UNIT8 THE RAMSEY MODEL

Structure

8.0 OBJECTIVES

After going through this Unit you should be able to:

- explain the Ramsey problem of optimal growth;
- compare the optimal growth problem of the central planner and that of the decentralized perfectly competitive economy;
- explain the role of government in the optimal growth framework and the corresponding concept of Ricardian Equivalence; and
- if you go through the Mathematical Appendix carefully you should be in a position to solve any standard dynamic optimization exercise.

8.1 INTRODUCTION

In our discussion of household's intertemporal consumption and saving decisions in the previous unit have, we considered optimization problems where the time horizon was finite. While this is a valid assumption for an individual, a household (or for that matter, the society as a whole) is generally infinitely lived. In other words, one can think of a household as consisting of individual members, each of whom are finitely lived, but new members are born in each time period who are an exact replica of the older members – so that the household as a dynasty continues to live forever.

An important economic question that is often asked in the context of an infinitely lived household is the following: how should the household allocate its resources between consumption and savings in every period so that the utility of each of its member are maximized? In other words, what are the optimal consumption path and optimal savings path for this household? To put it more generally, in an economy consisting of *identical* infinitely lived household, whaf should be the optimal savings/accumulation pattern that maximizes the utility of *each* of dynastic household? This latter question was first tackled by Frank Ramsey (1928), which was later generalized by Cass (1965) and Koopmans (1965). The model has subsequently been known as the Ramsey-Cass-Koopmans ***optimal growth*** model.

The Ramsey-Cass-Koopmans optimal growth model essentially involves a dynamic optimization exercise whereby an agent maximizes its objective function defined over a period of time (finite or infinite) subject to some **constraints.** The constraints are also dynamic in nature in the sense that they relate to values of the variables for different time periods. The technique for solving such dynamic optimization exercises has been specified in the Mathematical Appendix at the end of the Unit. In the text we shall simply state the conditions for optimization (without getting into the underlying technique) and explain the economic intuition behind each of these conditions.

8.2 PLANNER'S PROBLEM

In the original Ramsey formulation, the optimal growth problem was postulated as the problem of a social planner who seeks to maximize the welfare of each household subject to the economy's resource constraint at each point of time. The households are all assumed to be identical in terms of preferences and their welfare is represented by an infinite horizon utility function of the form $W = \int_0^{\infty} u(c_t) \exp^{-\rho t} dt$, where c_t is the ***per capita consumption*** in period t, $u(c_t)$ is the associated ***instantaneous utility*** in period t, and ρ is positive constant term which represents the ***subjective discount rate*** of the household, or its ***rate of time preference.*** Before we proceed further, it is important to explain the presence of the discount factor in the household's welfare function. The positive discount factor reflects the household's preference for present over future. In other words, it implies that a household puts more weight on consumption today vis-a-vis consumption tomorrow. Note that when $t = 0$ (i.e., the initial time period) $\exp^{-\rho t} = 1$. Subsequently when $t = 1$, $\exp^{-\rho t} = \exp^{-\rho}$; when $t = 2$, $\exp^{-\rho t} = \exp^{-2\rho}$ and so on, such that $1 > \exp^{-\rho} > \exp^{-2\rho}$ .... Therefore in the infinite-time utility function W, current utility at period zero is associated with the highest weight (unity), and each subsequent utility term is associated with lower and lower weights.

The social planner maximizes the integral W, but he is constrained by the fact that at each point of time, the economy's total consumption and total investment cannot exceed its total output.[1] To put it formally, $C_t + \frac{dK}{dt} = Y_t$, where C_t is ***aggregate consumption*** in period t; $\frac{dK}{dt}$ is the amount of investment in period t which augments the capital stock (K) of the economy and Y_t is the total output produced in period t.

Total population at time t is given by L_t, which is assumed to grow at a positive exogenous rate n: $\frac{1}{L}\frac{dL}{dt} = n$.

Output at any point of time is produced using the existing capital and labour at that point of time. Technology is represented by a neoclassical production function $Y_t = F(K_t, L_t)$, where F is continuous, concave and exhibits constant returns to scale

[1] We rule out international borrowing here.

