1, in place of one taken in the first order counterpart.

8.4.1 Homogeneous Equations

Consider the following second order constant coefficient equation

$$y_{t+2} + ay_{t+1} + by_t = 0 \qquad \text{…. (13)}$$

We need to find two solutions of the equation above.

If we make a guess that the solution takes the form $u_t = m^t$

In order for u_t to be a solution, we must have

$$m^t(m^2 + am + b) = 0$$

or, if $m \neq 0$,

$$m^2 + am + b = 0.$$

This is called the **characteristic (or auxiliary) equation** of the difference equation and its solutions are

$$-(1/2)a \pm \sqrt{((1/4)a^2 - b)}.$$

8.4.2 Behaviour of Solutions of Homogeneous Equations

Looking at the component $\sqrt{((1/4)a^2 - b)}$, we distinguish three cases:

i) Distinct real roots

If $a^2 > 4b$, the characteristic equation has distinct real roots, and the general solution of the homogeneous equation is

$$Am_1^t + Bm_2^t,$$

where m_1 and m_2 are the two roots.

ii) Repeated real root

If $a^2 = 4b$, then the characteristic equation has a single root, and the general solution of the homogeneous equation is

$$(A + Bt)m^t,$$

where $m = -(1/2)a$ is the root.

ii) Complex roots

If $a^2 < 4b$, then the characteristic equation has complex roots, and the general solution of the homogeneous equation is

$$Ar^t \cos(\theta t + \omega),$$

where A and ω are constants, $r = \sqrt{b}$, and $\cos\theta = -a/(2\sqrt{b})$, or, alternatively,

$C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t)$,

where $C_1 = A\cos\omega$ and $C_2 = -A\sin\omega$ (using the formula that $\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y)$.

When the characteristic equation has complex root, the solution **oscillates**. Ar^t is the **amplitude** (which depends on the initial conditions) at time t, and r is **growth factor**. $\theta/2\pi$ is the **frequency** of the oscillations and ω is the **phase** (which depends on the initial conditions).

If $|r| < 1$ then the oscillations are **damped**; if $|r| > 1$ then they are **explosive**.

Stability

We say that a system of differential equations is **stable** if its long-run behavior is not sensitive to the initial conditions.

Consider the second-order equation

$$y_{t+2} + ay_{t+1} + by_t = c_t.$$

Write the general solution as

$y_t = Au_t + Bv_t + u_t^*$,

where A and B are determined by the initial conditions.

This solution is **stable** if the first two terms approach 0 as $t \to \infty$, for all values of A and B. In this case, for *any* initial conditions, the solution of the equation approaches the particular solution u_t^*. If the first two terms approach zero for all A and B, then u_t and v_t must approach zero. You can take $A = 1$ and $B = 0$ to see that u_t approaches zero. On the other hand, take take $A = 0$ and $B = 1$ to see that v_t approaches 0. A necessary and sufficient condition for this to be so is that the moduli of the roots of the characteristic equation be both less than 1. Note that the modulus of a complex number $\alpha + \beta i$ is $+\sqrt{(\alpha^2 + \beta^2)}$, which is the absolute value of number if the number is real.

There are two cases:

- If the characteristic equation has complex roots then the modulus of each root is $\sqrt{b}$ (the roots are $\alpha \pm \beta i$, where $\alpha = -a/2$ and $\beta = \sqrt{(b - (1/4)a^2)}$). So for stability need $b < 1$.
- If the characteristic equation has real roots then the modulus of each root is its absolute value. So for stability we need the absolute values of each root to be less than 1, or $|-a/2 + \sqrt{(a^2/4 - b)}| < 1$ and $|-a/2 - \sqrt{(a^2/4 - b)}| < 1$.

8.4.3 Non-homogeneous Equations

To find the general solution of the original equation

$y_{t+2} + ay_{t+1} + by_t = c_t$

we need to find one of its solutions. Suppose that $b \neq 0$.

The form of a solution depends on c_t.

Suppose that $c_t = c$ for all t. Then $y_t = C$ is a solution if $C = c/(1 + a + b)$ and if $1 + a + b \neq 0$;

if $1 + a + b = 0$ then try $y_t = Ct$; if that does not yield a solution, we have to try $y_t = Ct^2$.