(CRS). We have explained the neoclassical production function in Unit 3, while discussing Solow growth model. The CRS property of the production function implies that per capita output (y) can be written as the function of the capital-labour ratio (k) in the following way:

$$y \equiv \frac{Y}{L} = \frac{F(K,L)}{L} = \frac{L\,F(K/L,1)}{L} = F\left(\frac{K}{L},1\right) \equiv f(k).$$

Moreover the marginal products of capital and labour can also written as the following functions of the capital-labour ratio:

$$\frac{\partial F}{\partial K} = f'(k)\ ;\ \frac{\partial F}{\partial L} = f(k) - kf'(k).$$

The continuity and concavity properties of $F(L,K)$ ensure that $f(k)$ is also continuous and concave. Additionally we assume that $f(0) = 0$; $f'(0) = \infty$; $f'(\infty) = 0$. The first of these assumptions implies that no production is possible with zero capital-labour ratio. The last two assumptions are known as the *Inada conditions* which state that when the capital-labour ratio tends to zero, the marginal product of capital tends, to infinity, while an infinitely high capital-labour ratio implies that the corresponding marginal product of capital approaches zero.

The aggregate resource constraint for the economy is, $C_t + \frac{dK}{dt} = Y_t$. Dividing both sides by L, we can write it as $\frac{C_t}{L_t} + \frac{1}{L_t}\frac{dK}{dt} = \frac{Y_t}{L_t}$, i.e., $c_t + \frac{dk}{dt} + nk_t = f(k_t)$ where the last equation denotes the resource constraint faced by the social planner in per capita terms.[2]

The economy starts with a given amount of capital stock and population; hence the initial capital-labour ratio is given, denoted by k_0. At every point of time the existing capital and labour stocks are fully employed. Hence the capital-labour ratio k also denotes the per capita capital stock. While referring to k we shall use these two terms interchangeably.

The complete dynamic optimization problem for the social planner can now be written in terms of the two time dependent variables per capita consumption (c_t) and per capita capital stock (k_t):

$$\text{Maximize } W = \int_0^\infty u(c_t)\exp^{-\rho t}\,dt \text{ subject to } \frac{dk}{dt} = f(k_t) - nk_t - c_t;\ k_0 \text{ given.}$$

The corresponding Hamiltonian function and the first order conditions in terms of the

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[2] Note that $\frac{dk}{dt} = \frac{1}{L}\frac{dK}{dt} - \frac{K}{L}\left(\frac{1}{L}\frac{dL}{dt}\right) = \frac{1}{L}\frac{dK}{dt} - nk$.

[3] See the Mathematical Appendix for the definitions of the Hamiltonian function, control, state and co-state variables in a dynamic optimization problem.

control variable (c_t), state variable (k_t) and co-state variable (λ_t) are given by:[3]

$$H = u(c_t)\exp^{-\rho t} + \lambda_t\left[f(k_t) - nk_t - c_t\right]$$

ia) $$u'(c_t)\exp^{-\rho t} = \lambda_t$$

iia) $$\frac{d\lambda}{dt} = -\lambda_t(f'(k_t) - n)$$

iiia) $$\frac{dk}{dt} = f(k_t) - nk_t - c_t$$

iva) $$\lim_{t\to\infty} \lambda_t k_t = 0$$

Notice that λ_t is the shadow price of capital (as explained in the Appendix) or the value of capital in utility terms. Hence condition (ia) states that along the optimal path the present discounted value of marginal utility from consumption would be equal to the shadow price of capital. The economic significance of this condition would become clear if you note that at any point of time one unit of output can be put to two different uses – one can either consume it or invest it which augments the capital stock. Now optimality condition requires that the returns in terms of utility from these two uses should be equal, which is precisely what condition (ia) states. Conditions (iia) and (iiia) respectively show how the co-state and the state variables change over time along the optimal path. The fourth condition, known as the ***transversality condition,*** implies that as the economy approaches its terminal time (which in this case is infinity), either the value of capital goes to zero (in which case it does not matter for the economy if it leaves a positive capital stock at the end), or, if the value of the capital stock is positive, then the economy uses up all its capital stock (i.e., k goes to zero).

Differentiating (ia) with respect to t and using (iia) to eliminate λ_t, we get the following differential equation: $\frac{dc}{dt} = \left(\frac{u'(c_t)}{-u''(c_t)}\right)(f'(k_t - n - \rho)$. Noting that the term $\frac{-cu''(c)}{u'(c)} \equiv \sigma$ denotes the elasticity of marginal utility with respect to consumption, the above equation can be written as:

va) $$\frac{dc}{dt} = \left(\frac{c_t}{\sigma}\right)(f'(k_t - n - \rho)$$

Equations (iiia) and (va) together form a system of differential equations which, along with the transversality condition, determine the movements of per capita consumption and per capita capital stock of the economy along the optimal path. The qualitative characterization of this path is shown in the phase diagram given at Fig. 8.1.

The phase diagram traces the lines/curves along which $\frac{dc}{dt} = 0$ and $\frac{dk}{dt} = 0$. The point of intersection of these two lines is called the ***steady state***. A steady state is equivalent to long run equilibrium whereby the values of the variable remain constant over time.

From (va), $\frac{dc}{dt}=0$ implies either $c=0$ or $f'(k)=n+\rho$. On the other hand, from (iiia), $\frac{dk}{dt}=0$ implies $c=f(k)-nk$. Plotting all these lines and curves in the c-k plane with c along the vertical axis and k along the horizontal axis, we get a diagram as shown in Fig. 8.1.

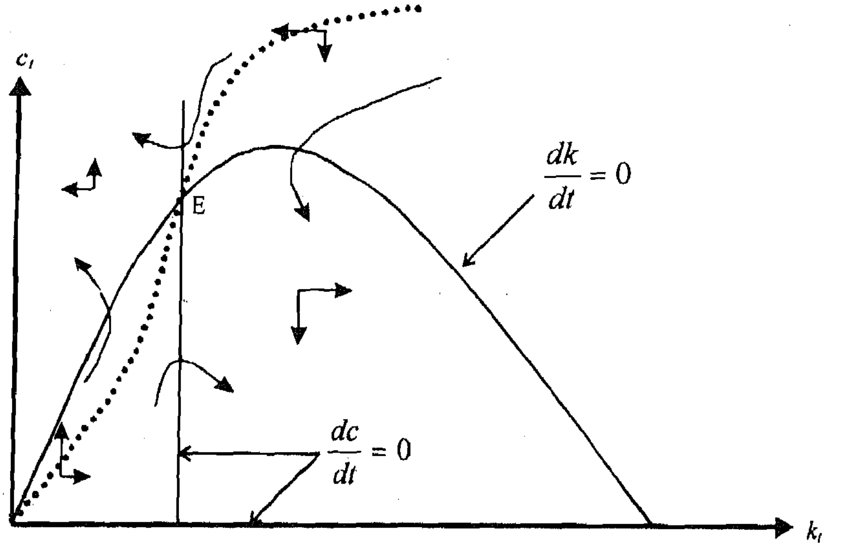

Fig. 8.1: **Phase Diagram**

The direction of arrows in the phase diagram shows the direction of movements of c and k when they are not on the $\frac{dc}{dt}=0$ and $\frac{dk}{dt}=0$ lines respectively. From Fig. 8.1 it is clear that there exits one steady state, denoted by point E in the diagram, which is associated with positive values of c and k. It can be shown (though we make no attempt to prove it here) that this steady state point is a 'saddle point' such that there exits a **unique** path which approaches the steady state point; all the other paths move away **from** the steady state point. This unique path is denoted by the dotted line in the **diagram.** This is the only path **which** satisfies all the first order conditions of **optimality** (conditions **(ia)-(iva)** given above) including the **transversality** condition. This path is therefore the optimal path which satisfies the social planner's dynamic optimization problem. Given any initial capital-labour **ratio,** the social planner will choose the initial per **capita consumption** so **as** to be on this optimal path, which will eventually take the economy to **its** long run equilibrium (or steady state) point E.