8.4.4 An Economic Application of Second Order Non-homogeneous Equation

We discuss now an economic example of a second order non-homogeneous equation. This is Samuelson's model of interaction between the multiplier and the accelerator. Consider the following macro-economic equations:

$C_t = C_0 + cY_{t-1}, 0 < c < 1.$

$I_t = I_0 + v(C_t - C_{t-1}); v > 0.$

$Y_t = C_t + I_t$

The symbols Y, C, I stand for national income, consumption and investment respectively. The first equation is the consumption function with a one-period lag; the second is the investment function of the accelerator type. $C_0 + I_0$ are the levels of autonomous consumption and investment. The marginal propensity to consume c and the accelerator coefficient v are assumed to be constant. The final equation is the condition of macro balance. The three equations together generate the following difference equation in Y

$$Y_t - c(1 + v) Y_{t-1} + cvY_{t-2} = C_0 + I_0 \qquad \dots (15)$$

The characteristic equation for the homogeneous part is

$m^2 - c(1 + v) m + cv = 0$

The roots are

$$m_1, m_2 = \frac{1}{2} \left(c(1 + v) \pm \sqrt{c^2(1+v)^2 - 4cv}\right) \qquad \dots (16)$$

Both m_1 and m_2 are positive because from the theory of quadratic equations we know $m_1 + m_2 = c(1 + v) > 0$ and $m_1m_2 = cv > 0$. Since $c(1 + v) - cv \neq 1$, the **particular** solution is $\frac{C_0 + I_0}{1-c}$. Three types of solution are possible depending on the values of c and v.

1) $c^2(1 + v)^2 > 4cv$

   or, $c(1+v)^2 > 4v$, the roots are real and distinct.

   Here,

   $$Y_t = A_1 m_1^t + A_2 m_2^t + \frac{C_0 + I_0}{1-c}; \; A_1, A_2 \neq 0 \text{ and constants}$$

2) $c(1+v)^2 = 4v$, the roots are real and equal with value $\frac{1}{2}c(1+v)$. In this case

   $$Y_t = (A_1 + A_2 t) \left(\frac{c(1+v)^2}{2}\right) + \frac{C_0 + I_0}{1-c};$$

3) $c(1+v)^2 < 4v$, the roots are complex. From (16) we see that the roots are of the form $(a \pm ib)$ with

   $$a = \frac{1}{2}c(1+v)$$

$b = \frac{1}{2}\sqrt{4cv - c^2(1+v)^2}$

$(\Theta \sqrt{c^2(1+v)^2 - 4cv} = \sqrt{i^2\left(4cv - c^2(1+v)^2\right)} = i\sqrt{4cv - c^2(1+v)^2} = ib)$

The modulus of the roots

$r = \sqrt{a^2 + b^2} = \sqrt{cv}$

The solution is

$Y_t = (\sqrt{cv})^2 (A_1 \cos(t\theta) + A_2 \sin(t\theta)) + \frac{C_0 + I_0}{1-c}$

where $\theta = \tan^{-1}\left(\frac{\sqrt{4cv - c^2(1+v)^2}}{c(1+v)}\right)$

In this case, we have a **cyclical** time path of national income Y. If $\sqrt{cv} < 1$, then $(\sqrt{cv})^t$ will tend to zero as t increases and Y_t will approach the value $\frac{C_0 + I_0}{1-c}$.

Thus, the condition for stability (damped oscillations in Y) is $\sqrt{cv} < 1$, that is, the product of the marginal propensity to consume and the accelerator coefficient should be less than unity.

Check Your Progress 2

1) Solve the following difference equations and determine whether the solution paths are convergent or divergent, oscillating or not.

   a) $y_{t+2} + 3y_{t+1} - (7/4)y_t = 9$.

   b) $y_{t+2} - 2y_{t+1} + 2y_t = 1$.

   c) $y_{t+2} - y_{t+1} + (1/4)y_t = 2$.

   d) $y_{t+2} + 2y_{t+1} + y_t = 9 \cdot 2^t$.

   e) $y_{t+2} - 3y_{t+1} + 2y_t = 3 \cdot 5^t + \sin((1/2)\pi t)$.

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2) Find the roots of the equation

   $y_t = ay_{t-1} + by_{t-2}$

Examine when the roots are

1) real, unequal

2) real, equal

3) complex

   What is the auxiliary or the characteristic equation of the equation above? What are the final forms of general solution of the equation in each case?

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3) Find the solutions of the equations:

a) $y_t + 4y_{t-2} = 0$, y = 12, 11 at t = 0, 1 respectively.

b) $y_t = 2y_{t-2} - \frac{4}{3}y_{t-2}$, y = 0, 1 at t = 0, 1 respectively.

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8.5 LET US SUM UP

In continuation with the theme on solving economic problems in a dynamic set up, the present unit took up 'time' as a discrete independent variable and examined tool of simple difference equations. In the process, we considered the solutions of first and second order linear difference equations covering homogeneous and non-homogeneous cases. To see the applications of these equations to economic problems, the time path of adjustment of macro-economic variable – national income – in case of the simple Keynesian multiplier model and Samuelson's multiplier-accelerator interaction model were discussed. We also examined the time path of adjustment of the price variable and looked into the conditions of dynamic stability of the different systems – i.e., whether over time, the economic variables – price or national income – converge to a stable equilibrium. Finally, the conditions when the systems become dynamically explosive – i.e., the variables move further and further away from the equilibrium value were examined.

8.6 KEY WORDS

Amortisation: It is the term associated with the method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Cobweb Model: A model where production or supply responds to price with a one-period lag. This model is often used to analyse the demand-supply mechanism for markets of agricultural commodities.

Constant Coefficient Difference Equation: A difference equation has constant coefficient if the coefficients a_i's associated with the y values are constant and do not change over time.

Difference Equation: A difference equation is an equation involving the values of an unknown function y(x) for different values of x. The independent variable – time in problems of economic dynamics – takes only discrete values. The form of the equation is, $y_t = a_1y_{t-1} + a_2y_{t-2} + \dots + a_ny_{n-1} + b$, where

a_1, a_2, …., a_n and b are constants, is an example of an n-th order linear, constant coefficient, difference equation.

Homogeneous Difference Equation: A difference equation is homogeneous if the constant term b is zero.

Linear Difference Equation: A difference equation is linear if (i) the dependent variable y is not raised to any power and (ii) there are no product terms.

Non-homogeneous Difference Equation: A difference equation is non-homogeneous if the constant term, b, is non-zero.

Order of a Difference Equation: It is determined by the maximum number of periods lagged.

8.7 SOME USEFUL BOOKS

Allen, R.G.D., 1959, *Mathematical Economics* (Second Edition) St. Martin's Press Inc., New York.

Baumol, W.J., 1974, *Economic Dynamics* (Second Edition) Macmillan, New York. Chapters 9, 10 and 11.

Chiang, Alpha C. 1984, *Fundamental Methods of Mathematical Economics* (Third Edition): Mc-Graw Hill International Edition, New Delhi.

8.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 8.1

2) i) Non-oscillatory; divergent

   ii) Oscillatory; convergent

3) Fig. 8.1 shows the time path for A > 0. Draw the corresponding figures for A < 0.

4) i) $y_t = -8\left(\frac{1}{3}\right)^t + 9$; ii) $y_t = -2\left(\frac{-1}{4}\right)^t + 4$

Check Your Progress 2

1)

   a) $A_1\left(\frac{1}{2}\right)^t + A_2\left(\frac{-7}{2}\right)^t + 4$. Nonconvergent oscillations.

   b) $\left(\sqrt{2}\right)^t\left(A_1\cos\left(\frac{\pi}{4}\right)t + A_2\sin\left(\frac{\pi}{4}\right)t\right) + 1$.Nonconvergent oscillation.

   c) $A_1\left(\frac{1}{2}\right)^t + A_2 t\left(\frac{1}{2}\right)^t + 8.$ Convergent, non-oscillating.

   d) The characteristic equation is $m^2 + 2m + 1 = (m+1)^2 = 0$, which has a double root of –1. So the general solution of the homogeneous equation is $y_t = \left(C_1 + C_2 t\right)(-1)^t$. A particular solution is obtained by inserting $u_t^* = A2^t$, which yields $A = 1$. So the general solution of the inhomogeneous equation is $y_t = (C_1 + C_2 t)(-1)^t + 2^t$.

e) By using the method of undetermined coefficients the constants A, B, and C in the particular solution
$u^* = A5^t + B\cos\left(\frac{\pi}{2}\right)t + C\sin\left(\frac{\pi}{2}\right)t$, we obtain $A = 1/4$, $B = 3/10$, and $C = 1/10$. So the general solution to the equation is
$y_t = C_1 + C_2 2^t + \left(\frac{1}{4}\right)5^t + \left(\frac{3}{10}\cos\left(\frac{\pi}{2}\right)t\right) + \left(\frac{1}{10}\sin\left(\frac{\pi}{2}\right)t\right).$

2) See Section 8.4 and answer.