At this juncture it is important to point out certain important features of the steady state point E. As was mentioned before, at the steady state both the per capita capital stock, k, and per capita consumption, c, are constant. Also, at this non-zero steady state, the k-value **and** the corresponding c-value **are** defined respectively by the equations: $f'(k)=n+\rho$ and $c=f(k)-nk$. The condition that $f'(k)=n+\rho$ is called the ***modified golden rule.*** It states that the steady state value of the capital-labour ratio is such that the marginal product of capital is **equal** to the **sum** of the rate of population growth and the **rate** of time preference. In any dynamic model of **capital accumulation and** growth the concept of golden rule plays an important role (see Unit 3). **The** ***golden rule*** itself refers ***to*** that steady **state** value of **capital-labour ratio** where $f'(k)=n$, i.e., the **marginal product** of capital is exactly **equal** to the rate of growth of **population. The significance** of the golden rule obtains from the fact **that** it is that steady state value of capital-labour ratio which corresponds to the maximum steady state value of per capita consumption. Note that when households' rate of time preference is zero, the modified golden rule coincides with the golden rule and the steady state of the Ramsey model is attained at the golden rule point. However, when people have positive rate of time preference, they are unwilling to sacrifice current consumption for future consumption beyond a point; as a result they accumulate less and reach a lower level of steady state consumption than in the golden rule.

8.3 DECENTRALISED HOUSEHOLDS' PROBLEM

The Ramsey model described above can be easily transformed into the problem of a decentralized competitive market economy consisting of many households – all with identical preferences. We shall see that under certain assumptions the optimal solution path in the two cases turns out to be identical.

Instead of assuming the existence of a central planner, let us consider a market economy where the households are price takers. There are two competitive factor markets which determine the wage rate (w_t) and the interest rate or the rate of return on capital (r_t) at each point of time.

There exist many identical competitive firms which hire in labour and capital from the households at the above mentioned wage rate and rental rate. Using labour and capital these firms produce the final output, using the same technology as the central planner. Competition ensures that the wage rate and the rental rate are equal to the respective marginal products of labour and capital at full employment, i.e., $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$.

Each household maximizes its welfare given as before by $W = \int_0^{\infty} u(c_t) \exp^{-\rho t} dt$. Each household starts with a certain amount of capital stock and labour stock. Additionally households can borrow from one another. Let $a_t \equiv k_t - b_t$ denote the per capita asset stock of an household at period t, which consists of the per capita capital stock owned by the household at period t minus the amount of debt (per capita) at period t. (If the household is a net lender, then b_t would be negative). Arbitrage condition in the asset market ensures that physical capital and lending earns the same rate of return. Hence the per capita income of the household at period t is $w_t + r_t a_t$, which the household spends on consumption and further asset accumulation. Thus the budget constraint faced by the household in per capita terms is given by: $c_t + \frac{da}{dt} + na_t = w_t + r_t a_t$.

The complete dynamic optimization problem for the household can now be written in terms of the two time dependent variables per capita consumption (c_t) and per capita asset stock (a_t):

Maximize $W = \int_0^{\infty} u(c_t) \exp^{-\rho t} dt$ subject to $\frac{da}{dt} = w_t + (r_t - n)a_t - c_t$; a_0 given.

The corresponding Hamiltonian function and the first order conditions in terms of the control variable (c_t), state variable (a_t) and co-state variable (λ_t) are given by:

$$H = u(c_t)\exp^{-\rho t} + \lambda_t\left[w_t + (r_t - n)a_t - c_t\right]$$

ib) $u'(c_t)\exp^{-\rho t} = \lambda_t$

iib) $\frac{d\lambda}{dt} = -\lambda_t(r_t - n)$

iiib) $\frac{da}{dt} = w_t + (r_t - n)a_t - c_t$

ivb) $\lim_{t\to\infty} \lambda_t a_t = 0$

Note that these conditions look quite similar to the first order conditions that we obtained for the central planner's problem (conditions (ia)-(iva) in the previous section). However, for the household's problem there is an additional condition which has to be satisfied if we want to get a meaningful solution. This condition is called the ***No-Ponzi-Game condition*** which we elaborate below. First note that we have allowed for intra-household borrowing which means that a household can maintain a consumption stream above its income by borrowing. Now if a household could go on borrowing indefinitely then it will be optimal for the household to always maintain an infinitely high (or maximum possible) consumption stream and finance such high level of consumption simply by borrowing more and more. Of course, such a strategy would also imply that the net per capita borrowing of the household would increase exponentially at the rate (r_t - n) (since the household has to borrow not only for consumption, but also to pay back the interest rate as well) and the present discounted value of the net debt of the household will approach infinity. Such a financing scheme is called *Ponzi-Game financing*[4]. In order to rule out such infinite indebtedness by any family we specify the condition that the even though the household can be temporarily a net debtor (i.e., the present discounted value of its asset is temporarily negative), over a sufficiently longer horizon, it must eventually repay all its debt and hold non-negative asset stock. Formally, as *t* goes to infinity, the present discounted value of the (per capita) asset stock of the household must be non-negative:

vb) $\lim_{t\to 0} a_t \exp^{-\int_0^t (r_\tau - n)d\tau} \geq 0 \cdot$

condition (vb) is called the No-Ponzi-Game condition. For a decentralized household conditions (ib)-(vb) specify the optimal path the solves the household's dynamic optimization problem.

As before, differentiating (ib) with respect to *t* and using (iib) to eliminate λ_t, and noting that the term $\frac{-cu''(c)}{u'(c)} \equiv \sigma$ denotes the elasticity of marginal utility with respect

4 Named after Charles Ponzi (1877-1949) who raised considerable amount of money promising a high rate of interest (50% for 45 days, 100% for 90 days). As long as new lenders were attracted to these returns, he was able to repay previous debt. In eight months he ended up with 10 million dollars of certificates and 14 million of debt. Criminal charges were finally levied on him in the US and died a pauper.

to consumption, we can derive the following differential equation:

vib) $$\frac{dc}{dt} = \left(\frac{c_t}{\sigma}\right)(r_t - n - \rho)$$

Equations (iiib) and (vib) together form a system of differential equations which, along with the transversality condition and the no-Ponzi-Game condition, determine the movements of per capita consumption and per capita asset stock of the household along the optimal path.

Note that while for the economy as a whole the wage rate and the rental rate are given by $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$, the price-taking households take these as given while optimizing. However we assume that the households are endowed with perfect foresight, which implies that though they ***do not know*** the relationship between w and r with the per capita capital stock, they can exactly ***guess*** the values of w and r at every point of time.

Also note that while the households can borrow from one another, in equilibrium for the economy as a whole net borrowing must be zero. Since households are all identical, this implies that net borrowing of each household would be zero, i.e., $a_t = k_t$.

Thus replacing $r_t = f'(k_t)$, $w_t = f(k_t) - k_t f'(k_t)$, and $a_t = k_t$ in equations (iiib) and (vib), we get a system of differential equation which is exactly identical to the system of differential equations that we obtained for the central planner. Hence their solution must also be the same. In other words, under the assumption of perfect foresight on the part of the households, the household's optimal consumption and accumulation paths are exactly identical to the consumption and accumulation paths that would be chosen by a social planner. To put it differently, the decentralized market economy's solution path coincides with the solution path of the centralized command economy, and the solution paths can once again be characterized by the diagram given at Fig. 8.1. This is a very strong result which shows the equivalence between the market equilibrium and the socially optimal solution. This equivalence result however depends crucially on the assumption of perfect foresight.

Check Your Progress 1

1) What are the steady state values of per capita consumption and per capita capital stock in the centralised version of the Ramsey optimal growth model?

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2) Would these values change if we consider a decentralized economy instead of a centralised planner's problem? Explain your answer.