Note that in the text $y_t = ay_{t-1} + by_{t-1}$

Here you have a slightly changed equation.

3) a) $y_t = 2^t\left[12\cos\left(\frac{t\pi}{2}\right) + \frac{11}{2}\sin\left(\frac{t\pi}{4}\right)\right].$

b) $y_t = \left(\frac{2}{\sqrt{3}}\right)^t\left(\frac{\sqrt{3}}{2\sin\theta}\right)\sin(t\theta)$

where $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

8.9 EXERCISES

1) Investigate the behaviour of price in a market, i.e., the stability of a system with demand and supply functions:

a) $D_t = 86 - 0.8\ P_t$

$S_t = -10 + 0.8\ P_{t-1}$

b) $D_t = 86 - 0.8\ P_t$

2) What is amortisation? Derive the exact relationship between the magnitude of the periodic payment B and the rate of interest r, the magnitude of the initial debt D_0 and the time horizon of the contract T.

3) Establish the stability condition of Samuelson's multiplier-accelerator interaction model.

4) Find the time path represented by the equation $y_t = 2\left(-\frac{4}{5}\right)^t + 9$.

5) Find the solution of the equation $y_{t+1} + \frac{1}{4}y_t = 5$ for $y_0 = 2$.

5) The demand and supply for cobweb model is given as $Q_{dt} = 19 - 6P_t$ and $Q_{st} = 6P_{t-1} - 5$. Find the intertemporal equilibrium price and comment on the stability of the equilibrium.

Answer or Hints to Exercises

1) Note that for the Cobweb model

$D_t = a - b\ P_t \qquad a, b > 0$

$S_t = \alpha + \beta P_{t-1}\ \alpha, \beta > 0, \alpha < 0$

$D_t = S_t$ for all t.

A specification of the initial price P_0 allows us to solve the equation.

$$P_t = \left(-\frac{\beta}{b}\right)P_{t-1} + \frac{a-\alpha}{b}$$

$$\text{as } P_t = \left(P_0 - \frac{a-\alpha}{b+\beta}\right)\left(-\frac{\beta}{b}\right)^t + \frac{a-\alpha}{b+\beta}$$

$$\text{or, } P_t = \left(P_0 - P^*\right)\left(-\frac{\beta}{b}\right)^t + P^*; P^* = \frac{a-\alpha}{b+\beta}$$

If $\left(-\frac{\beta}{b}\right)^t \to 0$ as $t \to \infty$, $P_t \to P^*$, the equilibrium value.

This is possible if and only if $\frac{\beta}{b} < 1$

i.e., $\beta < b$.

a) $\beta = 0.8$, $b = 0.8$. Hence $\beta = b$.

This results in uniform oscillation, as $\frac{1}{\beta} = \frac{1}{b}$.

or, the slope of the supply curve = the absolute slope of the demand curve.

b) $b = 0.8$, $\beta = 0.9$

Hence $\frac{\beta}{b} = 1$

or, $\beta > b$.

or, $\frac{1}{\beta} < \frac{1}{b}$.

i.e., the slope of the supply curve is less than the absolute value of the slope of the demand curve.

Hence, price diverges further and further away from the equilibrium and you come across an explosive and oscillatory situation.

c) B = 0.9, $\beta = 0.8$

$\frac{\beta}{b} < 1$ implies damped oscillation and the system is stable.

You should draw diagrams in each case and satisfy yourself.

2) See Example (b) in Sub-section 8.3.4.

3) Sub-section 8.4.4 and answer.

4) Since $b = -\frac{4}{5} < 0,$ the time path is oscillatory. As $|b| = \frac{4}{5} < 1$, the oscillation is damped and it converges to equilibrium level of 9.

5) $y_t = -2\left(-\frac{1}{4}\right)^t + 4.$

6) $\overline{p} = 2$; discuss on uniform oscillation.