8.4 GOVERNMENT IN RAMSEY MODEL: RICARDIAN EQUIVALENCE

The decentralized equilibrium that we discussed above does not incorporate government as a separate economic agent. It is however easy to introduce a role for the government in the de-centralized Râmsey model, whereby the government finances its own expenditure either by imposing a lump-sum tax on the household's income in every period. Since the government expenditure is unrelated to the households' consumption expenditure, the utility function of the households remain unchanged. However since the government is taking away a part of the household's income, the budget constraint of the household changes. To be more precise, if the government imposes a tax τ on the household income then the disposable income of the household reduces to $w_t + r_t a_t - \tau$; consequently the budget constraint of the household becomes: $\frac{da}{dt} = w_t + (r_t - n)a_t - c_t - \tau$. The rest of the optimal condition remains the same. In the phase diagram therefore, the $\frac{dk}{dt} = 0$ curve shifts downward by the amount τ, as shown in Fig. 8.2. The new steady state is obtained at the point E′ in Fig. 8.2, where the steady state value of the per capita capital stock remains the same, but steady state value of per capita consumption is lower. The corresponding optimal path of the household is once again denoted by the dotted line in the diagram.

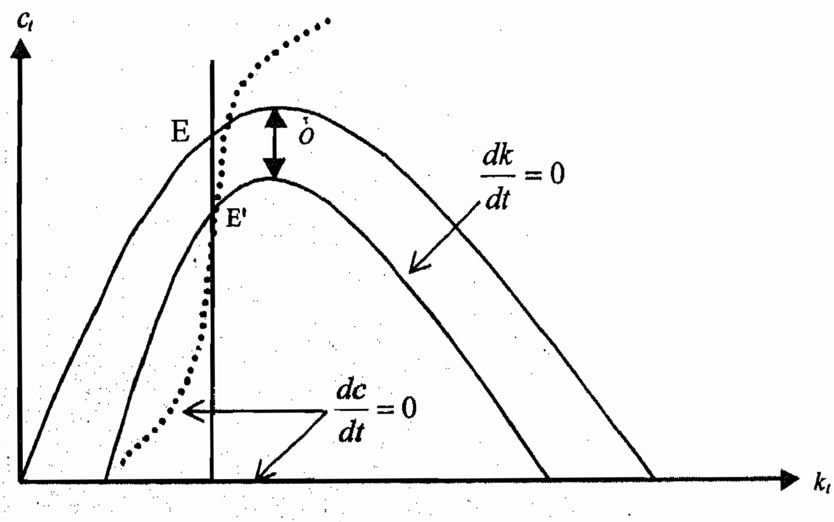

Fig. 8.2: Phase Diagram (with Government)

Now suppose the government decides to finance its expenditure by borrowing from the households, instead of taxation. Will that make any difference to the households' optima? At first glance it seems that if the government borrows from the households by distributing government bonds, then the government bonds would constitute an asset for the households (because the government would pay an interest on these loans). So the household's disposable income is not reduced; only the composition of its asset portfolio changes. So at first glance it seems that the household's optima under tax financing and under debt financing would be different. However, a more careful consideration of the problem reveals that the mode of financing of the government expenditure really **does not** make any difference to the household's optima. As long the government is forced to maintain a balanced budget in the long run, that is, it cannot go on borrowing indefinitely, at some point of time in future the government has to impose a tax in order to repay its debt incurred earlier. Since the households have perfect foresight, they would know that at some point of time they would have to pay the tax. Thus even though they are not paying the taxes today, they will adjust their current consumption and capital accumulation decisions accordingly which will enable them to bear the tax burden in future. This is a strong result which states that for optimizing households endowed with perfect foresight (or rational expectation), the mode of financing of government expenditure does not matter: tax financing and debt financing are exactly equivalent as far as the household optima is concerned. This result is known as the *Ricardian Equivalence Result.*

Check Your Progress 2

1) In the decentralized version of the optimal growth model, how would the equilibrium values of per capita capital stock and the per capita consumption change if the government imposes a lump sum tax to finance its own expenditure? In this context, explain the Ricardian Equivalence proposition.

8.5 LET US SUM UP

In this unit, we have looked at the Ramsey-Cass-Koopmans optimal growth model where infinitely lived individuals maximize their life-time utility. We identify the optimal consumption-accumulation path as the saddle path which takes the economy to its steady state in the long run. The steady state point is identified with the modified golden rule point. Although the problem is identified as a social planner's problem, its basic conclusion remains unchanged even when the economy is characterized by competitive decentralized households. Moreover, introducing a government sector which finances its expenditure through lump sum taxation does not have any effect on the steady state values of the model.

8.6 KEY WORDS

Infinite Horizon Dynamic Optimization Problem: An optimization exercise where the objective function is defined over several time periods starting from time zero to infinity, and where the relevant variables are dynamically linked from one period to another through a dynamic (difference or differential) equation.

Rate of Time Preference: The rate at which future utility is discounted vis-à-vis current utility.

Steady State: A long run equilibrium point where all the relevant dynamic variables remain constant over time.

Phase Diagram: A diagram which plots the dynamic (difference or differential) equations in order to characterize the direction of movements of the variables over time.

Golden Rule: When the marginal product of capital in an economy is equal to the rate of population growth. This is also the point where the steady state value of per capita consumption is at its maximum.

Modified Golden Rule: When the marginal product of capital in an economy is equal to the sum of the rate of population growth and the rate of time preference.

No-Ponzi-Game Condition: A condition that rules out the situation where an individual keeps on borrowing to meet previous debt obligations and eventually in the log run ends up with negative assets.

Ricardian Equivalence: A situation where the government's policy to influence some economic outcome is nullified by the actions of the private agents so that there is a equivalence between the pre-policy (competitive) and post-policy economy.

8.7 SOME USEFUL BOOKS

The standard references for the optimal growth model of section 8.2 are the following two classic articles by Ramsey and Cass:

Ramsey, Frank P., 1928, "A Mathematical Theory of Savings", *Economic Journal*, Vol. 38, pp. 543-559. Reprinted in Sen, Amartya (ed): *Growth Economics*, Penguin, 1970.

Cass, David, 1965, "Optimum Growth in an Aggregate Model of Capital Accumulation", *Review of Economic Studies*, Vol. 32, pp. 233-240.

All the relevant topics, including the Ricardian Equivalence proposition, can be found in the following textbook:

Blanchard, Olivier and Stanley Fischer, 1989, *Lectures on Macroeconomics*, MIT Press, chapter 2.

8.8 ANSWER/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Read Section 8.2 and write your answer.

2) Read Section 8.3 and write your answer.

Check Your Progress 2

3) Read Section 8.4 and write your answer.

8.9 MATHEMATICAL APPENDIX

Dynamic Optimisation in Continuous Time (Optimal Control)

A.1 Finite Horizon Problem

Consider the following optimisation problem which is defined over a finite time horizon from 0 to T:

$$\text{Max. } W = \int_0^T F(u_t, x_t, t)\,dt, \text{ subject to } \frac{dx}{dt} = f(u_t, x_t, t)$$
$$x_0 = \bar{x}; \quad u_t \in U. \qquad \ldots(1)$$

The objective function here is an ***integral***, and our task is to find out a time path of the time dependent variable u from the choice set U, (i.e., to choose a $u \in U$ for each point of time t starting from 0 to T) such that this integral is maximized.

But our choice is not unconstrained. (Had it been so, a simple point-by-point static optimization exercise would have given us the required solution path). Note that the F function (which can be called the ***instantaneous*** objective function) depends not only on u but also on another time dependent variable x. And our choice of u at each point of time affects the next period's value of x through the given differential equation. Thus our choice of u affects the instantaneous objective function directly, as well as indirectly through x. So when we are choosing the optimal time path for u that maximizes the integral W, we have to take into account how x is changing due to our choice of u – this explains the presence of a constraint function in the form of a differential equation in x.

Since we are choosing the variable u directly (i.e., we have direct control over this variable) it is called the ***control variable***. The other time dependant variable, x, which changes over time due to our choice of u, but we do not have direct control over its change, is called the ***state variable***. The function f in the given differential equation that describes the change in x over time is called the ***state transition function***.

Pontryagin's Maximum Principle: Let u_t^* be a solution path to the problem specified in (1), and let x_t^* be the associated path for the state variable, where u_t^* is a piece-wise continuous function of t and x_t^* is a strictly continuous but piece-wise differentiable in t. Then there exists a strictly continuous and piece-wise differentiable variable λ_t, and a function H defined as:

$$H(u,x,\lambda,t) = F(u_t,x_t,t) + \lambda_t f(u_t,x_t,t),$$

such that

i) H is maximised with respect to u at u^* for all $t \in [0,T]$

ii) $$\left.\frac{\partial H}{\partial x}\right|_{(u^*,x^*,\lambda,t)} = -\frac{d\lambda}{dt}$$

iii) $$\left.\frac{\partial H}{\partial \lambda}\right|_{(u^*,x^*,\lambda,t)} = \frac{dx}{dt}$$

iv) $\lambda_T x_T = 0$

The function H is called the ***Hamiltonian Function*** associated with the given dynamic optimizations problem, and the newly introduced time dependent variable λ is called the ***co-state variable*** associated with the state variable x.

The co-state variable λ_t measures the change in the value of the objective function W associated with an infinitesimal change in the state variable x at time t. If there were an exogenous tiny increment to the state variable at time t, and if the problem were modified optimally there after, then the increment in the total value of the objective would be λ_t. Thus it is the marginal valuation of the state variable at time t, and is therefore often referred to as the ***shadow price*** of the state variable at time t.

Pontryagin's Maximum Principle gives us four ***first order necessary conditions*** for the optimization problem defined in (1). The first three F.O.N.C's are defined in terms of the Hamiltonian function. Note that if the Hamiltonian function is non-linear in u, then (i) can be replaced by the condition $\left.\frac{\partial H}{\partial u}\right|_{(u^*,x,\lambda,t)} = 0$, **provided the second order check** $\left.\frac{\partial^2 H}{\partial u^2}\right|_{(u^*,x,\lambda,t)} < 0$ **is verified.**

The last condition of the Maximum Principle (condition (iv)), which specifies a terminal condition for λ, is called the ***Transversality Condition***.

A.2 Infinite Horizon Problems

Consider the following infinite horizon dynamic optimization exercise:

$$\text{Max. } W = \int_0^{\infty} F(u_t, x_t, t)\,dt, \text{ subject to } \frac{dx}{dt} = f(u_t, x_t, t)$$
$$x_0 = \overline{x};\ u_t \in U. \qquad \ldots(2)$$

This problem is almost the same as the problem specified in (1); only now the time horizon is no longer finite.

When the time horizon stretches from 0 to ∞, additional complications may arise: Firstly, the integral W may not converge. In that case it does not make sense to talk about optimal paths, because any paths (u_t, x_t) will eventually make the integral infinitely large, and no comparison between one set of (u_t, x_t) with another is possible. Hence for the dynamic optimization exercise to be meaningful, we must impose additional restrictions so that the integral converges to a finite value.

Secondly, there is some controversy regarding the appropriate transversality condition in infinite horizon problems. Usually some limiting conditions, which are similar to the transversality conditions applied in the finite horizon problems, are applied.

Consider problem (2) and suppose the integral converges. The first order non-essential conditions for this problem are given in the following theorem:

Let u_t^* be a solution path to the problem specified in (5), and let x_t^* be the associated path for the state variable, where u_t^* is a piece-wise continuous function of t and x_t^* is strictly continuous but piece-wise differentiable in t. Then there exists a strictly continuous and piece-wise differentiable variable λ_t, and a function H defined as:

$$H(u, x, \lambda, t) = F(u_t, x_t, t) + \lambda_t f(u_t, x_t, t),$$

such that

i) H is maximised with respect to u at u^* for all $t \in [0, \infty)$

ii) $$\left.\frac{\partial H}{\partial x}\right|_{(u^*, x^*, \lambda, t)} = -\frac{d\lambda}{dt}$$

iii) $$\left.\frac{\partial H}{\partial \lambda}\right|_{(u^*, x^*, \lambda, t)} = \frac{dx}{dt}$$

iv) $$\lim_{t \to \infty} \lambda_t x_t = 0$$